

Part VI.

Models for Context-Free

Languages

Context-Free Grammar (CFG)

Gist: A *grammar* is based on a finite set of grammatical rules, by which it generates strings of its language.

Illustration: Start nonterminal S

Grammar G :

Nonterminals: A, B, S

Terminals: a, b, c, d

Rules:

- $S \rightarrow AB,$
- $A \rightarrow aAb,$
- $A \rightarrow ab,$
- $B \rightarrow bBa,$
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S
↓
 \overbrace{AB}

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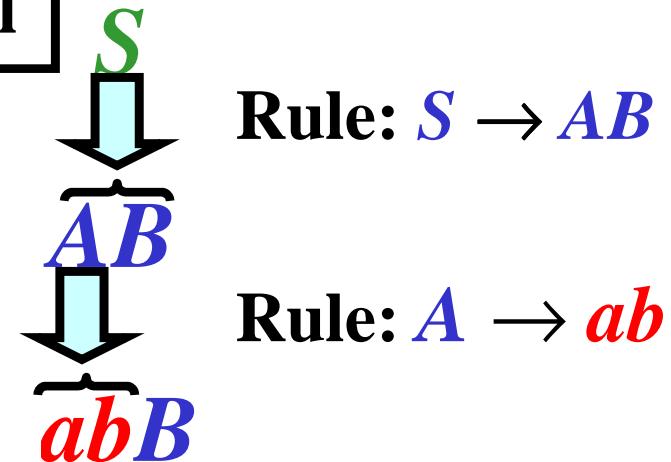
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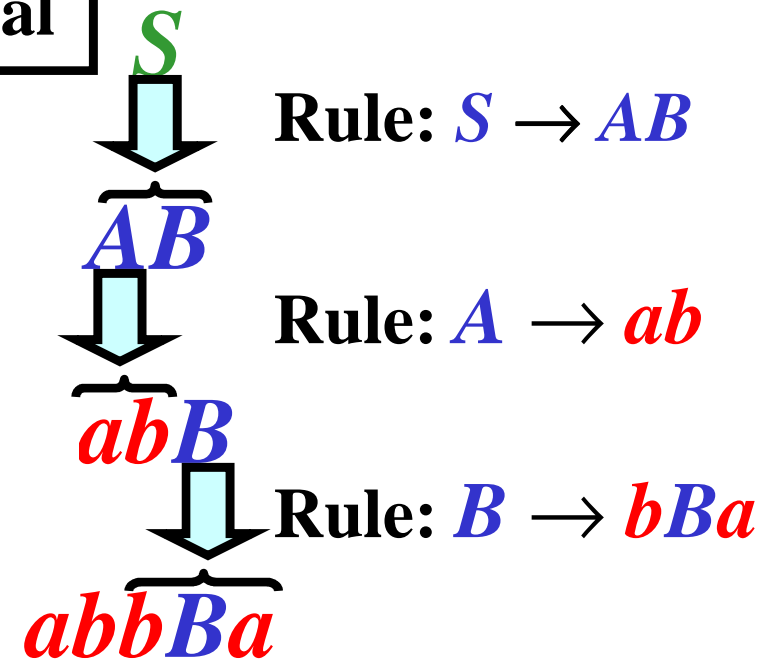
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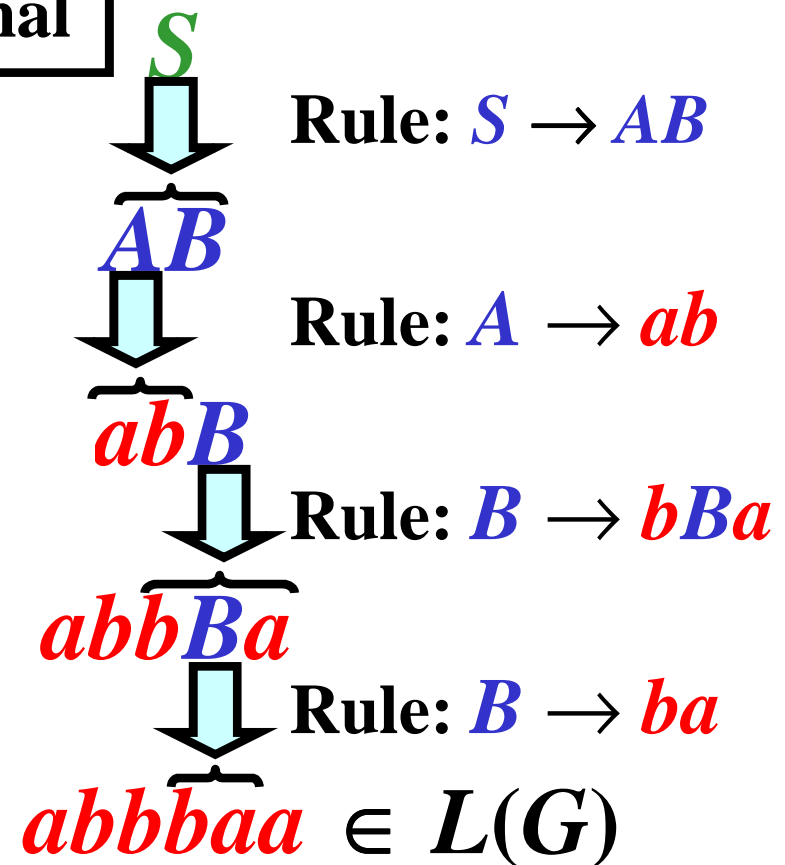
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Context-Free Grammar: Definition

Definition: A *context-free grammar* (CFG) is a quadruple $G = (N, T, P, S)$, where

- N is an alphabet of *nonterminals*
- T is an alphabet of *terminals*, $N \cap T = \emptyset$
- P is a finite set of *rules* of the form $A \rightarrow x$,
where $A \in N$, $x \in (N \cup T)^*$
- $S \in N$ is the *start nonterminal*

Mathematical Note on Rules:

- Strictly mathematically, P is a relation from N to $(N \cup T)^*$
 - Instead of $(A, x) \in P$, we write $A \rightarrow x \in P$
-
- $A \rightarrow x$ means that A can be replaced with x
 - $A \rightarrow \varepsilon$ is called *ε -rule*

Convention

- A, \dots, F, S : nonterminals
- S : the start nonterminal
- a, \dots, d : terminals
- U, \dots, Z : members of $(N \cup T)$
- u, \dots, z : members of $(N \cup T)^*$
- π : sequence of productions

A subset of rules of the form:

$$A \rightarrow x_1, A \rightarrow x_2, \dots, A \rightarrow x_n$$

can be simply written as:

$$A \rightarrow x_1 \mid x_2 \mid \dots \mid x_n$$

Derivation Step

Gist: A change of a string by a rule.

Definition: Let $G = (N, T, P, S)$ be a CFG. Let $u, v \in (N \cup T)^*$ and $p = A \rightarrow x \in P$. Then, uAv directly derives uxv according to p in G , written as $uAv \Rightarrow uxv [p]$ or, simply, $uAv \Rightarrow uxv$.

Note: If $uAv \Rightarrow uxv$ in G , we also say that G makes a *derivation step* from uAv to uxv .

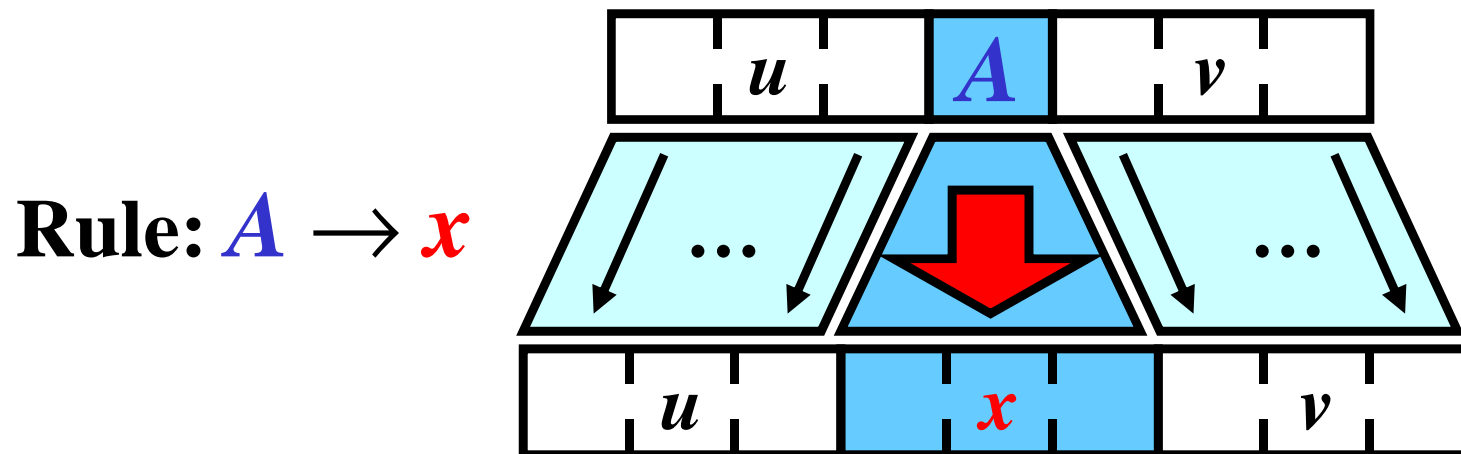


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Sequence of Derivation Steps 1/2

Gist: Several consecutive derivation steps.

Definition: Let $u \in (N \cup T)^*$. G makes a *zero-step derivation* from u to u ; in symbols,

$$u \Rightarrow^0 u [\varepsilon] \text{ or, simply, } u \Rightarrow^0 u$$

Definition: Let $u_0, \dots, u_n \in (N \cup T)^*$, $n \geq 1$, and $u_{i-1} \Rightarrow u_i [p_i]$, $p_i \in P$, for all $i = 1, \dots, n$; that is

$$u_0 \Rightarrow u_1 [p_1] \Rightarrow u_2 [p_2] \dots \Rightarrow u_n [p_n]$$

Then, G makes n *derivation steps* from u_0 to u_n ,

$$u_0 \Rightarrow^n u_n [p_1 \dots p_n] \text{ or, simply, } u_0 \Rightarrow^n u_n$$

Sequence of Derivation Steps 2/2

If $u_0 \Rightarrow^n u_n [\pi]$ for some $n \geq 1$, then u_0 *properly derives* u_n in G , written as $u_0 \Rightarrow^+ u_n [\pi]$.

If $u_0 \Rightarrow^n u_n [\pi]$ for some $n \geq 0$, then u_0 *derives* u_n in G , written as $u_0 \Rightarrow^* u_n [\pi]$.

Example: Consider

$aAb \Rightarrow a**Bb** \quad [1: A \rightarrow **aBb**],$ and

$aa**Bbb** \Rightarrow aa**cbb** \quad [2: B \rightarrow **c**].$

Then, $aAb \Rightarrow^2 aa**cbb** [1 \ 2],$

$aAb \Rightarrow^+ aa**cbb** [1 \ 2],$

$aAb \Rightarrow^* aa**cbb** [1 \ 2]$

Generated Language

Gist: *G generates a terminal string w by a sequence of derivation steps from S to w*

Definition: Let $G = (N, T, P, S)$ be a CFG. The *language generated by G , $L(G)$* , is defined as

$$L(G) = \{w: w \in T^*, S \Rightarrow^* w\}$$

Illustration:

$G = (N, T, P, S)$, let $w = a_1 a_2 \dots a_n$; $a_i \in T$ for $i = 1..n$

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Illustration:

$G = (N, T, P, S)$, let $w = a_1 a_2 \dots a_n$; $a_i \in T$ for $i = 1..n$

if $S \Rightarrow \dots \Rightarrow \dots \Rightarrow \underbrace{a_1 a_2 \dots a_n}_w$ then $w \in L(G)$;

otherwise, $w \notin L(G)$

Context-Free Language (CFL)

Gist: A language generated by a CFG.

Definition: Let L be a language. L is a *context-free language* (CFL) if there exists a context-free grammar that generates L .

Example:

$G = (N, T, P, S)$, where $N = \{S\}$, $T = \{a, b\}$,
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 $S \Rightarrow aSb$ [1] $\Rightarrow ab$ [2] \nearrow
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 \vdots

$L(G) = \{a^n b^n : n \geq 0\}$

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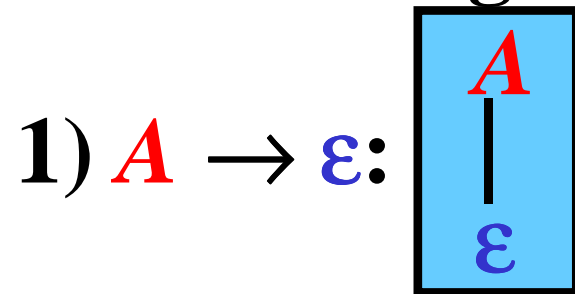
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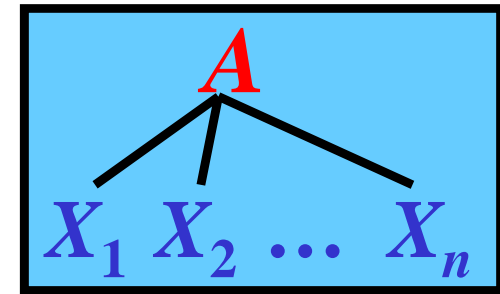
$L = \{a^n b^n : n \geq 0\}$ is a CFL.

Rule Tree

- Rule tree graphically represents a rule



2) $A \rightarrow X_1 X_2 \dots X_n$:

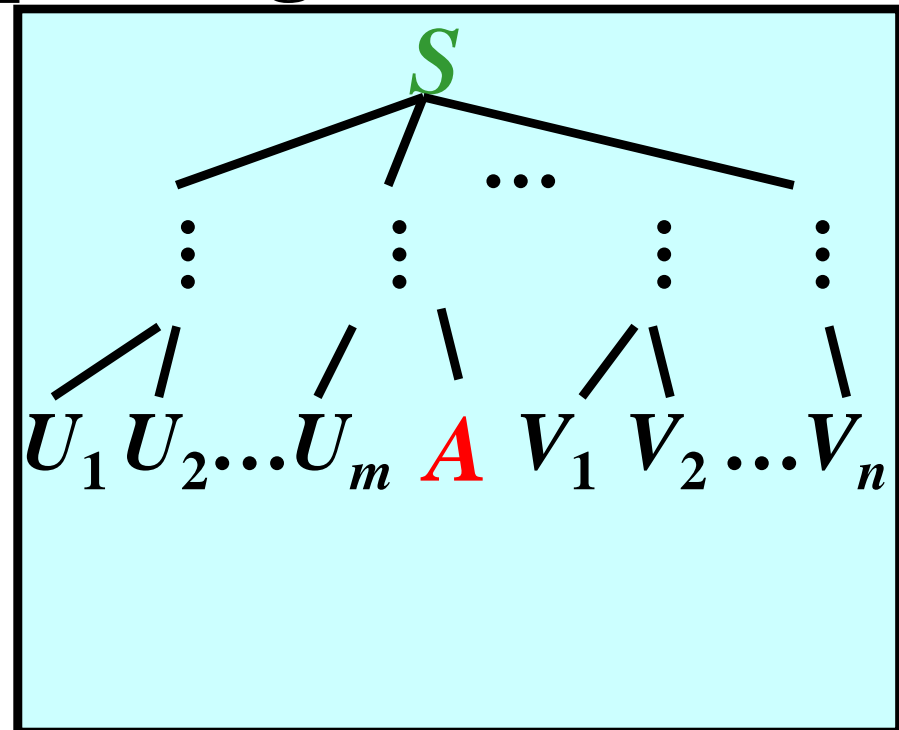


- Derivation tree corresponding to a derivation

$S \Rightarrow \dots$

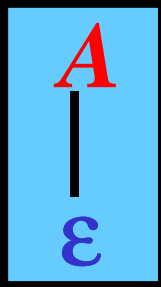
\vdots

$\Rightarrow U_1 U_2 \dots U_m A V_1 V_2 \dots V_n$

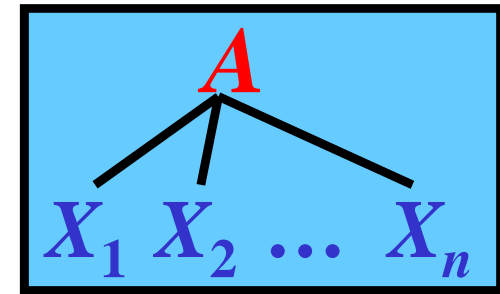


Rule Tree

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1) $A \rightarrow \epsilon$: 

2) $A \rightarrow X_1 X_2 \dots X_n$:



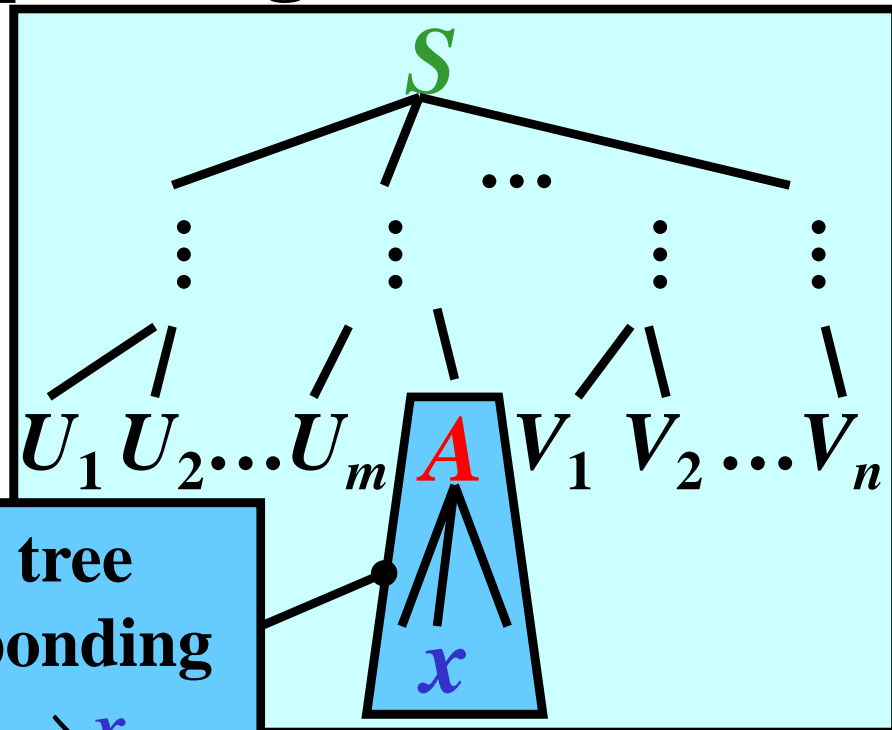
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\vdots

$\Rightarrow U_1 U_2 \dots U_m A V_1 V_2 \dots V_n$

$\Rightarrow U_1 U_2 \dots U_m x V_1 V_2 \dots V_n$



Rule tree
corresponding
to $A \rightarrow x$

Derivation Tree: Example

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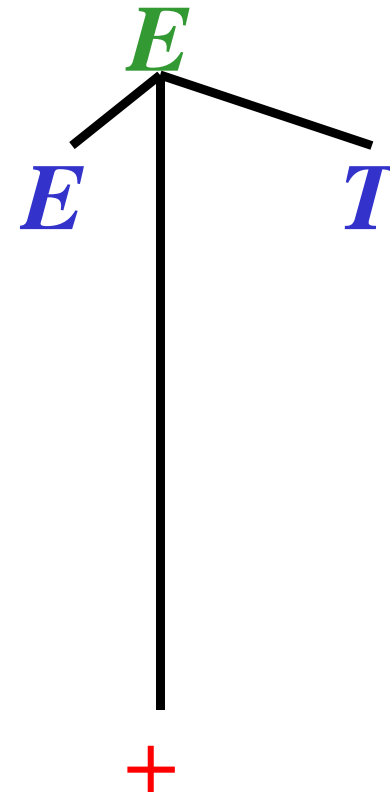
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Derivation:

$\underline{\mathbf{E}} \Rightarrow \mathbf{E} + \underline{\mathbf{T}} \quad [\mathbf{1}]$

Derivation tree:



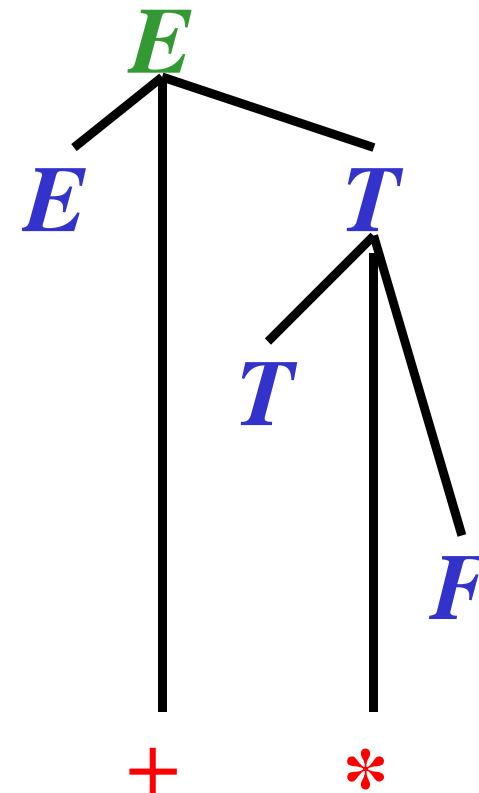
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Derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow \mathbf{E} + \underline{\mathbf{T}} && [\mathbf{1}] \\ &\Rightarrow \mathbf{E} + \underline{\mathbf{T}} * \mathbf{F} && [\mathbf{3}] \end{aligned}$$

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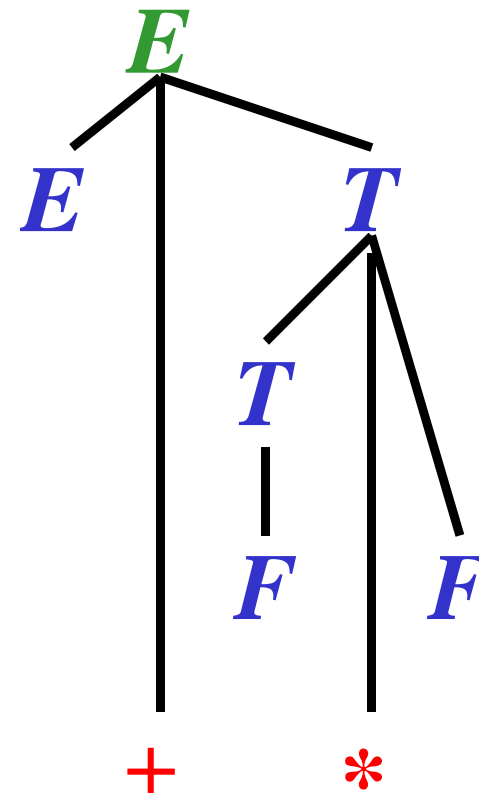
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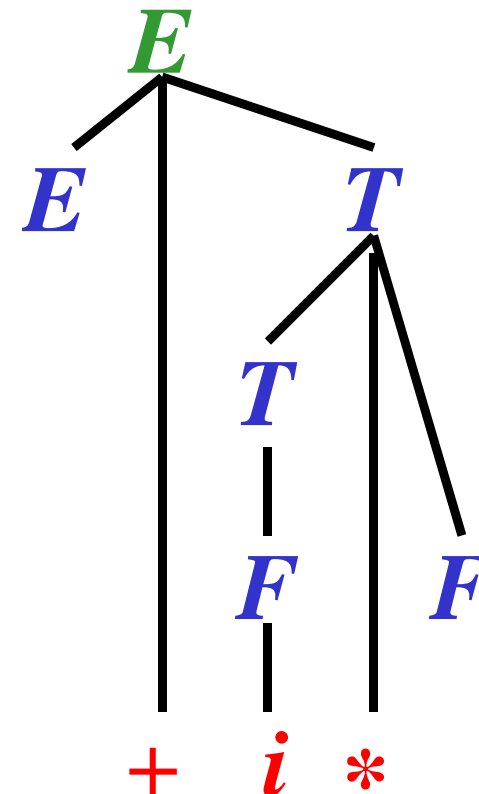
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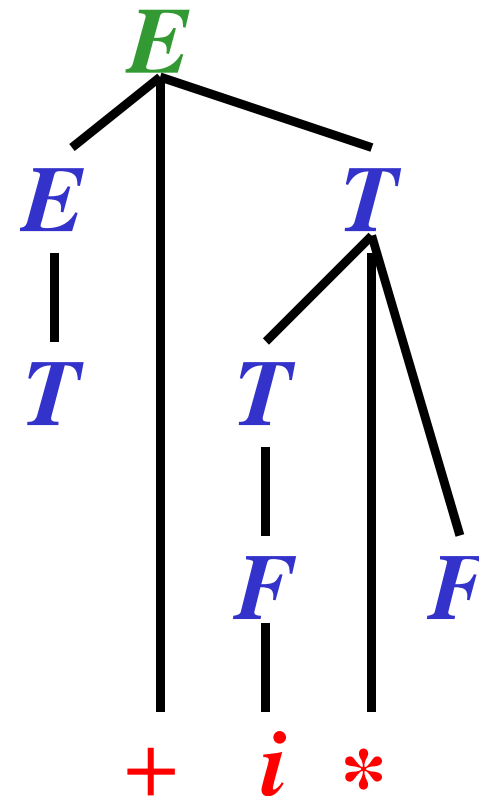
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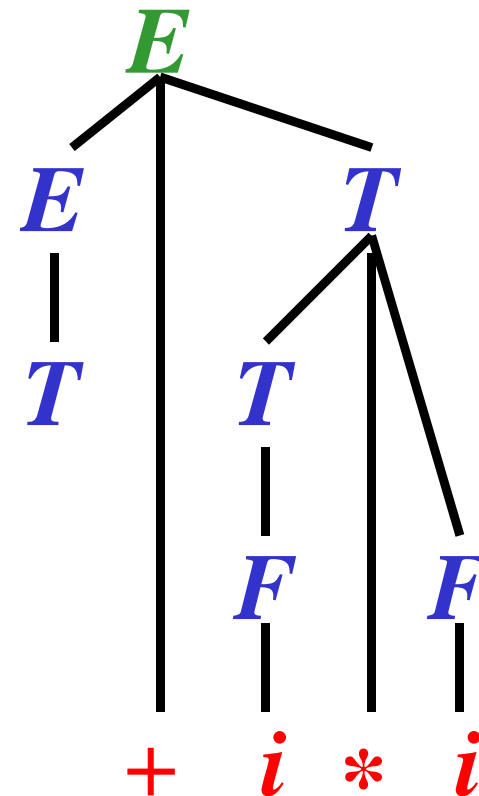
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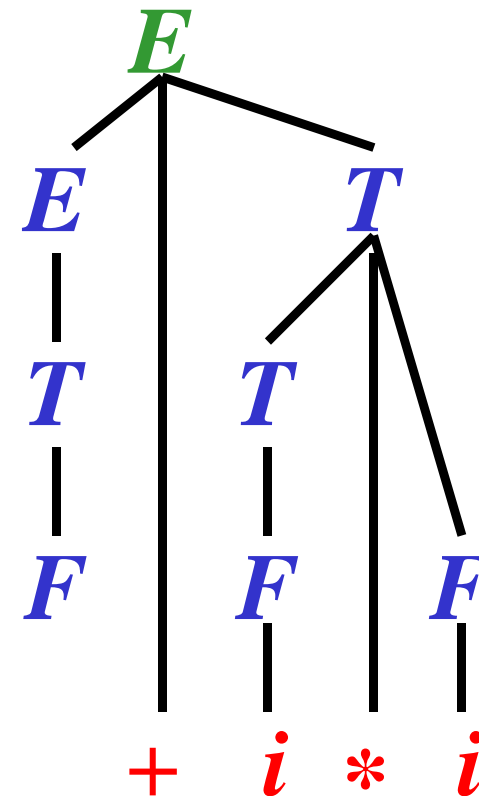
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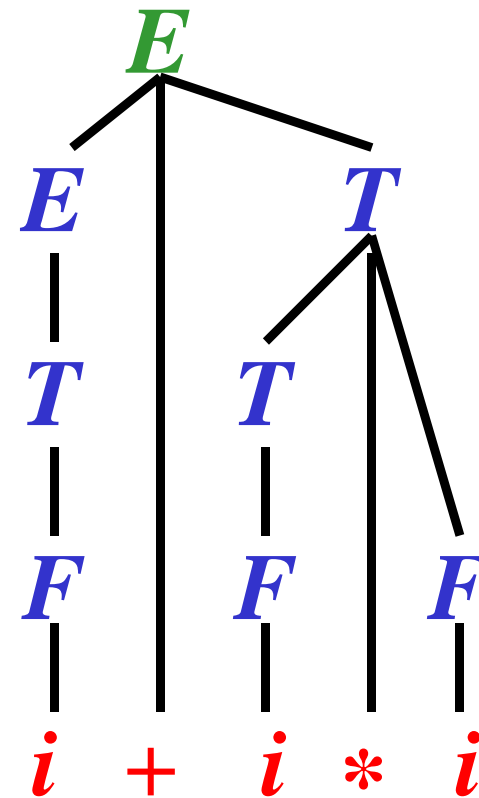
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Derivation tree:



Leftmost Derivation

Gist: During a *leftmost derivation step*, the **leftmost nonterminal is rewritten**.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $u \in T^*$, $v \in (N \cup T)^*$. Let $p = A \rightarrow x \in P$ be a rule. Then, uAv directly derives uxv in the *leftmost way* according to p in G , written as

$$uAv \Rightarrow_{lm} uxv [p]$$

Note: We define \Rightarrow_{lm}^+ and \Rightarrow_{lm}^* by analogy with \Rightarrow^+ and \Rightarrow^* , respectively.

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Leftmost derivation:

Derivation tree:

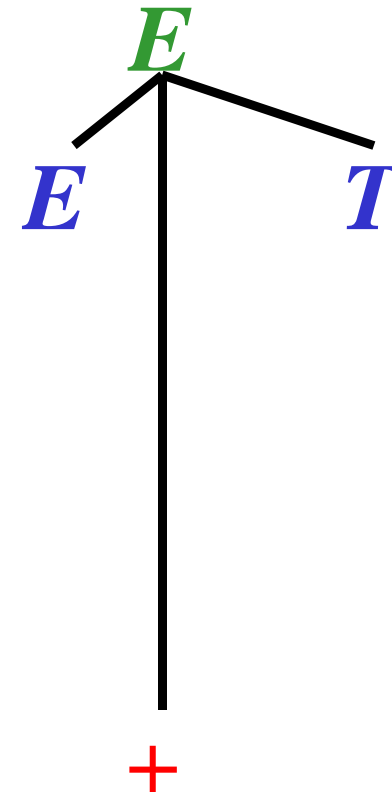
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$\underline{\mathbf{E}} \Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} \quad [\mathbf{1}]$

Derivation tree:



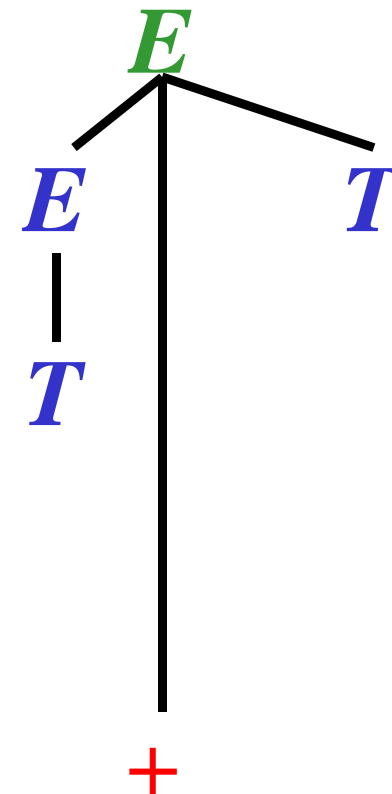
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} & [\mathbf{1}] \\ &\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} & [\mathbf{2}] \end{aligned}$$

Derivation tree:



Leftmost Derivation: Example

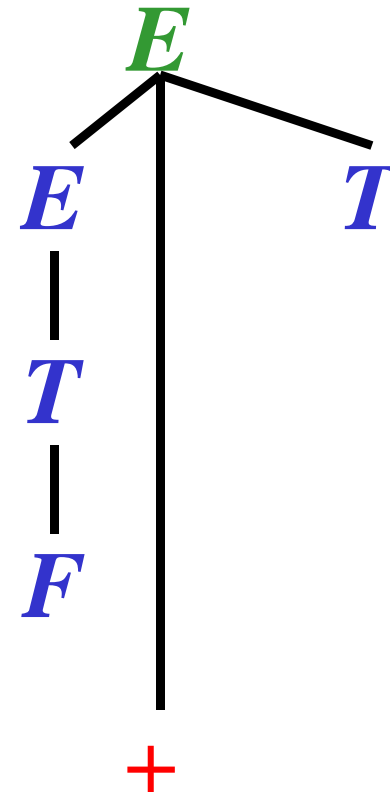
$$G = (N, T, P, \mathbf{E}), \text{ where } N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}, T = \{\mathbf{i}, +, *, (,)\},$$

$$P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E}+\mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T}*\mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$$

Leftmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{lm} \underline{E} + T & [1] \\ &\Rightarrow_{lm} \underline{T} + T & [2] \\ &\Rightarrow_{lm} \underline{F} + T & [4] \end{aligned}$$

Derivation tree:



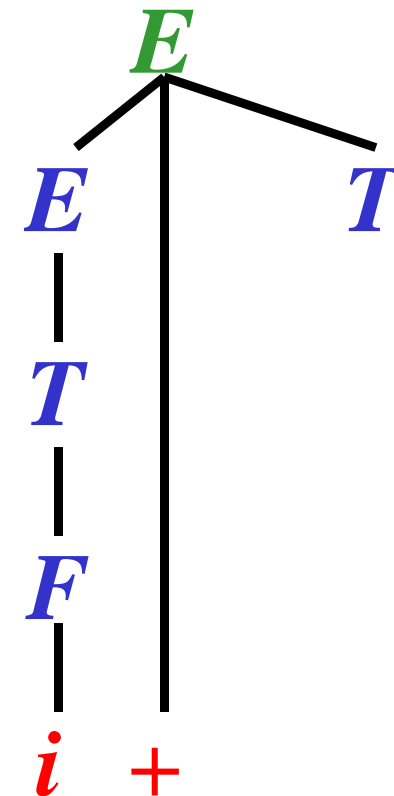
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} & [\mathbf{1}] \\ &\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} & [\mathbf{2}] \\ &\Rightarrow_{lm} \underline{\mathbf{F}} + \mathbf{T} & [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} & [\mathbf{6}] \end{aligned}$$

Derivation tree:



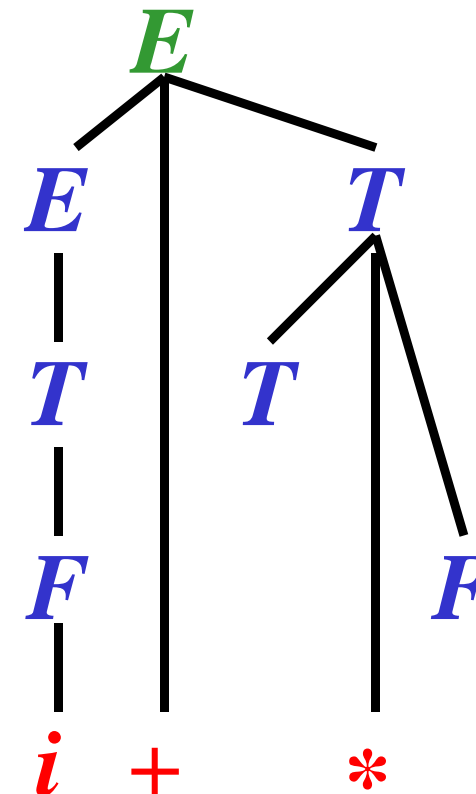
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} & [\mathbf{1}] \\ &\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} & [\mathbf{2}] \\ &\Rightarrow_{lm} \underline{\mathbf{F}} + \mathbf{T} & [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} & [\mathbf{6}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} * \mathbf{F} & [\mathbf{3}] \end{aligned}$$

Derivation tree:



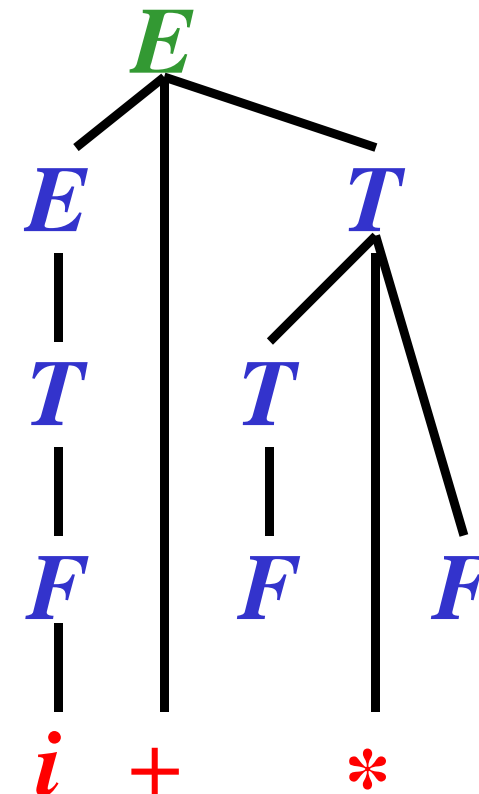
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} && [\mathbf{1}] \\ &\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} && [\mathbf{2}] \\ &\Rightarrow_{lm} \underline{\mathbf{F}} + \mathbf{T} && [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} && [\mathbf{6}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} * \mathbf{F} && [\mathbf{3}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{F}} * \mathbf{F} && [\mathbf{4}] \end{aligned}$$

Derivation tree:



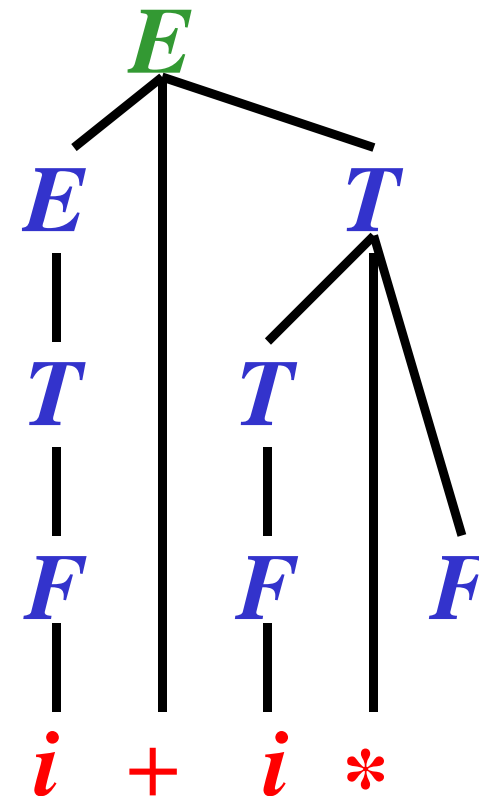
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} && [\mathbf{1}] \\ &\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} && [\mathbf{2}] \\ &\Rightarrow_{lm} \underline{\mathbf{F}} + \mathbf{T} && [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} && [\mathbf{6}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} * \mathbf{F} && [\mathbf{3}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{F}} * \mathbf{F} && [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \mathbf{i} * \underline{\mathbf{F}} && [\mathbf{6}] \end{aligned}$$

Derivation tree:



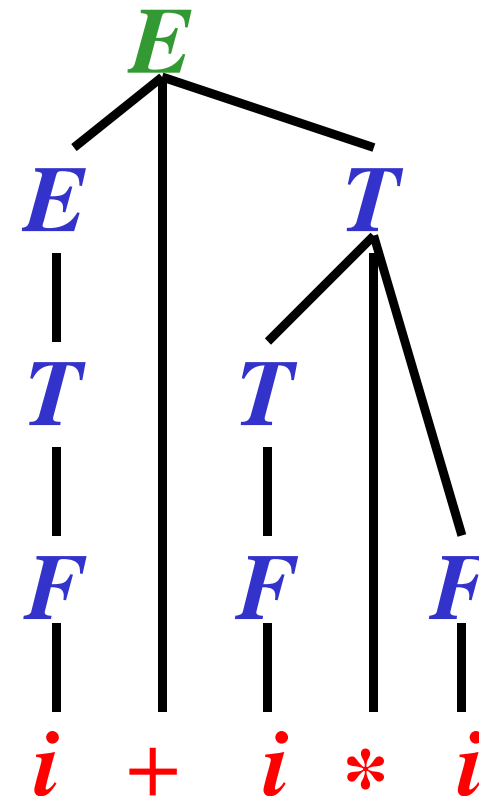
Leftmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{ \begin{array}{lll} \mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}, & \mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}, & \mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}, \\ \mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}, & \mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E}), & \mathbf{6}: \mathbf{F} \rightarrow \mathbf{i} \end{array} \}$

Leftmost derivation:

$$\begin{aligned} \underline{\mathbf{E}} &\Rightarrow_{lm} \underline{\mathbf{E}} + \mathbf{T} & [\mathbf{1}] \\ &\Rightarrow_{lm} \underline{\mathbf{T}} + \mathbf{T} & [\mathbf{2}] \\ &\Rightarrow_{lm} \underline{\mathbf{F}} + \mathbf{T} & [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} & [\mathbf{6}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{T}} * \mathbf{F} & [\mathbf{3}] \\ &\Rightarrow_{lm} \mathbf{i} + \underline{\mathbf{F}} * \mathbf{F} & [\mathbf{4}] \\ &\Rightarrow_{lm} \mathbf{i} + \mathbf{i} * \underline{\mathbf{F}} & [\mathbf{6}] \\ &\Rightarrow_{lm} \mathbf{i} + \mathbf{i} * \mathbf{i} & [\mathbf{6}] \end{aligned}$$

Derivation tree:



Rightmost Derivation

Gist: During a *rightmost derivation step*, the **rightmost nonterminal is rewritten**.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $u \in (N \cup T)^*$, $v \in T^*$. Let $p = A \rightarrow x \in P$ be a rule. Then, uAv directly derives uxv in the *rightmost way* according to p in G , written as

$$uAv \Rightarrow_{rm} uxv [p]$$

Note: We define \Rightarrow_{rm}^+ and \Rightarrow_{rm}^* by analogy with \Rightarrow^+ and \Rightarrow^* , respectively.

Rightmost Derivation: Example

$G = (N, T, P, \mathbf{E})$, where $N = \{\mathbf{E}, \mathbf{F}, \mathbf{T}\}$, $T = \{\mathbf{i}, +, *, (,)\}$,
 $P = \{$
 $\mathbf{1}: \mathbf{E} \rightarrow \mathbf{E} + \mathbf{T}$,
 $\mathbf{2}: \mathbf{E} \rightarrow \mathbf{T}$,
 $\mathbf{3}: \mathbf{T} \rightarrow \mathbf{T} * \mathbf{F}$,
 $\mathbf{4}: \mathbf{T} \rightarrow \mathbf{F}$,
 $\mathbf{5}: \mathbf{F} \rightarrow (\mathbf{E})$,
 $\mathbf{6}: \mathbf{F} \rightarrow \mathbf{i}$
 $\}$

Rightmost derivation:

Derivation tree:

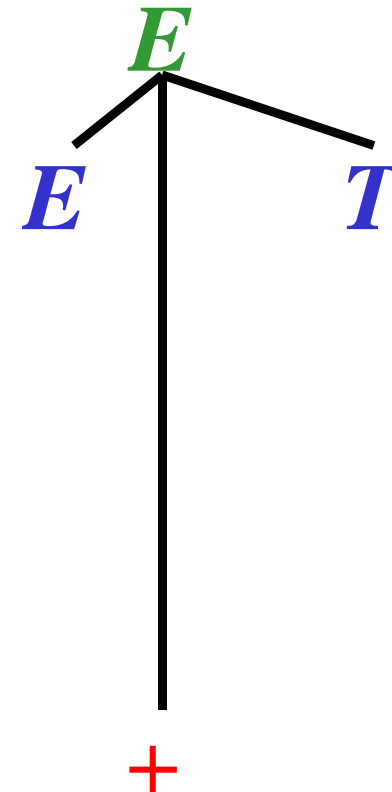
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\underline{i}, +, *, (,)\}$,
 $P = \{ \quad \underline{1}: \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \underline{2}: \underline{E} \rightarrow \underline{T}, \quad \underline{3}: \underline{T} \rightarrow \underline{T} * \underline{F}, \quad \underline{4}: \underline{T} \rightarrow \underline{F}, \quad \underline{5}: \underline{F} \rightarrow (\underline{E}), \quad \underline{6}: \underline{F} \rightarrow \underline{i} \quad \}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} \underline{E} + \underline{T} \quad [\underline{1}]$

Derivation tree:



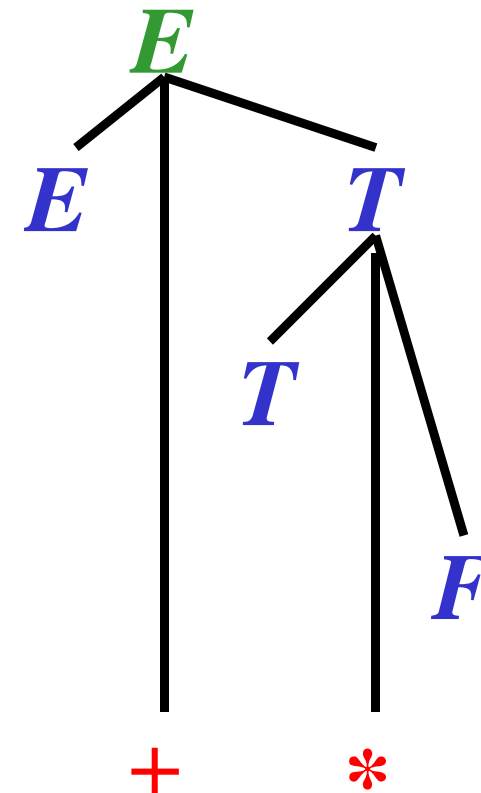
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\underline{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F}, \quad \text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \underline{i} \}$

Rightmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} \quad [1] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} \quad [3] \end{aligned}$$

Derivation tree:



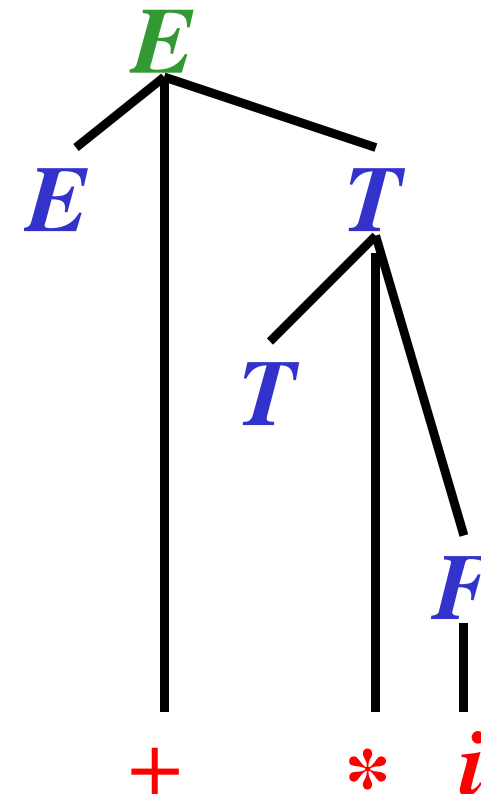
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\dot{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F}, \quad \text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \dot{i} \}$

Rightmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} && [\text{1}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} && [\text{3}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \dot{i} && [\text{6}] \end{aligned}$$

Derivation tree:



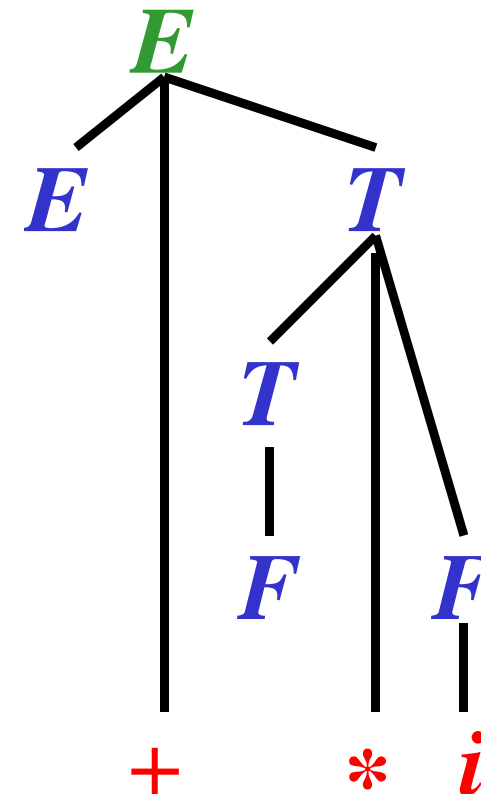
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\dot{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F}, \quad \text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \dot{i} \}$

Rightmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} && [\text{1}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} && [\text{3}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \dot{i} && [\text{6}] \\ &\Rightarrow_{rm} \underline{E} + \underline{F} * \dot{i} && [\text{4}] \end{aligned}$$

Derivation tree:



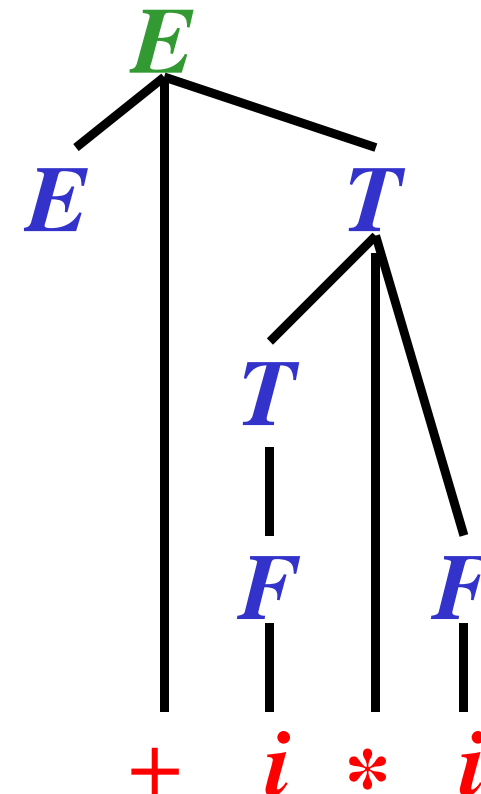
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\dot{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F},$
 $\text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \dot{i} \}$

Rightmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} && [\text{1}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} && [\text{3}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \dot{i} && [\text{6}] \\ &\Rightarrow_{rm} \underline{E} + \underline{F} * \dot{i} && [\text{4}] \\ &\Rightarrow_{rm} \underline{E} + \dot{i} * \dot{i} && [\text{6}] \end{aligned}$$

Derivation tree:



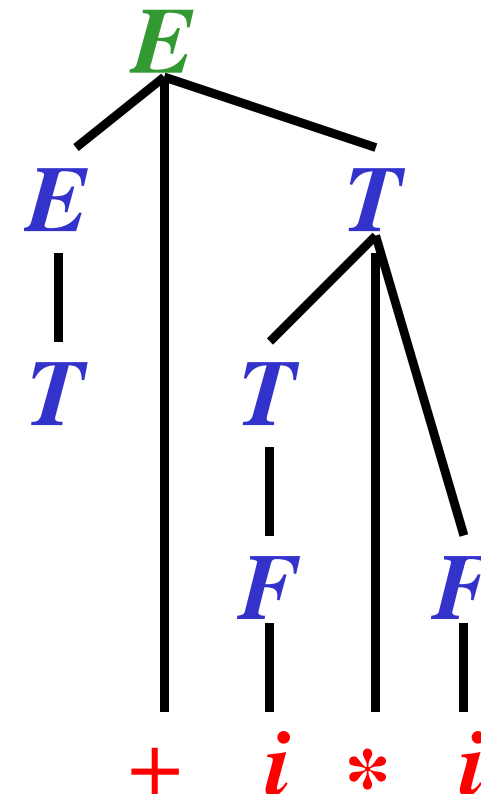
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\underline{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F},$
 $\quad \text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \underline{i} \quad \}$

Rightmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} && [\text{1}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} && [\text{3}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{i} && [\text{6}] \\ &\Rightarrow_{rm} \underline{E} + \underline{F} * \underline{i} && [\text{4}] \\ &\Rightarrow_{rm} \underline{E} + \underline{i} * \underline{i} && [\text{6}] \\ &\Rightarrow_{rm} \underline{T} + \underline{i} * \underline{i} && [\text{2}] \end{aligned}$$

Derivation tree:



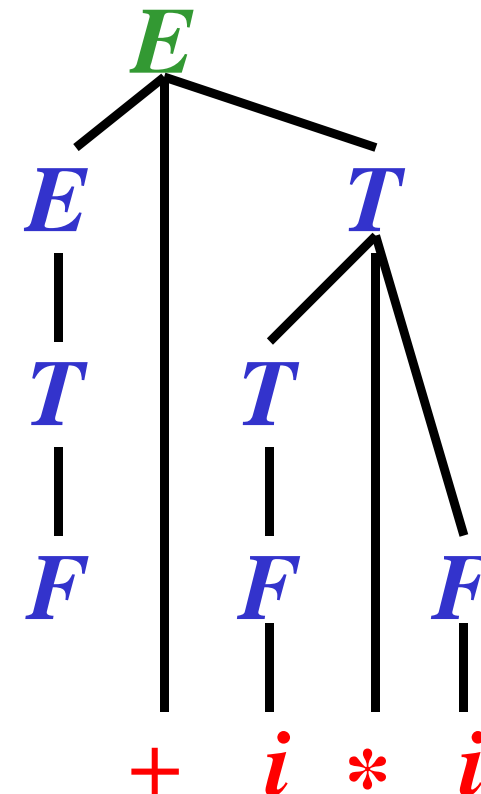
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\underline{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F},$
 $\quad \text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \underline{i} \}$

Rightmost derivation:

$$\begin{aligned} \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} && [\text{1}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} && [\text{3}] \\ &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{i} && [\text{6}] \\ &\Rightarrow_{rm} \underline{E} + \underline{F} * \underline{i} && [\text{4}] \\ &\Rightarrow_{rm} \underline{E} + \underline{i} * \underline{i} && [\text{6}] \\ &\Rightarrow_{rm} \underline{T} + \underline{i} * \underline{i} && [\text{2}] \\ &\Rightarrow_{rm} \underline{F} + \underline{i} * \underline{i} && [\text{4}] \end{aligned}$$

Derivation tree:



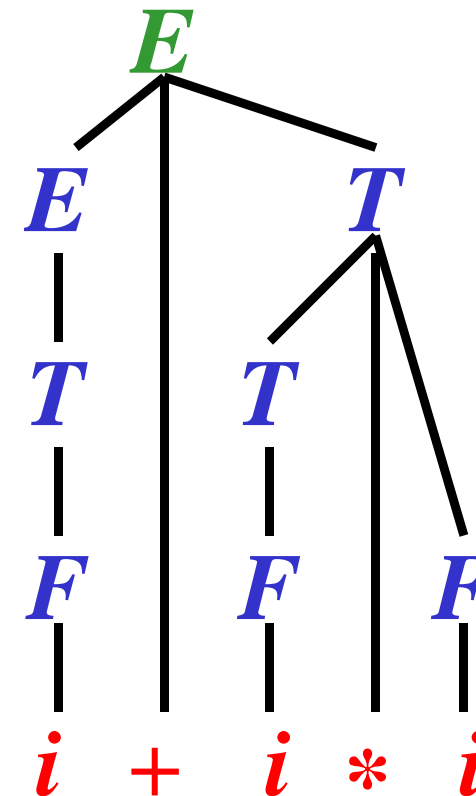
Rightmost Derivation: Example

$G = (N, T, P, \underline{E})$, where $N = \{\underline{E}, \underline{F}, \underline{T}\}$, $T = \{\underline{i}, +, *, (,)\}$,
 $P = \{ \text{1: } \underline{E} \rightarrow \underline{E} + \underline{T}, \quad \text{2: } \underline{E} \rightarrow \underline{T}, \quad \text{3: } \underline{T} \rightarrow \underline{T} * \underline{F}, \quad \text{4: } \underline{T} \rightarrow \underline{F}, \quad \text{5: } \underline{F} \rightarrow (\underline{E}), \quad \text{6: } \underline{F} \rightarrow \underline{i} \}$

Rightmost derivation:

$$\begin{aligned}
 \underline{E} &\Rightarrow_{rm} \underline{E} + \underline{T} && [\text{1}] \\
 &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{F} && [\text{3}] \\
 &\Rightarrow_{rm} \underline{E} + \underline{T} * \underline{i} && [\text{6}] \\
 &\Rightarrow_{rm} \underline{E} + \underline{F} * \underline{i} && [\text{4}] \\
 &\Rightarrow_{rm} \underline{E} + \underline{i} * \underline{i} && [\text{6}] \\
 &\Rightarrow_{rm} \underline{T} + \underline{i} * \underline{i} && [\text{2}] \\
 &\Rightarrow_{rm} \underline{F} + \underline{i} * \underline{i} && [\text{4}] \\
 &\Rightarrow_{rm} \underline{i} + \underline{i} * \underline{i} && [\text{6}]
 \end{aligned}$$

Derivation tree:



Derivations: Summary

- Let $A \rightarrow x \in P$ be a rule.
-

1) Derivation:

Let $u, v \in (N \cup T)^*$: $uAv \Rightarrow uxv$

Note: Any nonterminal is rewritten

2) Leftmost derivation:

Let $u \in T^*, v \in (N \cup T)^*$: $uAv \Rightarrow_{\text{lm}} uxv$

Note: Leftmost nonterminal is rewritten

3) Rightmost derivation:

Let $u \in (N \cup T)^*, v \in T^*$: $uAv \Rightarrow_{\text{rm}} uxv$

Note: Rightmost nonterminal is rewritten

Reduction of the Number of Derivations

Gist: Without any loss of generality, we can consider only leftmost or rightmost derivations.

Theorem: Let $G = (N, T, P, S)$ be a CFG. The next three languages coincide

$$(1) \{w: w \in T^*, S \Rightarrow_{lm}^* w\}$$

$$(2) \{w: w \in T^*, S \Rightarrow_{rm}^* w\}$$

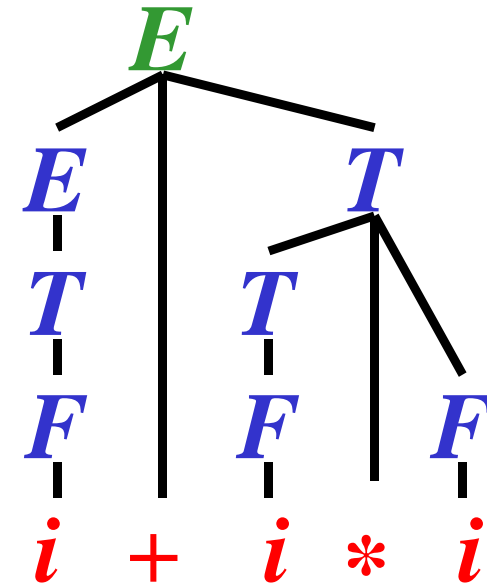
$$(3) \{w: w \in T^*, S \Rightarrow^* w\} = L(G)$$

Introduction to Ambiguity

$G_{expr1} = (N, T, P, E)$, where

$N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$
 1: $E \rightarrow E+T$, 2: $E \rightarrow T$,
 3: $T \rightarrow T*F$, 4: $T \rightarrow F$,
 5: $F \rightarrow (E)$, 6: $F \rightarrow i$
 $\}$

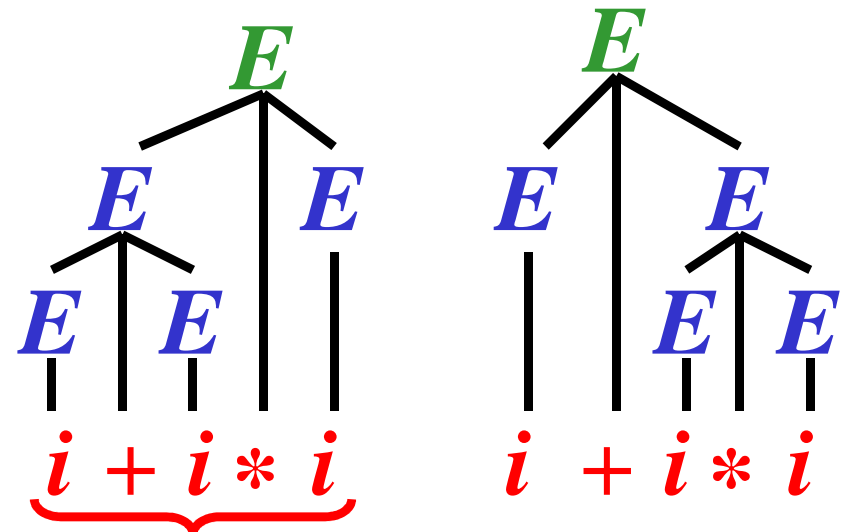


Theory: ☹️ **Practice:** 😊

$G_{expr2} = (N, T, P, E)$, where

$N = \{E\}$, $T = \{i, +, *, (,)\}$,

$P = \{$
 1: $E \rightarrow E+E$, 2: $E \rightarrow E*E$,
 3: $E \rightarrow (E)$, 4: $E \rightarrow i$
 $\}$



Theory: 😊 **Practice:** ☹️

Note: $L(G_{expr1}) = L(G_{expr2})$

Improper during compilation

Grammatical Ambiguity

Definition: Let $G = (N, T, P, S)$ be a CFG. If there exists $x \in L(G)$ with more than one derivation tree, then G is *ambiguous*; otherwise, G is *unambiguous*.

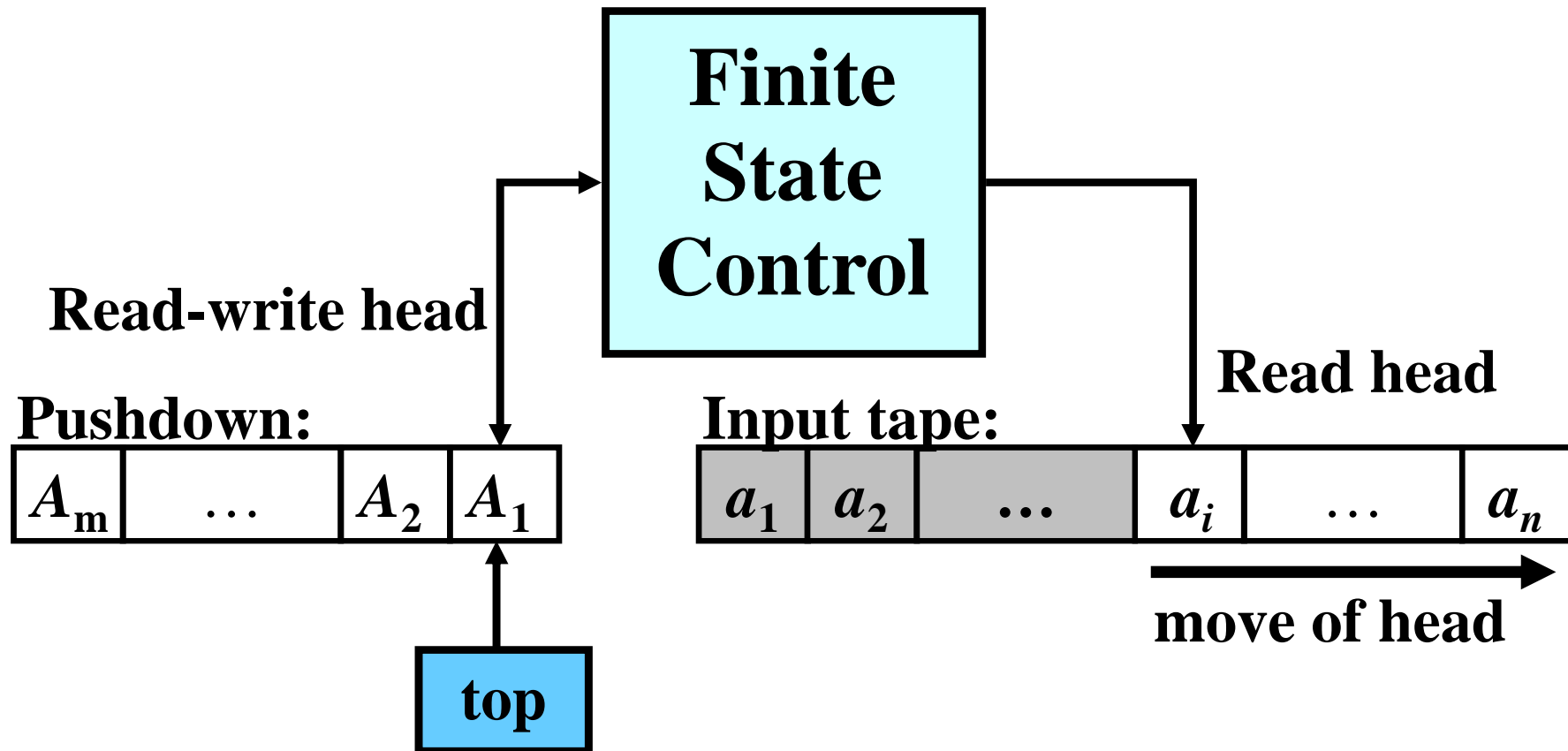
Definition: A CFL, L , is *inherently ambiguous* if L is generated by no unambiguous grammar.

Example:

- G_{expr1} is **unambiguous**, because for every $x \in L(G_{expr1})$ there exists **only one derivation tree**
- G_{expr2} is **ambiguous**, because for $i+i*i \in L(G_{expr2})$ there exist **two derivation trees**
- $L_{expr} = L(G_{expr1}) = L(G_{expr2})$ is **not inherently ambiguous** because G_{expr1} is unambiguous

Pushdown Automata (PDA)

Gist: An FA extended by a pushdown store.



Pushdown Automata: Definition

Definition: A *pushdown automaton* (PDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where

- Q is a *finite set of states*
- Σ is an *input alphabet*
- Γ is a *pushdown alphabet*
- R is a *finite set of rules* of the form: $Apa \rightarrow wq$
where $A \in \Gamma, p, q \in Q, a \in \Sigma \cup \{\varepsilon\}, w \in \Gamma^*$
- $s \in Q$ is the *start state*
- $S \in \Gamma$ is the *start pushdown symbol*
- $F \subseteq Q$ is a set of *final states*

Notes on PDA Rules

Mathematical note on rules:

- Strictly mathematically, R is a finite relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
 - Instead of $(Apa, wq) \in R$, however, we write $Apa \rightarrow wq \in R$
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- Strictly mathematically, R is a finite relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
- Instead of $(Apa, wq) \in R$, however, we write $Apa \rightarrow wq \in R$

- **Interpretation of $Apa \rightarrow wq$:** if the current state is p , current input symbol is a , and the topmost symbol on the pushdown is A , then M can read a , replace A with w and change state p to q .
- **Note:** if $a = \varepsilon$, no symbol is read

Graphical Representation

 represents $q \in Q$

 represents the initial state $s \in Q$

 represents a final state $f \in F$

 $\xrightarrow{A/w, a}$  denotes $Apa \rightarrow wq \in R$

Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

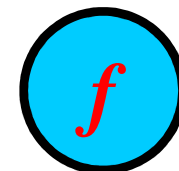
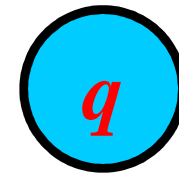
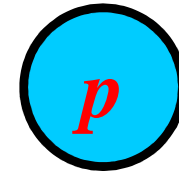
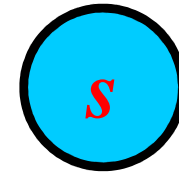
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Graphical Representation: Example

$$M = (Q, \Sigma, \Gamma, R, s, S, F)$$

where:

- $Q = \{s, p, q, f\};$

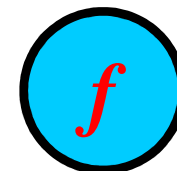
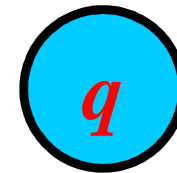
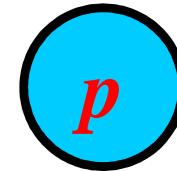
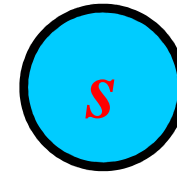


Graphical Representation: Example

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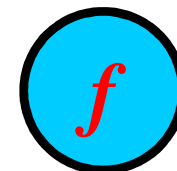
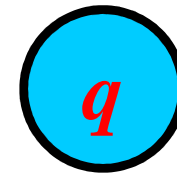
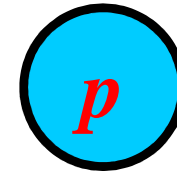
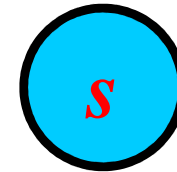


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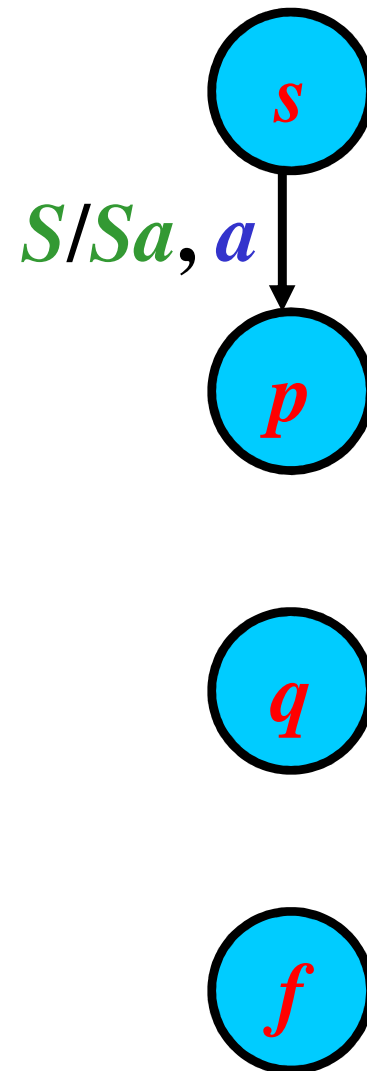


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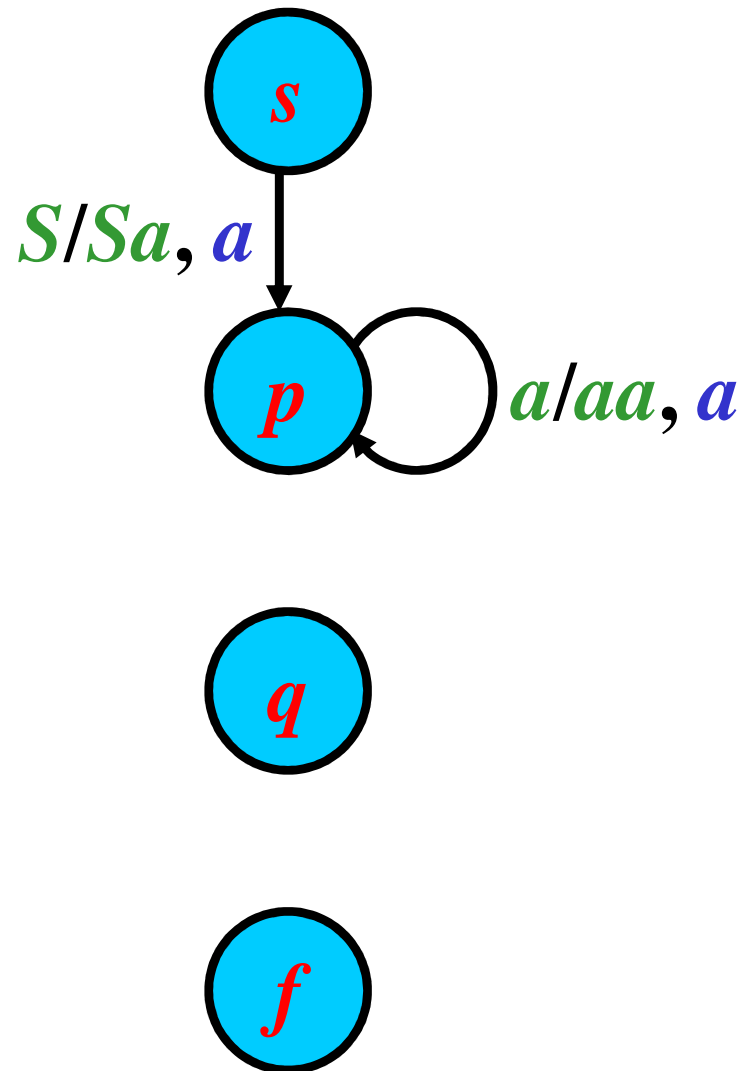


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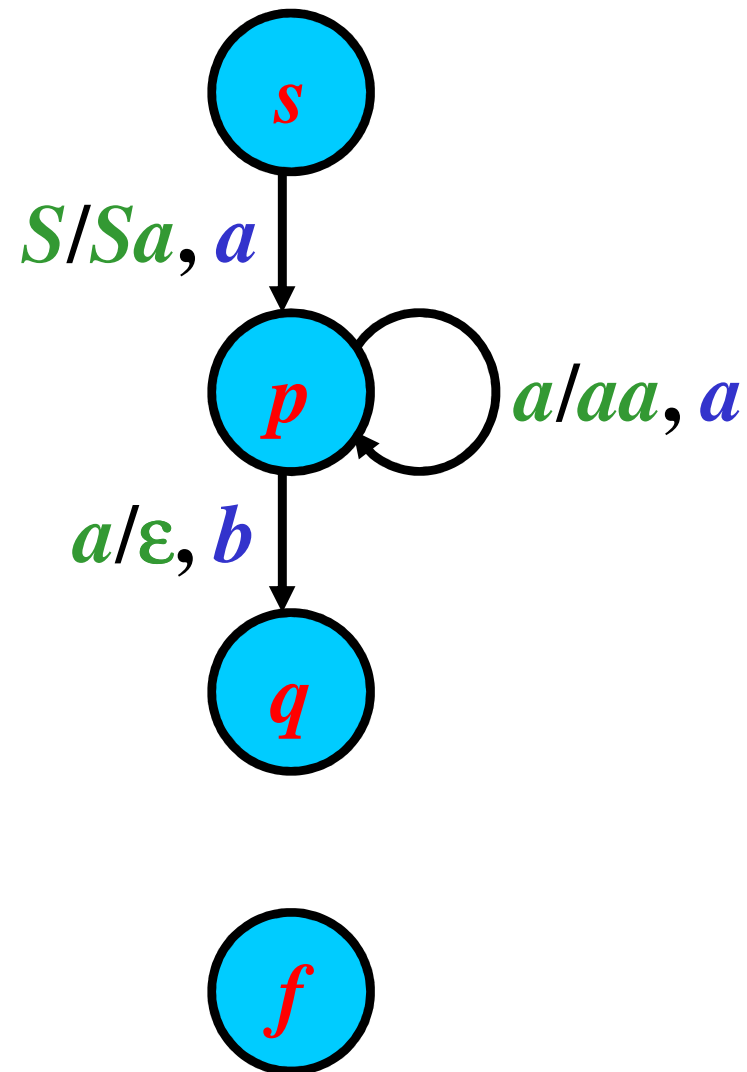


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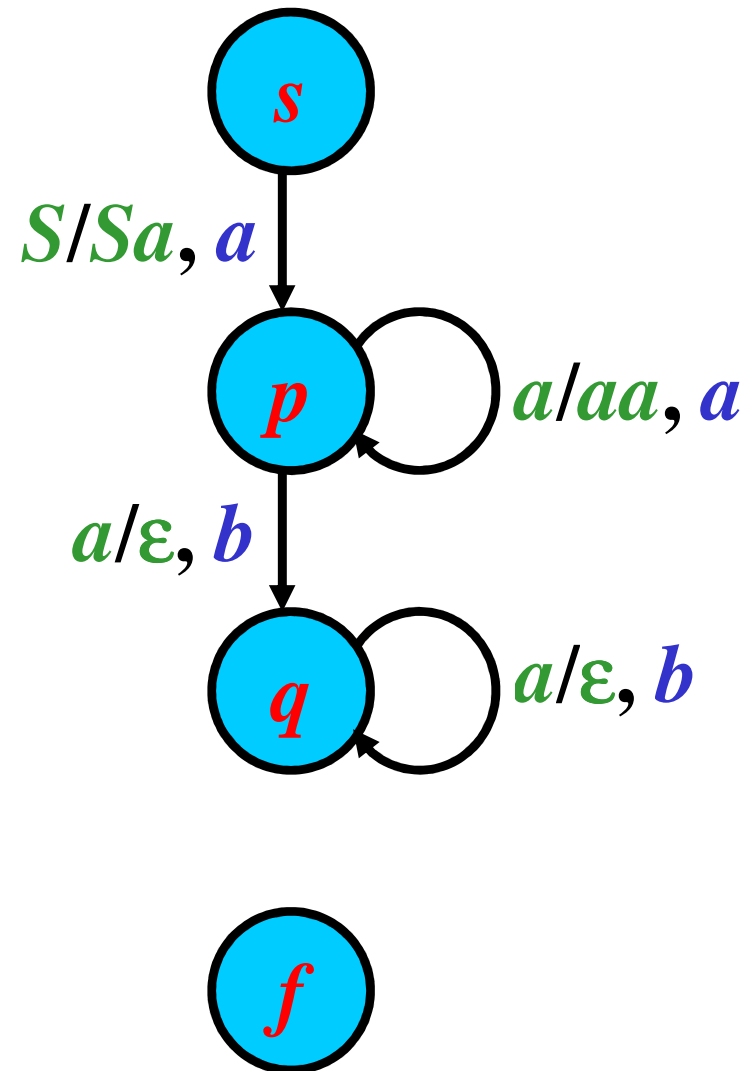


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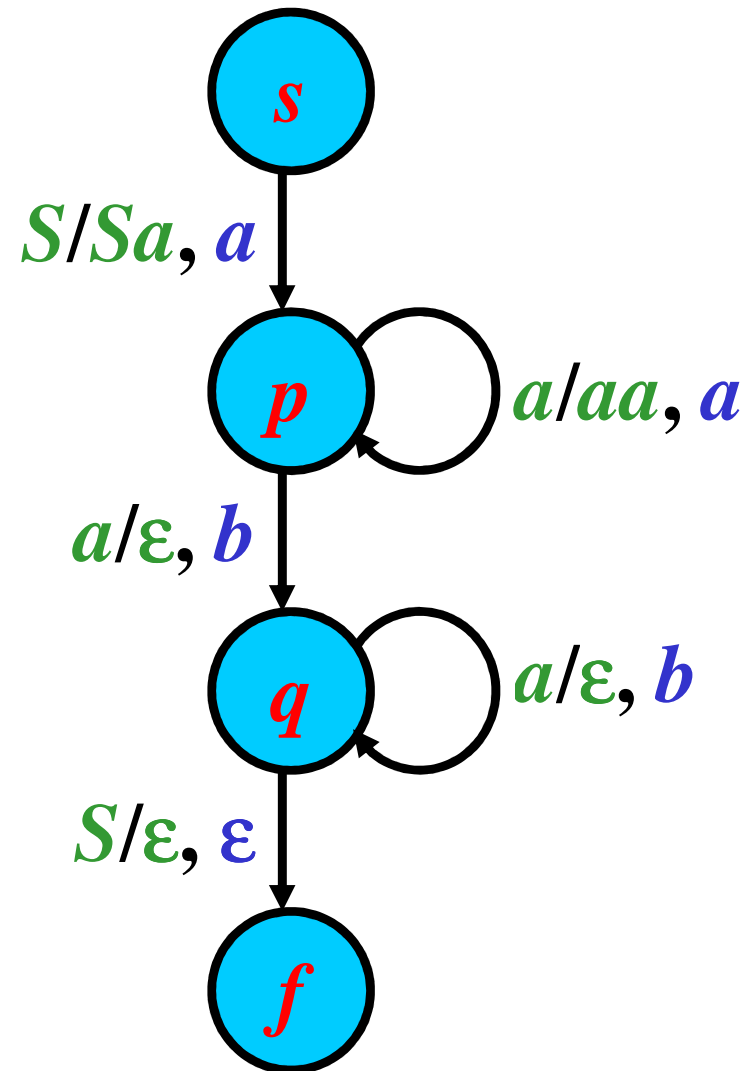


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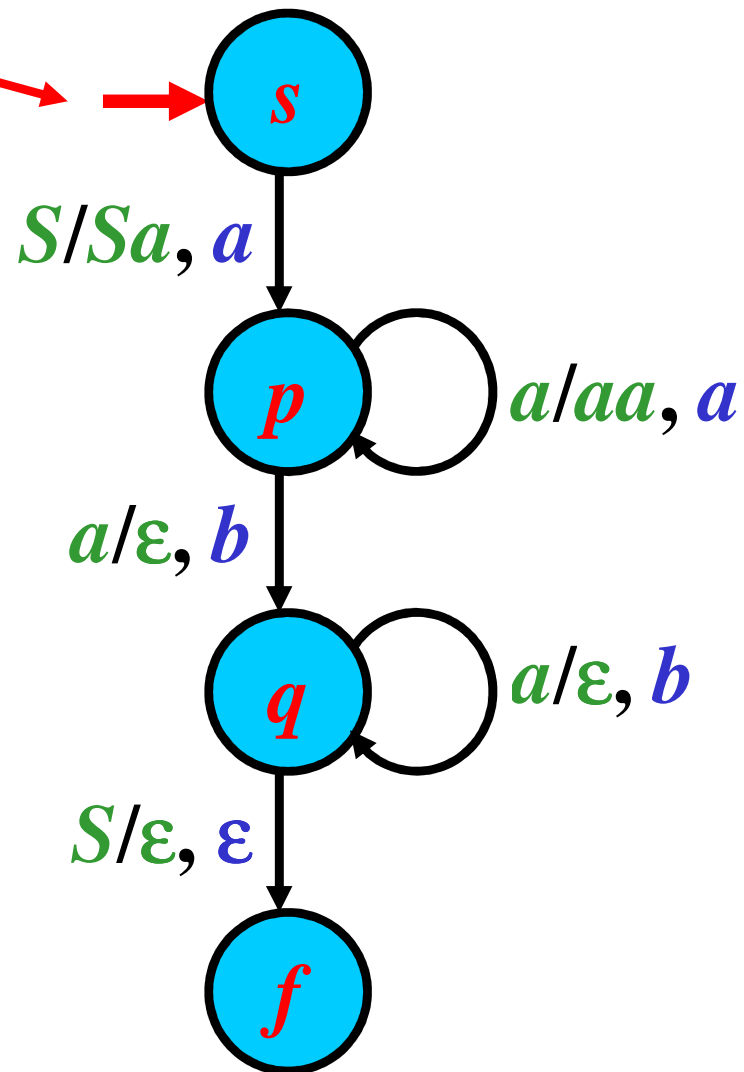


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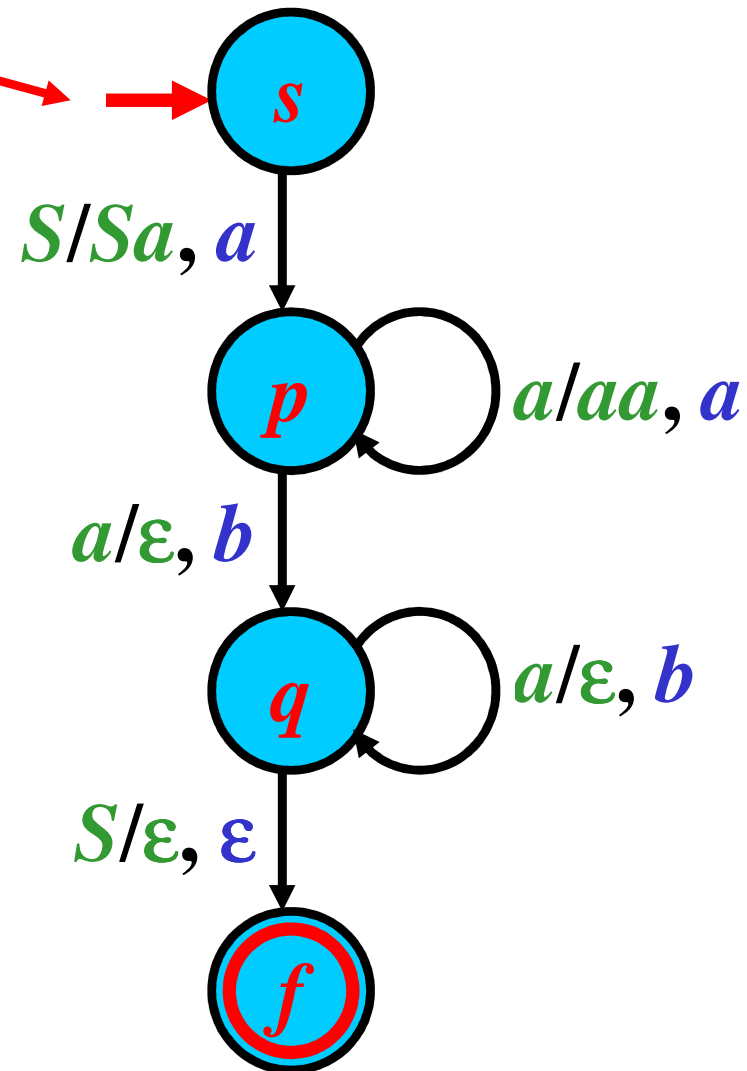


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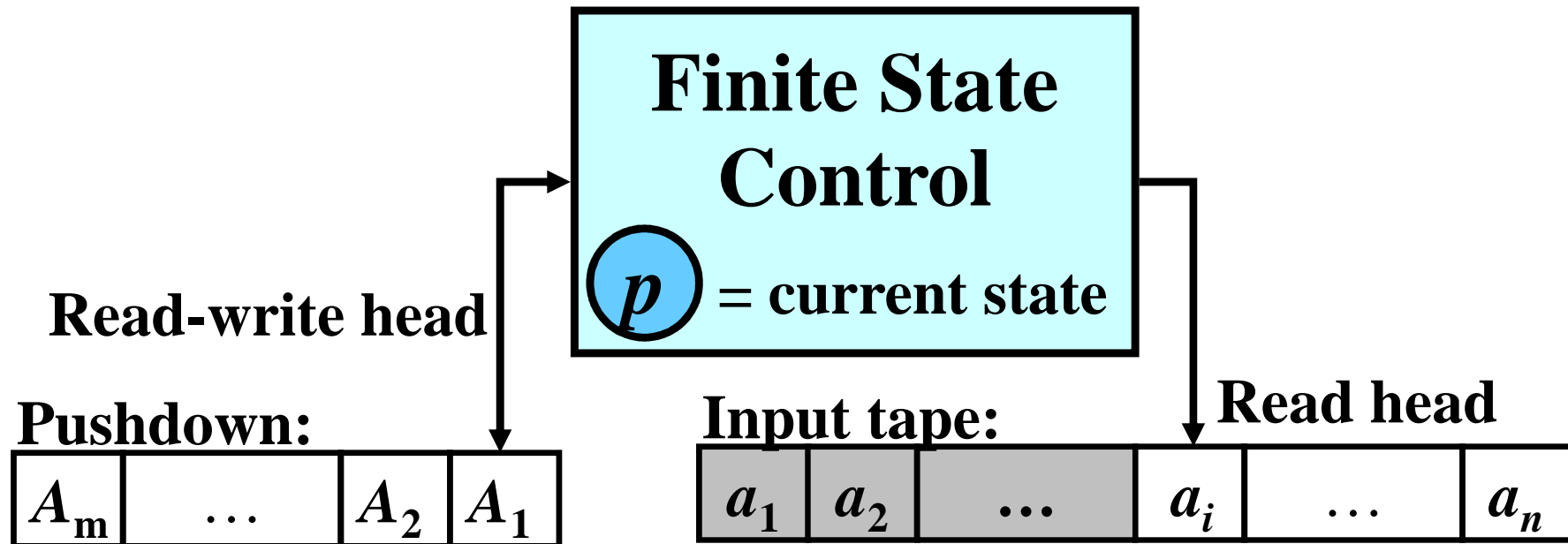
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PDA Configuration

Gist: Instantaneous description of PDA

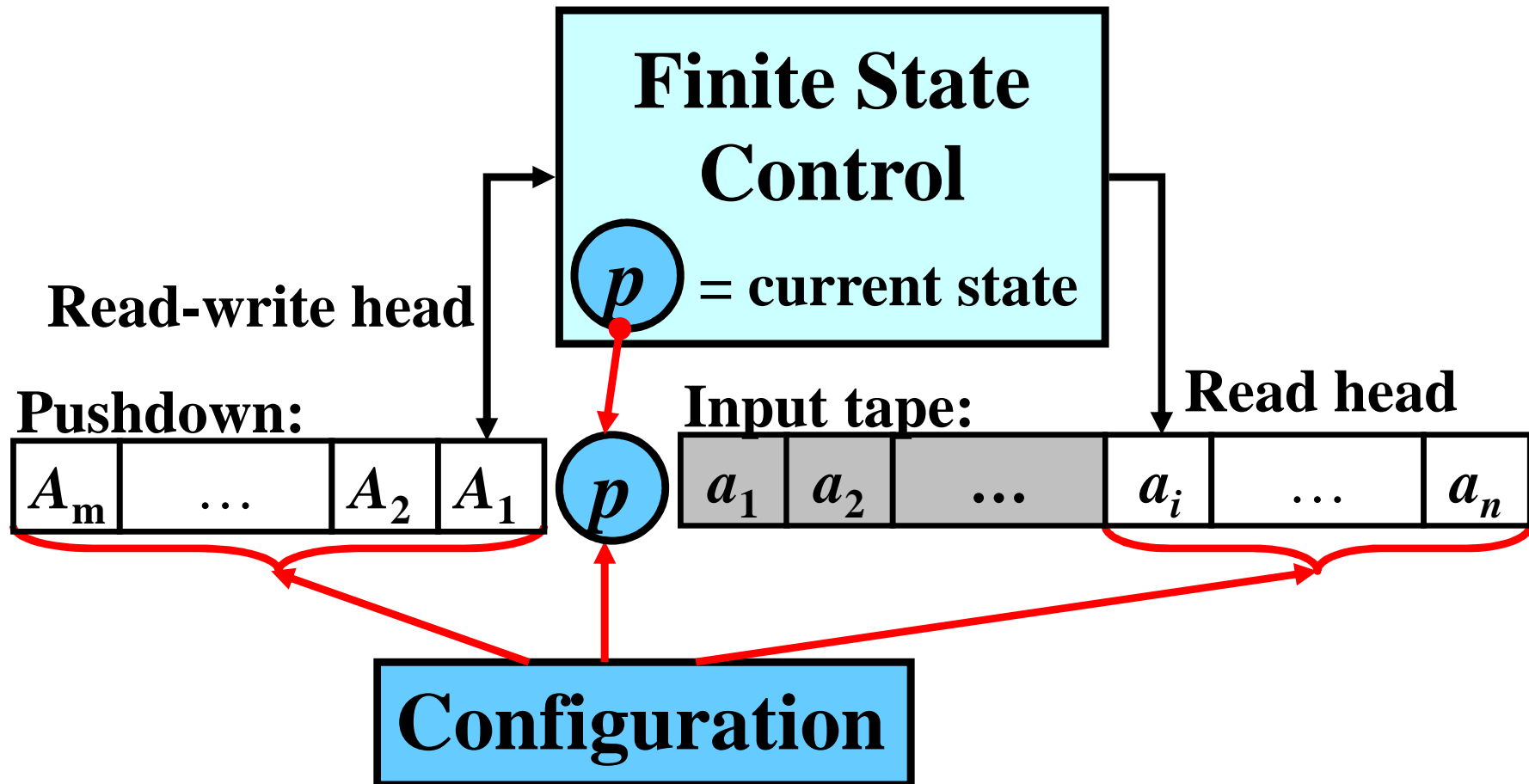
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A configuration of M is a string $\chi \in \Gamma^ Q \Sigma^*$*



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Move

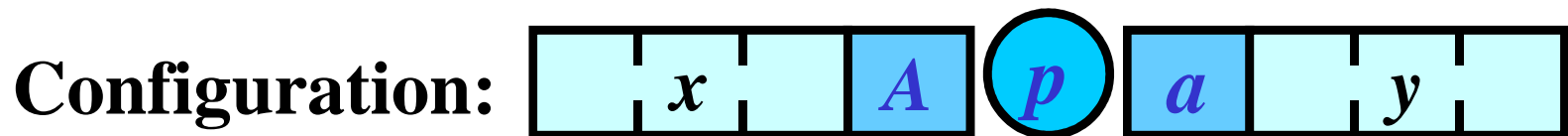
Gist: A computational step made by a PDA

Definition: Let $xApay$ and $xwqy$ be two configurations of a PDA, M , where

$x, w \in \Gamma^*$, $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $y \in \Sigma^*$.

Let $r = Apa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xApay$ to $xwqy$ according to r , written as $xApay \vdash xwqy [r]$ or, simply, $xApay \vdash xwqy$.

Note: if $a = \epsilon$, no input symbol is read



Move

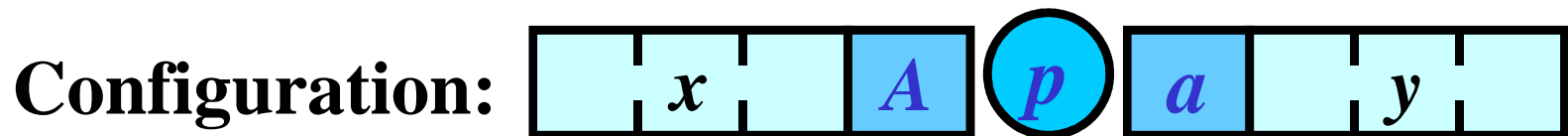
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Rule: $Ap a \rightarrow wq$

Move

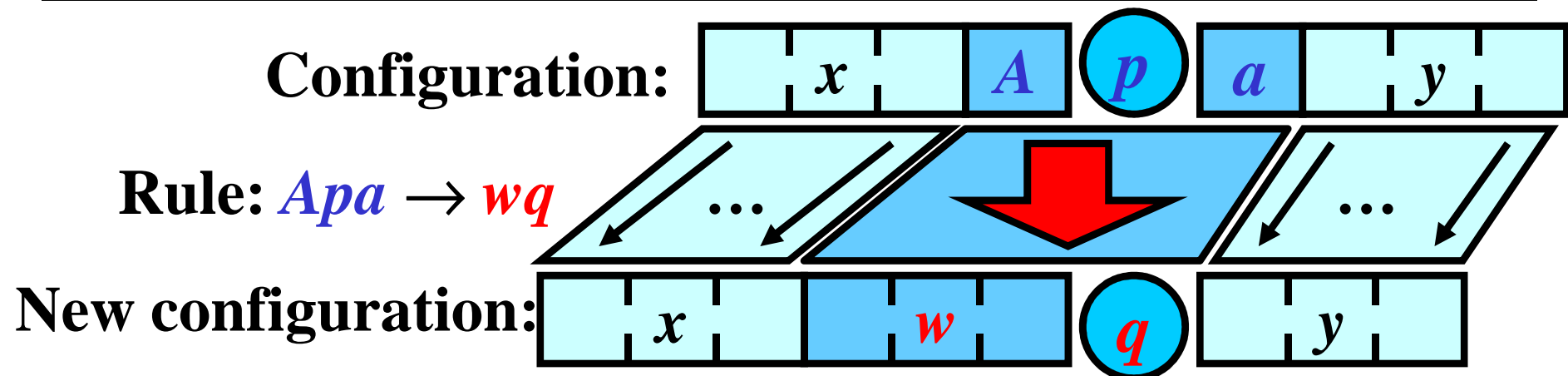
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Sequence of Moves 1/2

Gist: Several consecutive computational steps

Definition: Let χ be a configuration. M makes *zero moves* from χ to χ ; in symbols,

$$\chi \vdash^0 \chi [\varepsilon] \text{ or, simply, } \chi \vdash^0 \chi$$

Definition: Let $\chi_0, \chi_1, \dots, \chi_n$ be a sequence of configurations, $n \geq 1$, and $\chi_{i-1} \vdash \chi_i [r_i]$, $r_i \in R$, for all $i = 1, \dots, n$; that is,

$$\chi_0 \vdash \chi_1 [r_1] \vdash \chi_2 [r_2] \dots \vdash \chi_n [r_n]$$

Then M makes *n moves* from χ_0 to χ_n ,

$$\chi_0 \vdash^n \chi_n [r_1 \dots r_n] \text{ or, simply, } \chi_0 \vdash^n \chi_n$$

Sequence of Moves 2/2

If $\chi_0 \vdash^n \chi_n [\rho]$ for some $n \geq 1$, then

$\chi_0 \vdash^+ \chi_n [\rho]$ or, simply, $\chi_0 \vdash^+ \chi_n$

If $\chi_0 \vdash^n \chi_n [\rho]$ for some $n \geq 0$, then

$\chi_0 \vdash^* \chi_n [\rho]$ or, simply, $\chi_0 \vdash^* \chi_n$

Example: Consider

$A\textcolor{blue}{A}pabc \vdash A\textcolor{red}{B}qbc$ [1: $\textcolor{blue}{A}pa \rightarrow \textcolor{red}{B}q$], and

$A\textcolor{blue}{B}qbc \vdash A\textcolor{red}{B}Crc$ [2: $\textcolor{blue}{B}qb \rightarrow \textcolor{red}{B}Cr$].

Then, $A\textcolor{blue}{A}pabc \vdash^2 A\textcolor{red}{B}Crc$ [1 2],

$A\textcolor{blue}{A}pabc \vdash^+ A\textcolor{red}{B}Crc$ [1 2],

$A\textcolor{blue}{A}pabc \vdash^* A\textcolor{red}{B}Crc$ [1 2]

Accepted Language: Three Types

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.

- 1) The *language that M accepts by final state*, denoted by $L(M)_f$, is defined as

$$L(M)_f = \{w: w \in \Sigma^*, Ssw \xrightarrow{*} zf, \mathbf{z} \in \Gamma^*, \mathbf{f} \in \mathbf{F}\}$$

- 2) The *language that M accepts by empty pushdown*, denoted by $L(M)_\epsilon$, is defined as

$$L(M)_\epsilon = \{w: w \in \Sigma^*, Ssw \xrightarrow{*} zf, \mathbf{z} = \epsilon, \mathbf{f} \in Q\}$$

- 3) The *language that M accepts by final state and empty pushdown*, denoted by $L(M)_{f\epsilon}$, is defined as

$$L(M)_{f\epsilon} = \{w: w \in \Sigma^*, Ssw \xrightarrow{*} zf, \mathbf{z} = \epsilon, \mathbf{f} \in \mathbf{F}\}$$

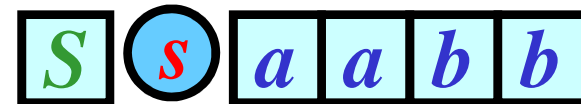
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 $Sq \rightarrow f\}$
- $F = \{f\}$

Question: $aabb \in L(M)_{f_\epsilon}?$



$Ssaabb$

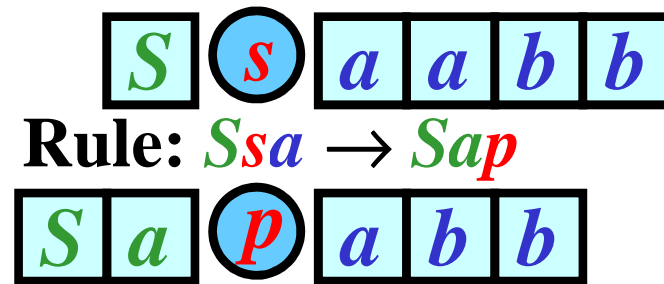
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$Ssaabb \vdash Sapabb$

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where:

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Question: $aabb \in L(M)_{f_\epsilon}?$

S	s	a	a	b	b
---	---	---	---	---	---

Rule: $Ssa \rightarrow Sap$

S	a	p	a	b	b
---	---	---	---	---	---

Rule: $apa \rightarrow aap$

S	a	a	p	b	b
---	---	---	---	---	---

$Ssaabb \vdash Sapabb \vdash Saapbb$

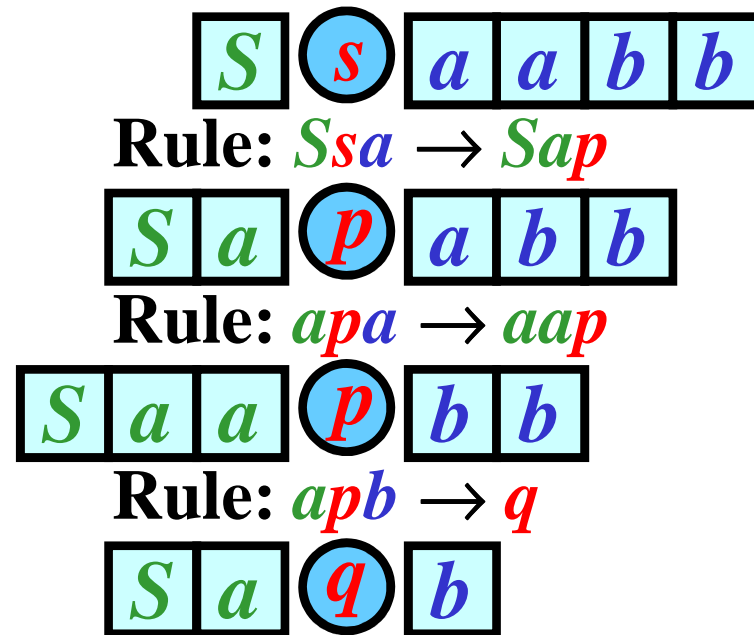
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Question: $aabb \in L(M)_{f\epsilon}?$



$Ssaabb \mid\!-\! Sapabb \mid\!-\! Saapbb \mid\!-\! Saqb$

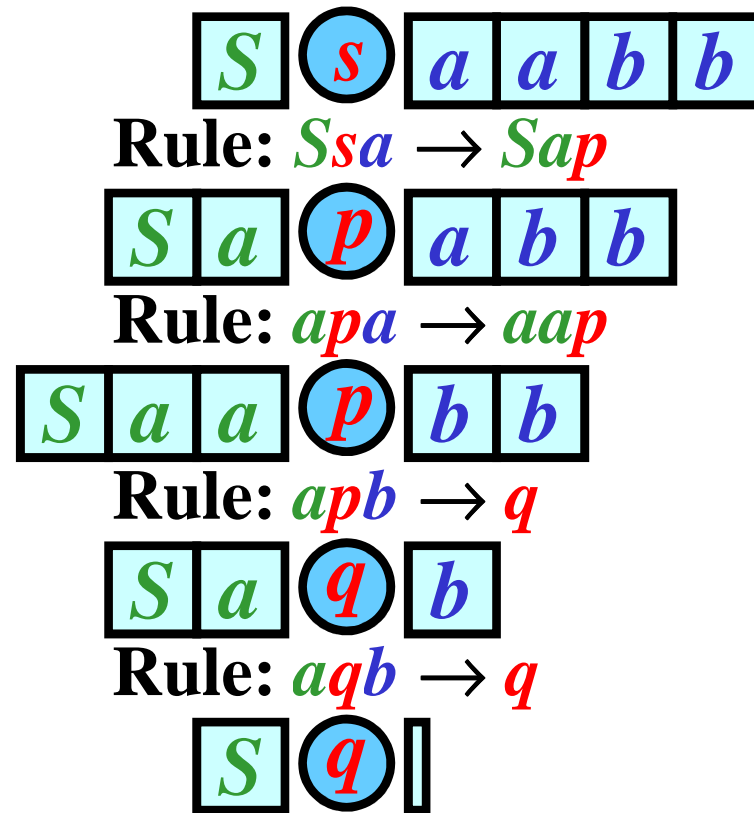
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Question: $aabb \in L(M)_{f\epsilon}?$



$Ssaabb \mid - Sapabb \mid - Saapbb \mid - Saqb \mid - Sq$

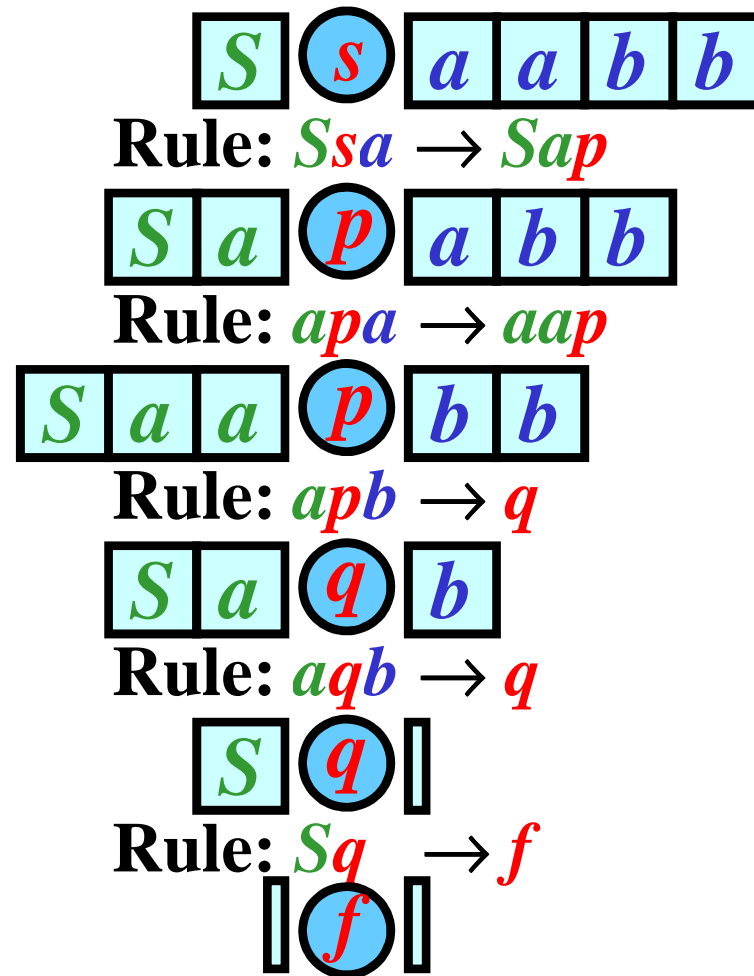
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Question: $aabb \in L(M)_{f\epsilon}$?

S s a a b b

Rule: $Ssa \rightarrow Sap$

S a p a b b

Rule: $apa \rightarrow aap$

S a a p b b

Rule: $apb \rightarrow q$

S a q b

Rule: $aqb \rightarrow q$

S q

Rule: $Sq \rightarrow f$

f

Empty
pushdown

$Ssaabb \mid - Sapabb \mid - Saapbb \mid - Saqb \mid - Sq \mid - f$

PDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

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Question: $aabb \in L(M)_{f\epsilon}$?

S s a a b b

Rule: $Ssa \rightarrow Sap$

S a p a b b

Rule: $apa \rightarrow aap$

S a a p b b

Rule: $apb \rightarrow q$

S a q b

Rule: $aqb \rightarrow q$

S q

Rule: $Sq \rightarrow f$

Final state

f

Answer: YES

Empty
pushdown

$Ssaabb \mid - \mid Sapabb \mid - \mid Saapbb \mid - \mid Saqb \mid - \mid Sq \mid - \mid f$

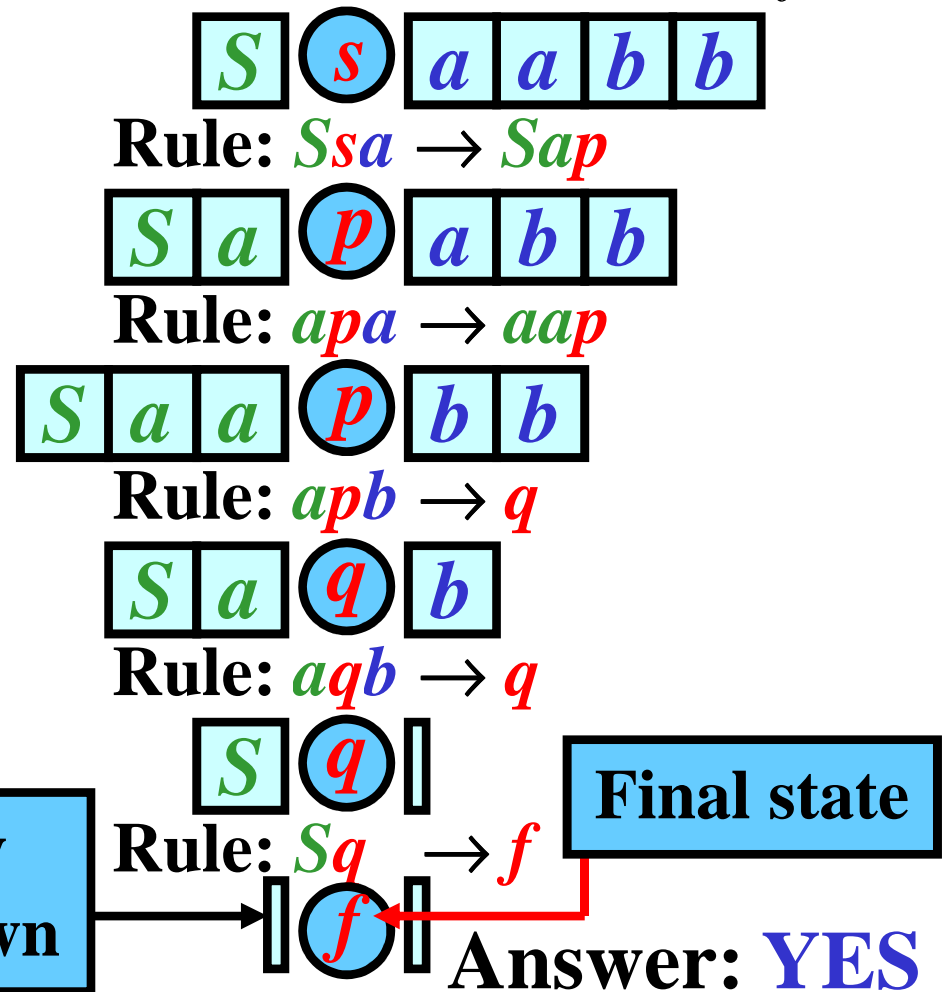
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Question: $aabb \in L(M)_{f\epsilon}$?



$Ssaabb \mid - Sapabb \mid - Saapbb \mid - Saqb \mid - Sq \mid - f$

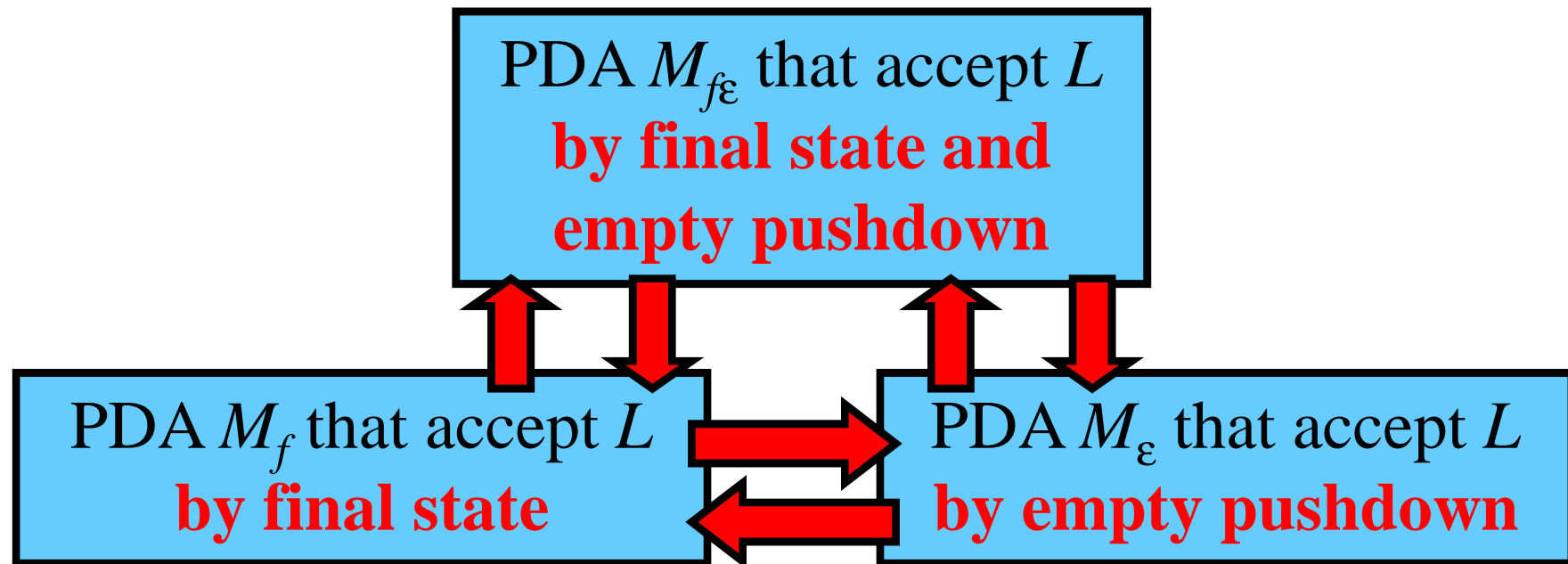
Note: $L(M)_f = L(M)_\epsilon = L(M)_{f\epsilon} = \{a^n b^n : n \geq 1\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for a PDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for a PDA M_ε

Note: There exist these conversions:



Deterministic PDA (DPDA)

Gist: Deterministic PDA makes no more than one move from any configuration.

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. M is a *deterministic PDA* if for each rule $Apa \rightarrow wq \in R$, it holds that $R - \{Apa \rightarrow wq\}$ contains no rule with the left-hand side equal to Apa or Ap .

Illustration:

Configuration:



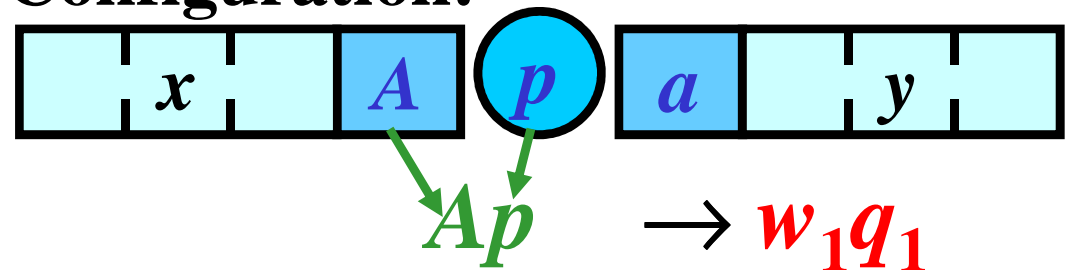
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Illustration:

Configuration:



Deterministic PDA (DPDA)

Gist: Deterministic PDA makes no more than one move from any configuration.

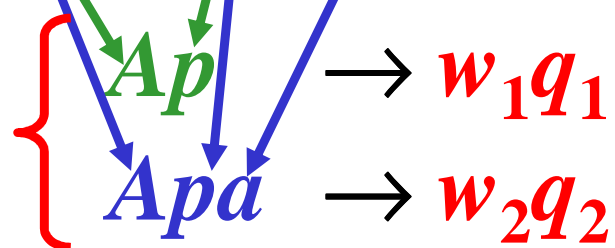
Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. M is a *deterministic PDA* if for each rule $Apa \rightarrow wq \in R$, it holds that $R - \{Apa \rightarrow wq\}$ contains no rule with the left-hand side equal to Apa or Ap .

Illustration:

Configuration:



No more than one rule of the forms



PDAs are Stronger than DPDAs

Theorem: There exists no DPDA M_{f_ε} that accepts
$$L = \{xy: x, y \in \Sigma^*, y = \textit{reversal}(x)\}$$

Proof: See page 431 in [Meduna: Automata and Languages]

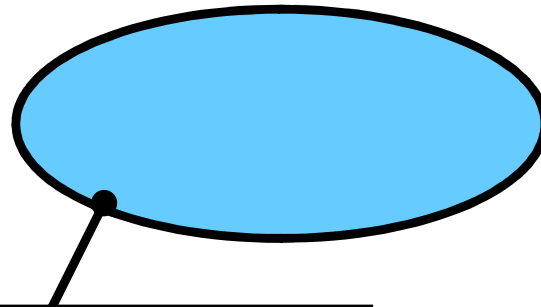
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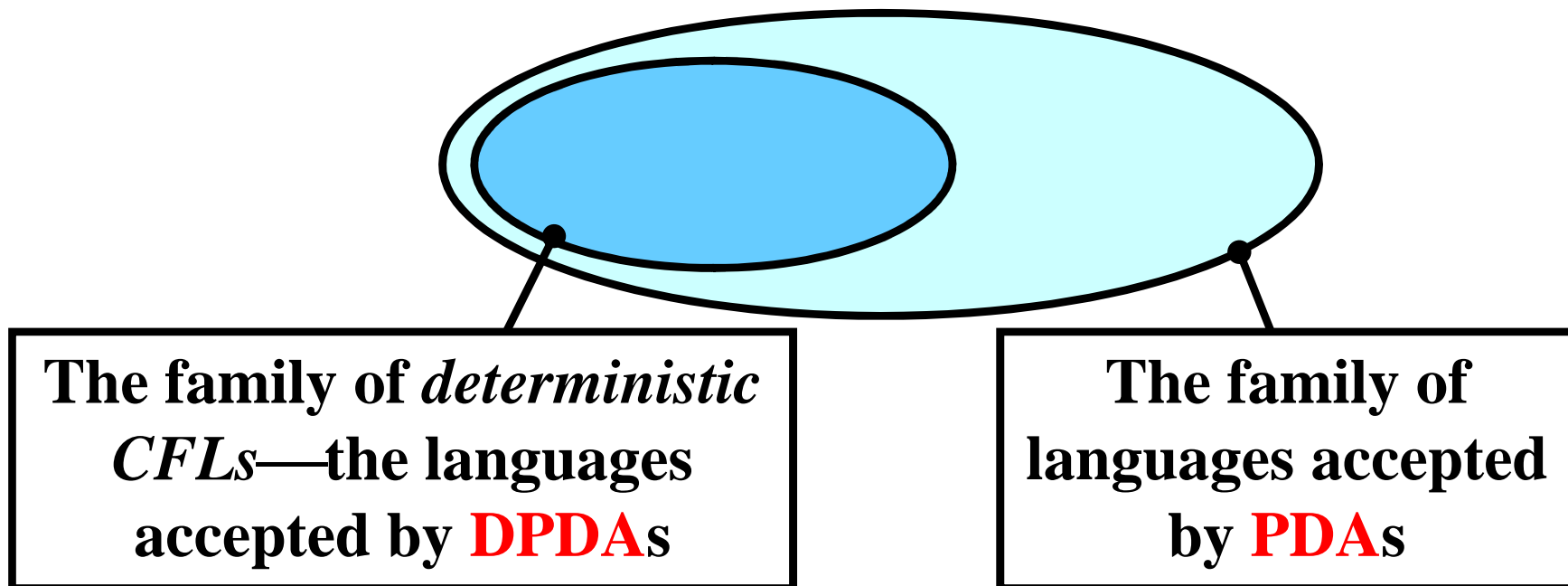
The family of *deterministic CFLs*—the languages accepted by **DPDAs**

PDAs are Stronger than DPDAs

Theorem: There exists no DPDA $M_{f\epsilon}$ that accepts
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Proof: See page 431 in [Meduna: Automata and Languages]

Illustration:



PDAs are Stronger than DPDAs

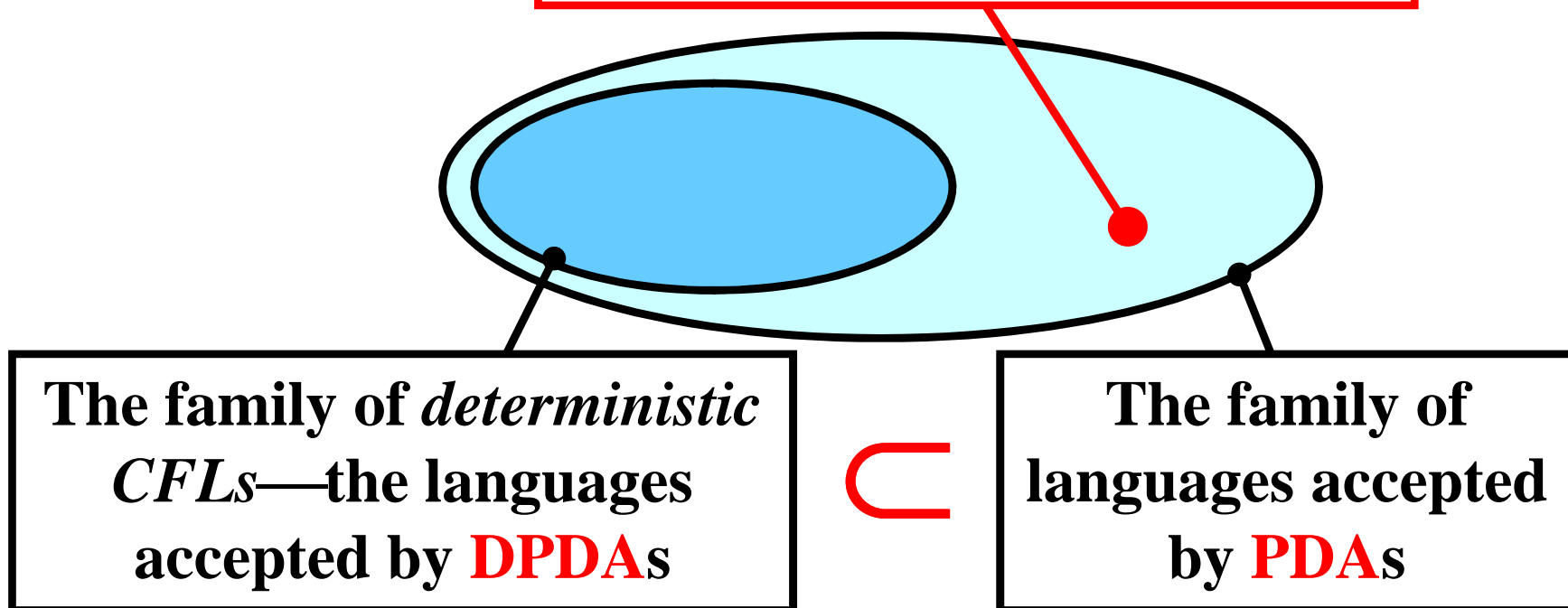
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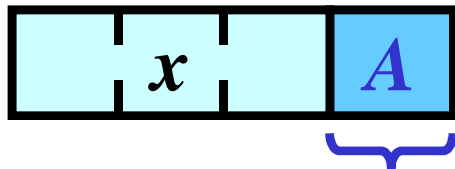
Extended PDA (EPDA)

Gist: The pushdown top of an EPDA represents a string rather than a single symbol.

Definition: An Extended Pushdown automaton (EPDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where $Q, \Sigma, \Gamma, s, S, F$ are defined as in an PDA and R is a finite set of rules of the form: $\mathbf{v}pa \rightarrow wq$, where $\mathbf{v}, w \in \Gamma^*, p, q \in Q, a \in \Sigma \cup \{\epsilon\}$

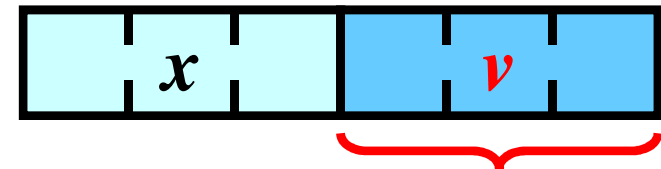
Illustration:

Pushdown of PDA:



PDA has a **single symbols** as the pushdown top

Pushdown of EPDA:



EPDA has **a string** as the pushdown top


Move in EPDA

Definition: Let $x\mathbf{v}pay$ and $xwqy$ be two configurations of an EPDA, M , where $x, \mathbf{v}, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$. Let $r = \mathbf{v}pa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $x\mathbf{v}pay$ to $xwqy$ according to r , written as $x\mathbf{v}pay \dashv\vdash xwqy [r]$ or $x\mathbf{v}pay \dashv\vdash xwqy$.



Move in EPDA

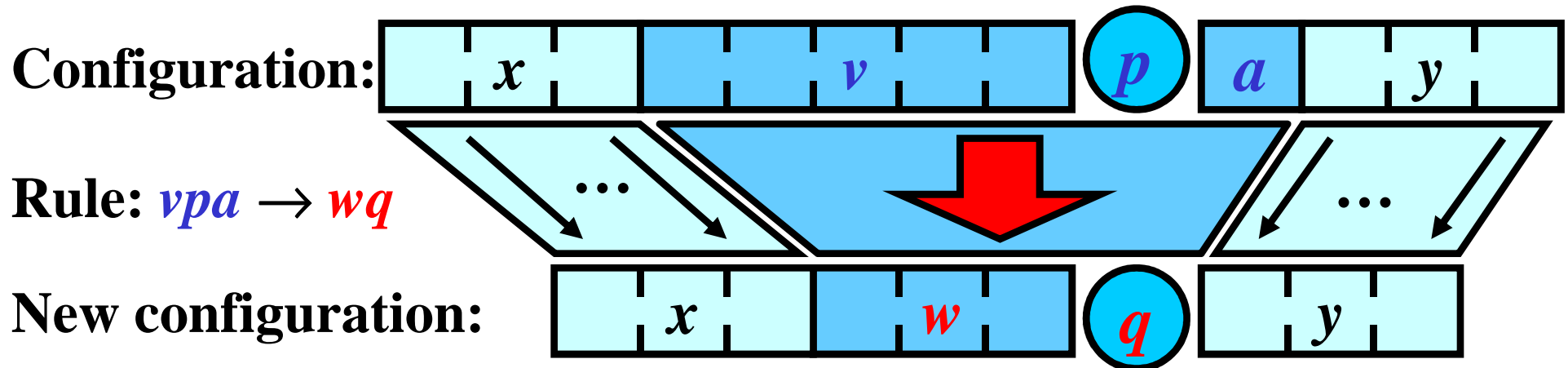
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Configuration: 

Rule: $\mathbf{v}pa \rightarrow \mathbf{w}q$

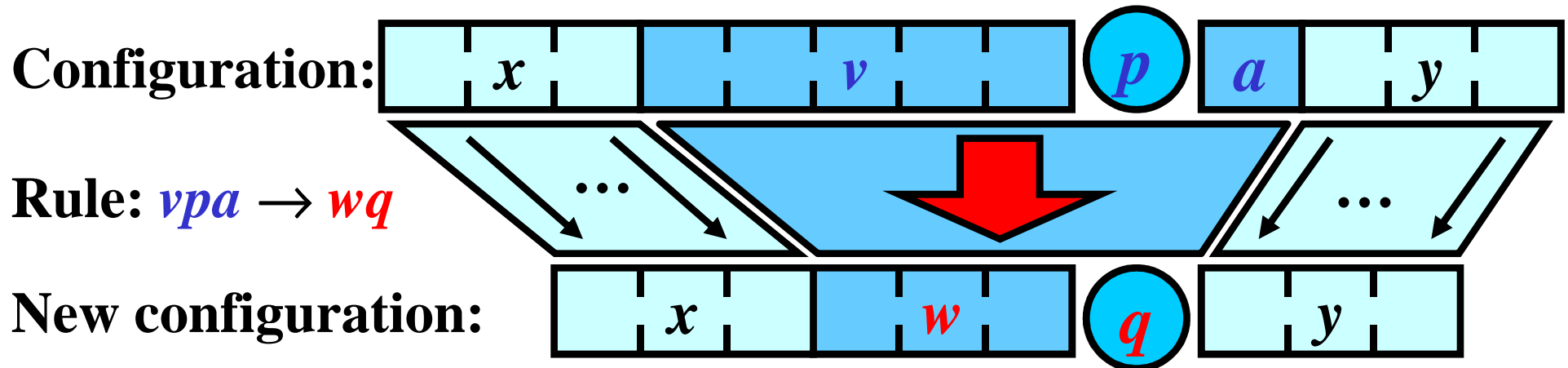
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Note: \vdash^n , \vdash^+ , \vdash^* , $L(M)_f$, $L(M)_\varepsilon$, and $L(M)_{f\varepsilon}$ are defined analogously to the corresponding definitions for PDA.

EPDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

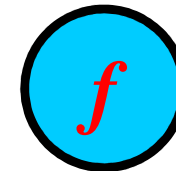
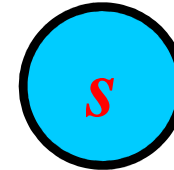
where:

EPDA: Example

$$M = (Q, \Sigma, \Gamma, R, s, S, F)$$

where:

- $Q = \{s, f\};$

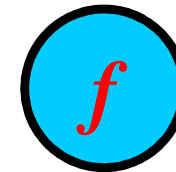
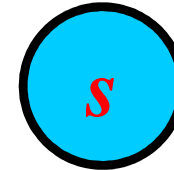


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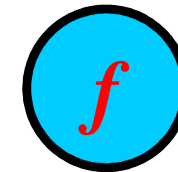
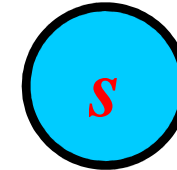


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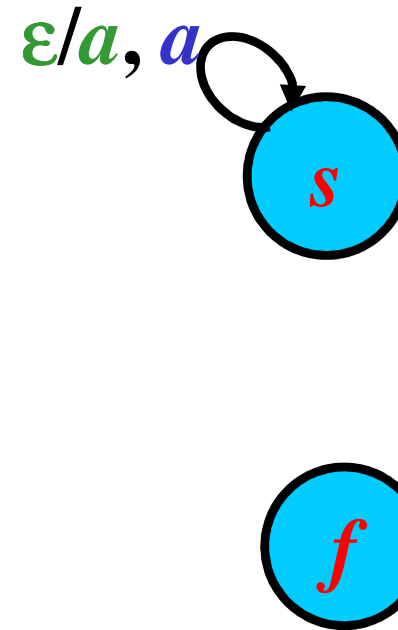


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- $Q = \{s, f\};$
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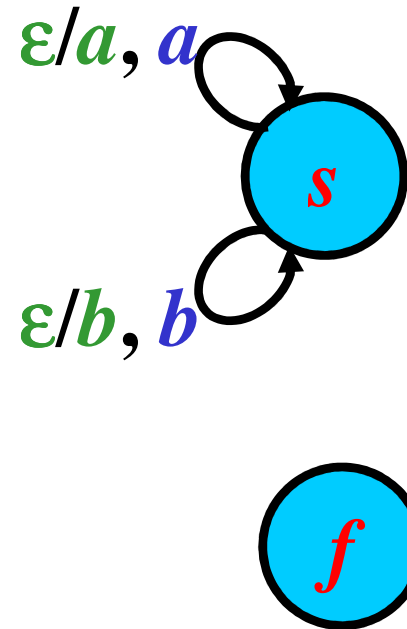


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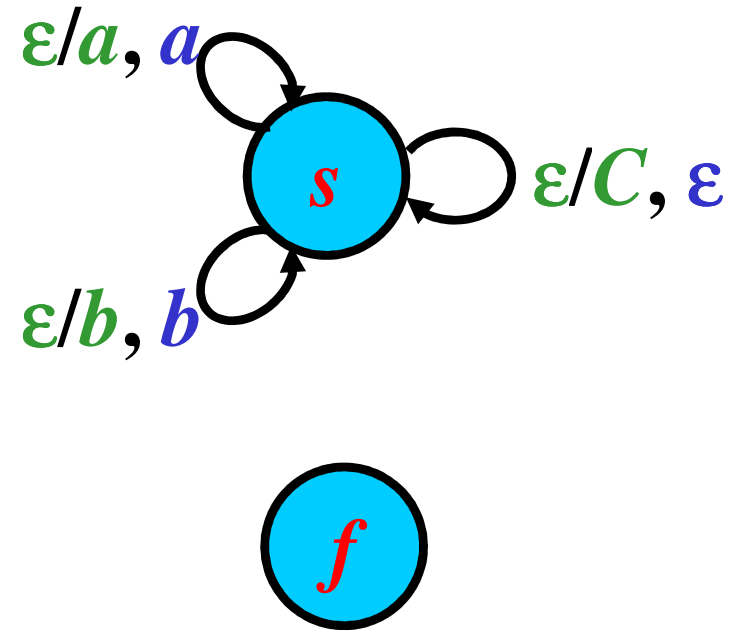


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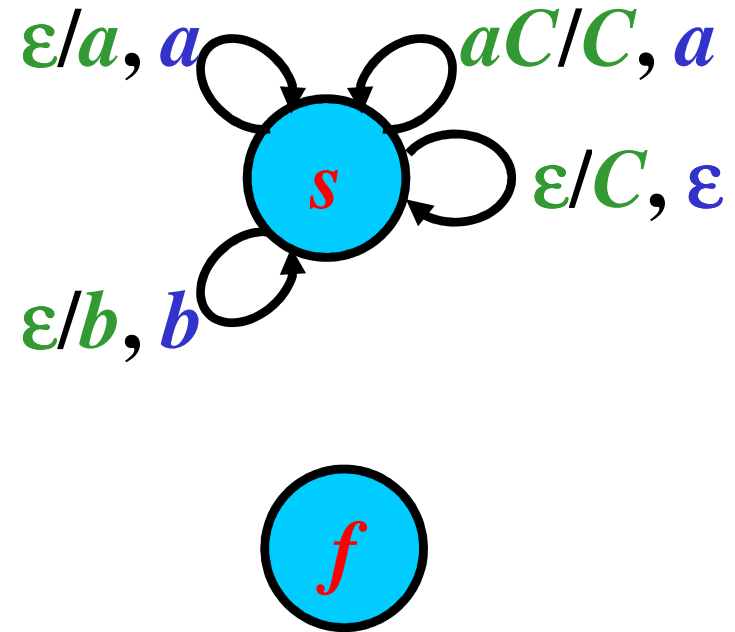


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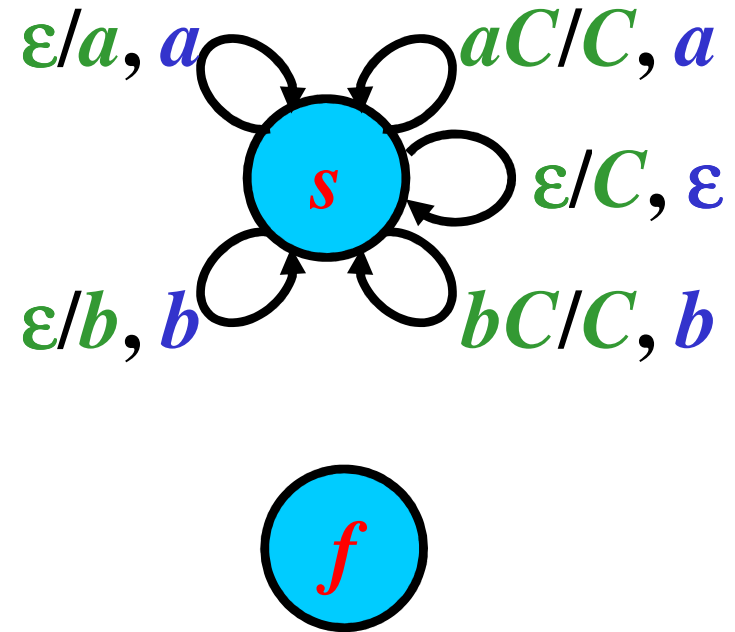


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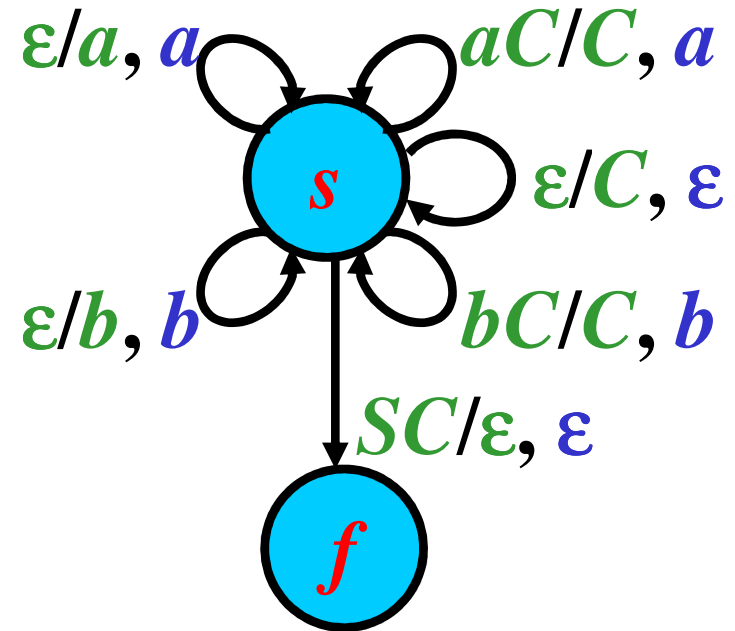


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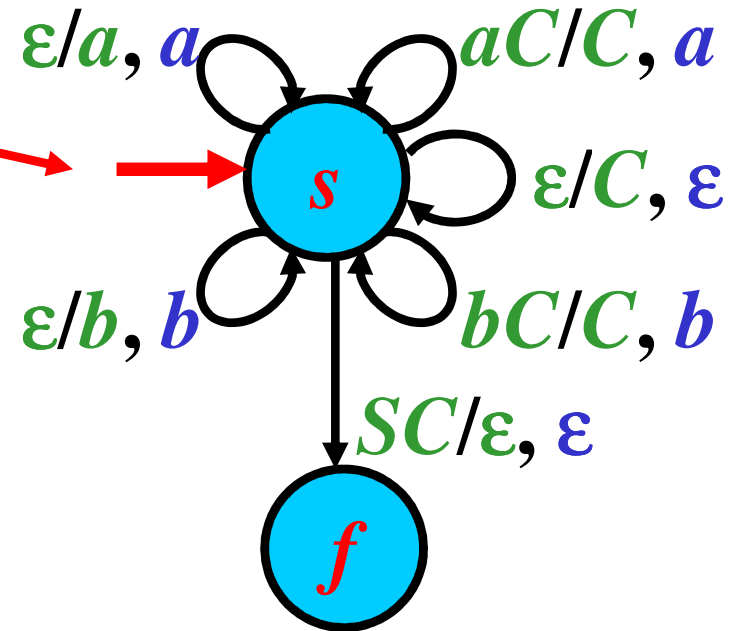


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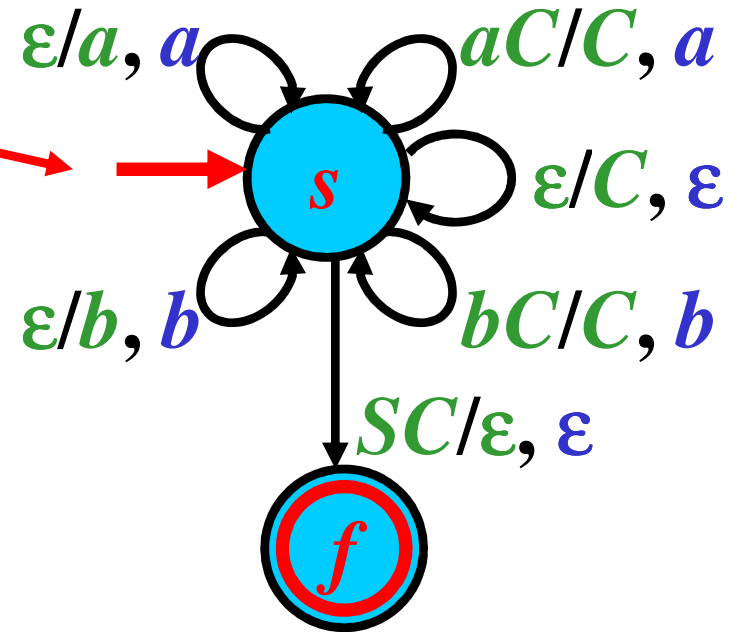


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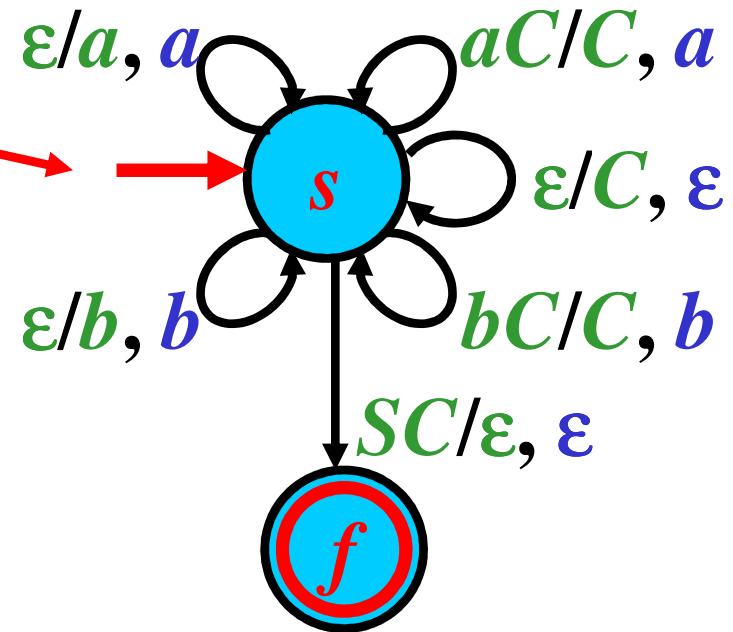


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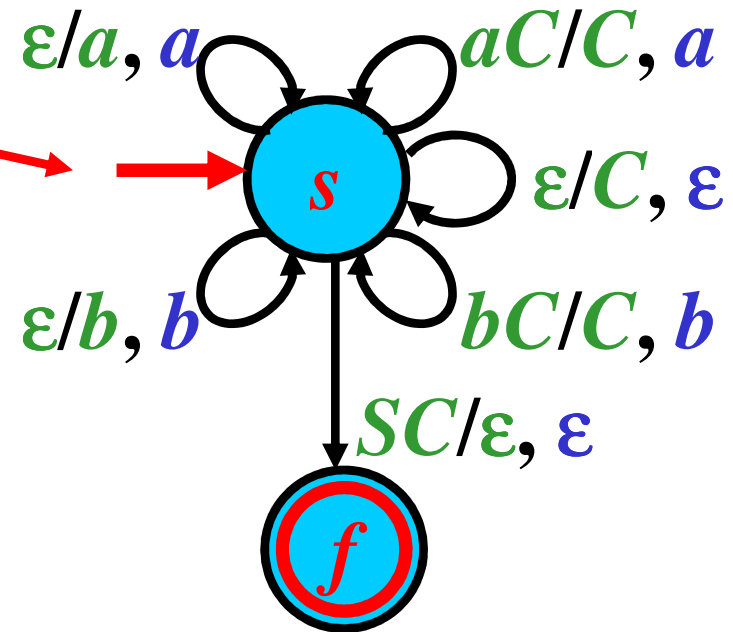
Question: $abba \in L_{f\varepsilon}(M)?$

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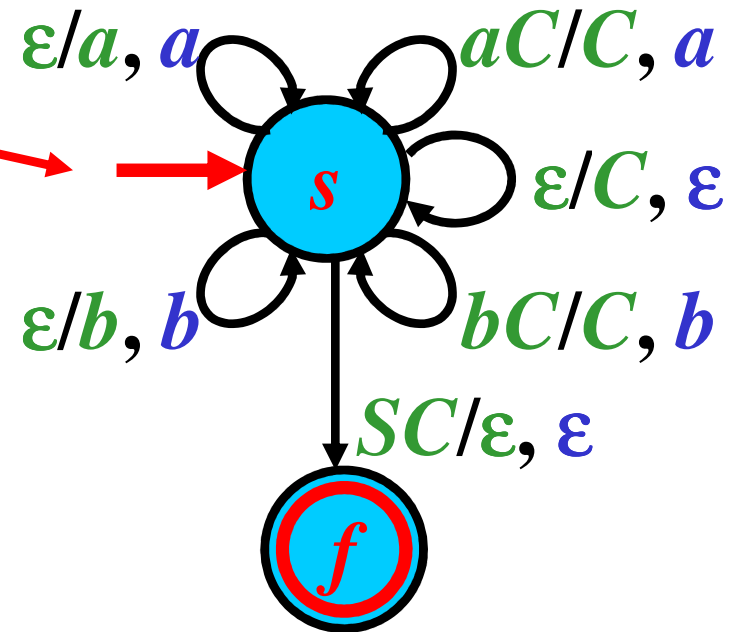
S s $abba$

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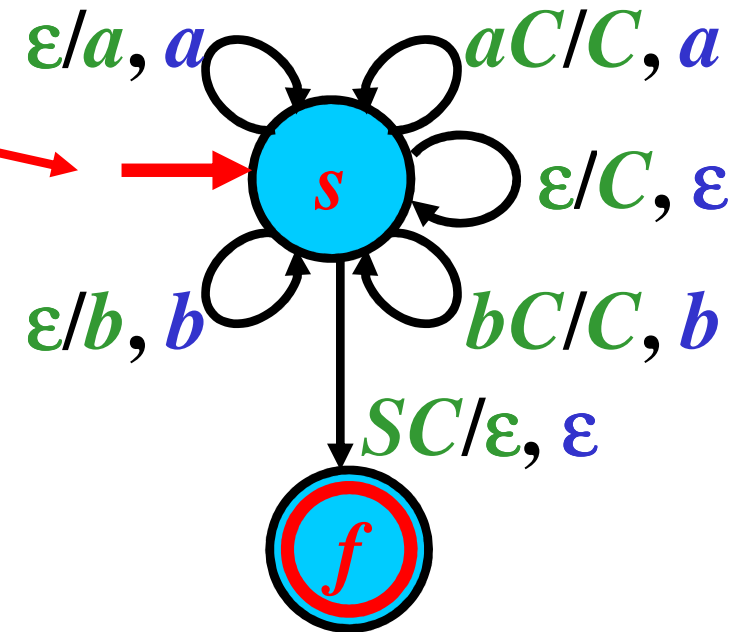
$Ssabba \vdash Ssabba$

EPDA: Example

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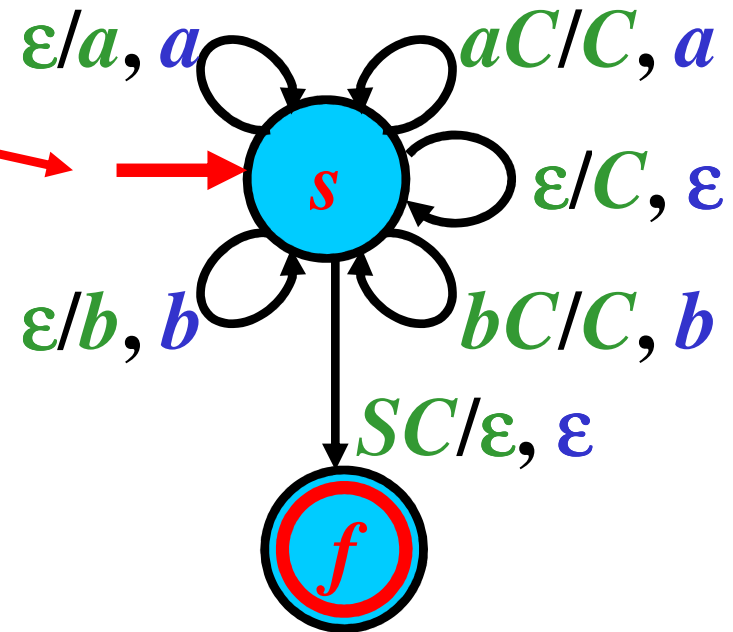
$S\underline{s}abba \mid - S\underline{a}s\underline{b}ba \mid - S\underline{a}b\underline{s}ba$

EPDA: Example

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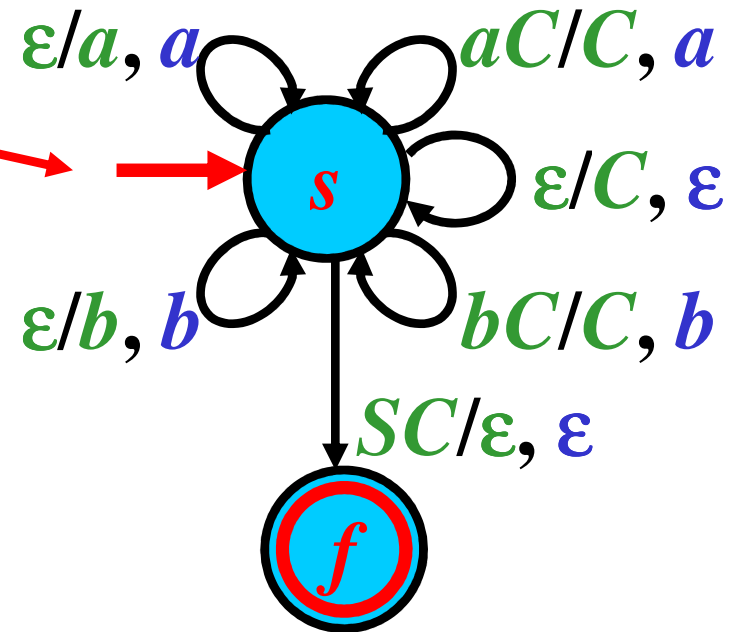
$S\underline{s}abba \mid - S\underline{a}sbba \mid - S\underline{a}b\underline{s}ba$
 $\mid - S\underline{a}bC\underline{s}ba$

EPDA: Example

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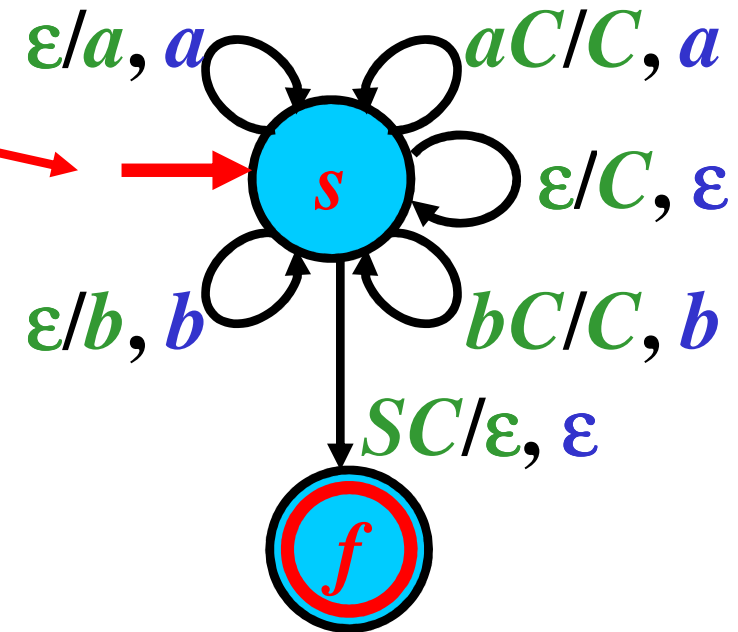
$Ssabba \vdash Sasbba \vdash Sab\underline{s}ba$
 $\vdash Sab\underline{C}sba \vdash Sa\underline{C}sa$

EPDA: Example

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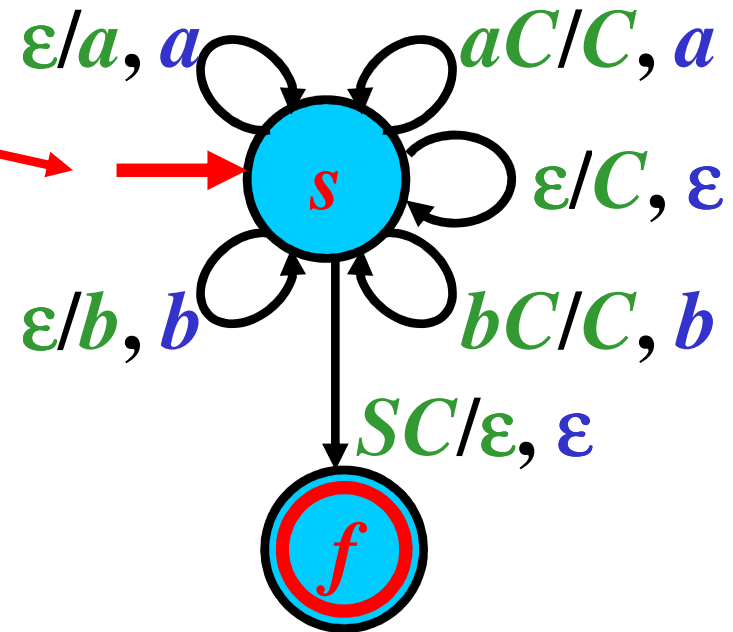
$Ssabba \vdash Sa**s**bba \vdash Sab**s**ba$
 $\vdash SabC**s**ba \vdash SaC**s**a$
 $\vdash SC**s**$

EPDA: Example

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Question: $abba \in L_{f\epsilon}(M)$?

$Ssabba \vdash Sa\textcolor{red}{s}bba \vdash Sab\textcolor{red}{s}ba$
 $\vdash SabC\textcolor{red}{s}ba \vdash SaC\textcolor{red}{s}a$
 $\vdash SC\textcolor{red}{s} \vdash f$

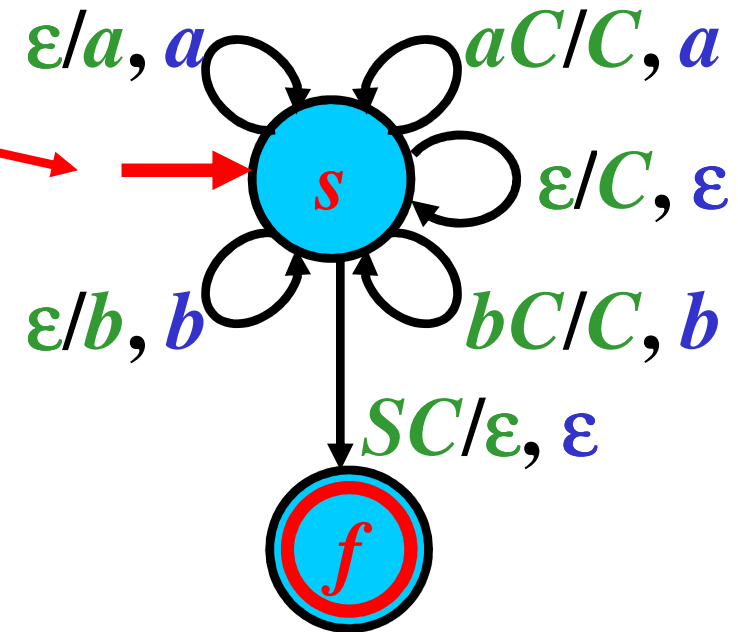
Answer: YES

EPDA: Example

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Question: $abba \in L_{f\varepsilon}(M)$?

$S\underline{s}abba \mid - Sa\underline{s}bba \mid - Sab\underline{s}ba$
 $\mid - Sab\underline{C}sba \mid - Sa\underline{C}sa$
 $\mid - \underline{SC}s \mid - f$

Answer: **YES**

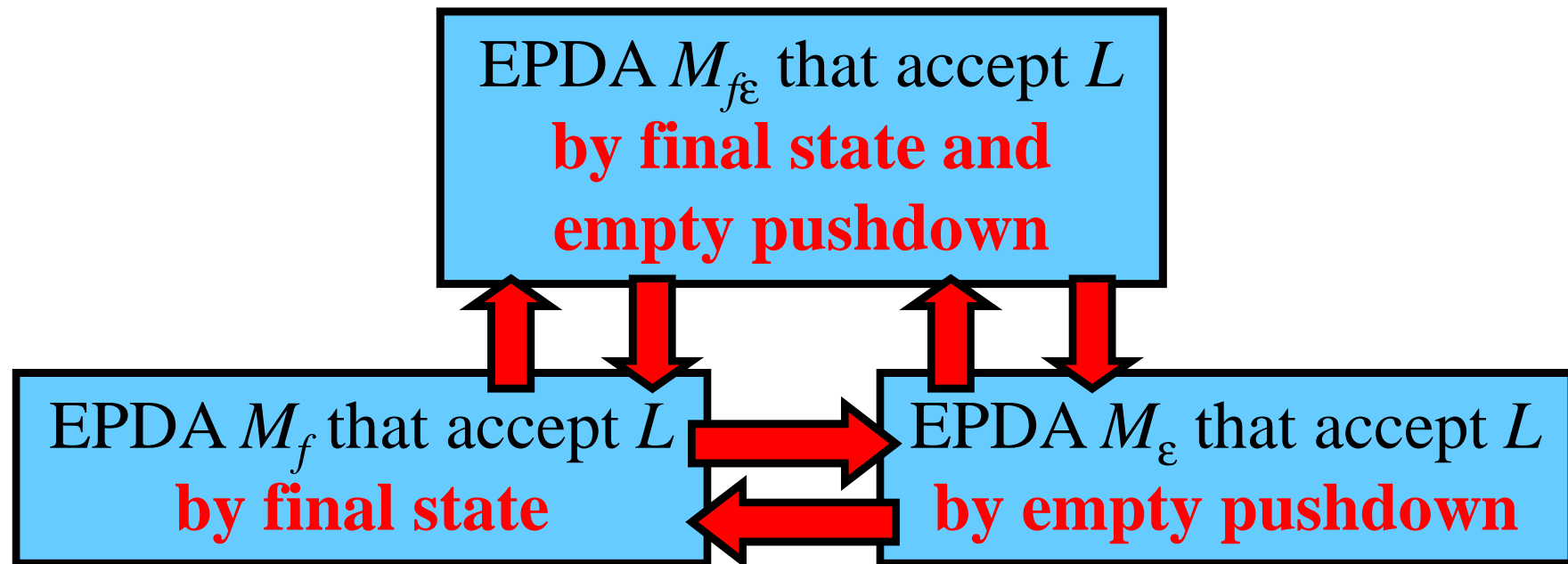
Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{xy : x, y \in \Sigma^*, y = \text{reversal}(x)\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for an EPDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for an EPDA M_ε

Note: There exist these conversion:



EPDAs and PDAs are Equivalent

Theorem: For every EPDA M , there is a PDA M' ,
and $L(M)_f = L(M')_f$.

Proof: See page 419 in [Meduna: Automata and Languages]

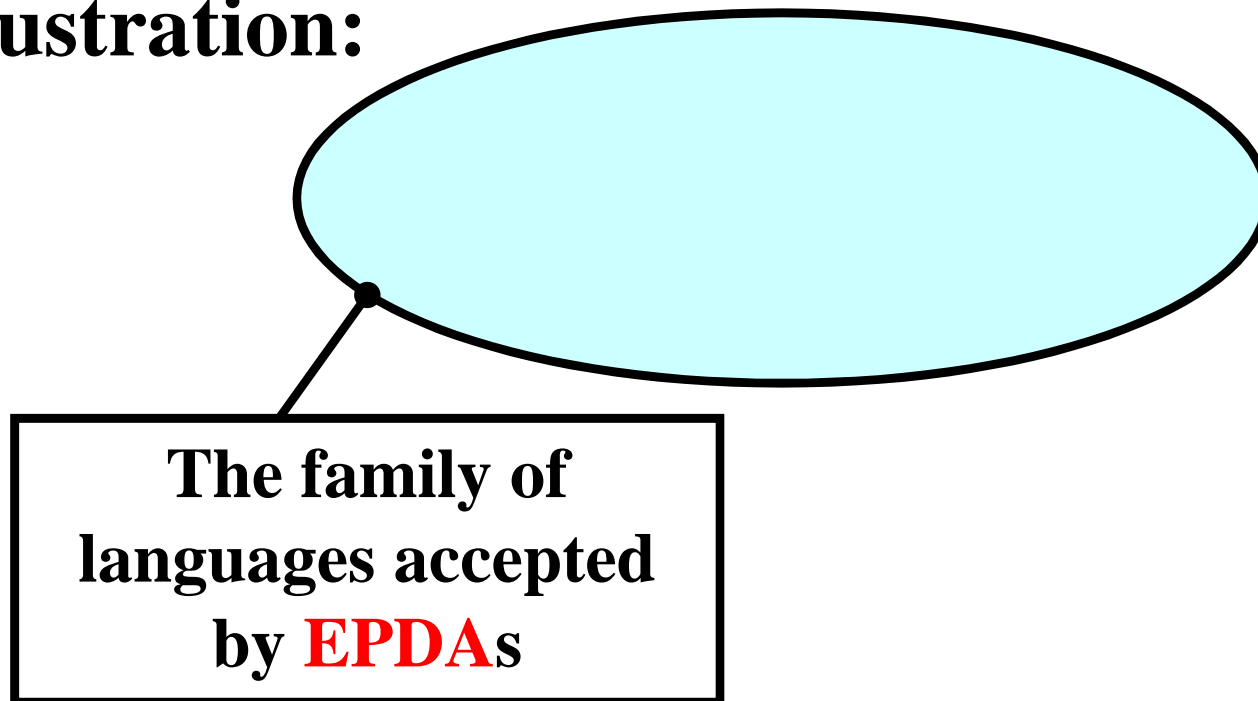
Illustration:

EPDAs and PDAs are Equivalent

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Illustration:

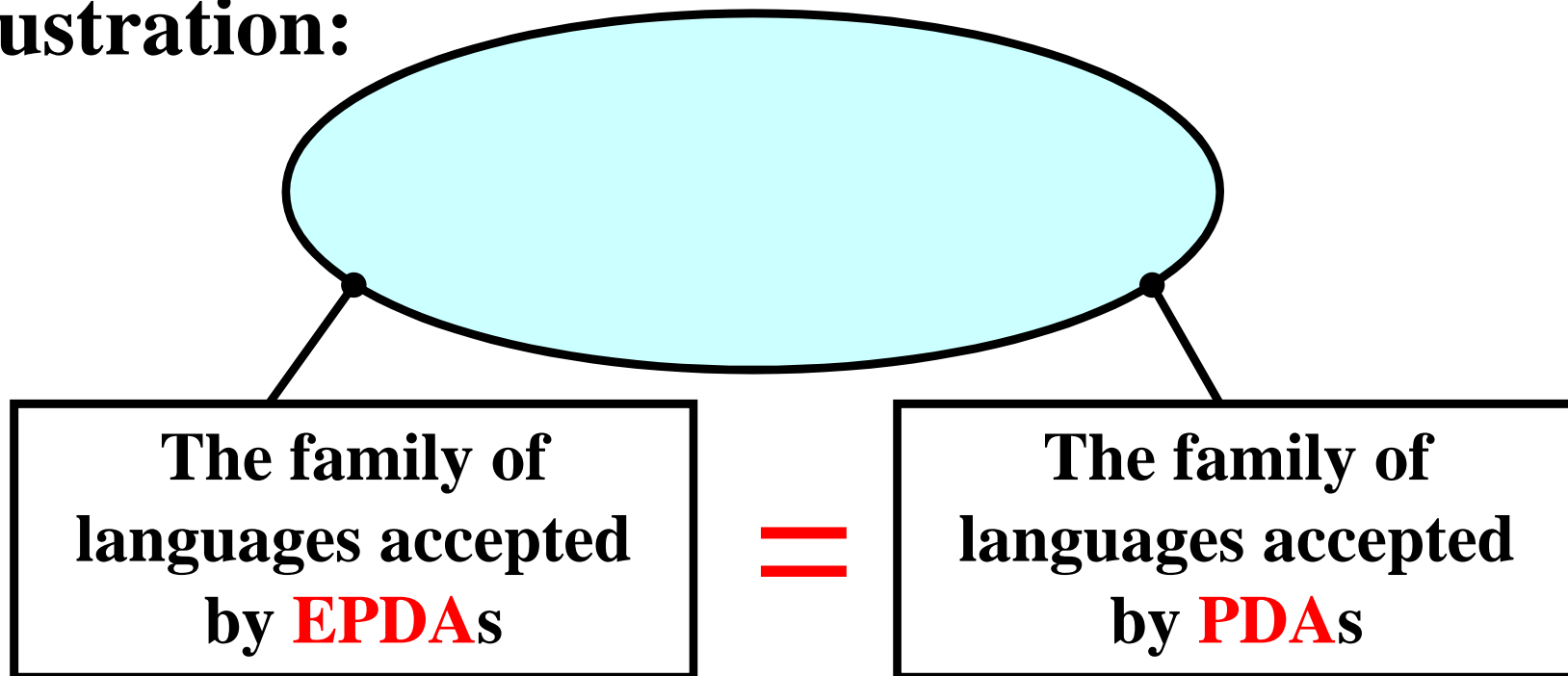


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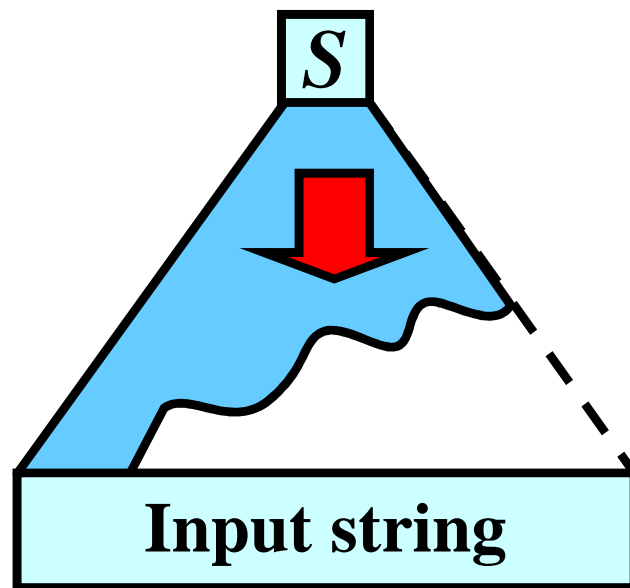


EPDAs and PDAs as Parsing Models for CFGs

Gist: An EPDA or a PDA can simulate the construction of a derivation tree for a CFG

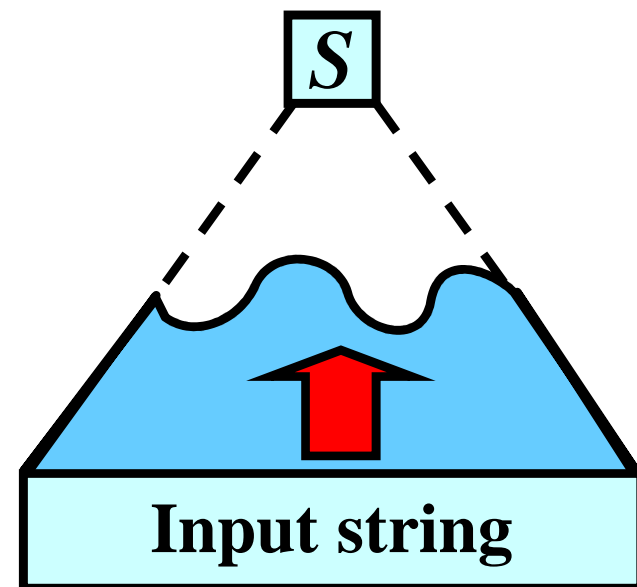
• Two basic approaches:

1) Top-Down Parsing



From S towards the input string

2) Bottom-Up Parsing

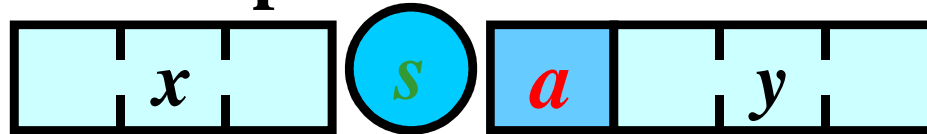


From the input string towards S

EPDAs as Models of Bottom-Up Parsers 1/2

Gist: An EPDA M underlies a bottom-up parser

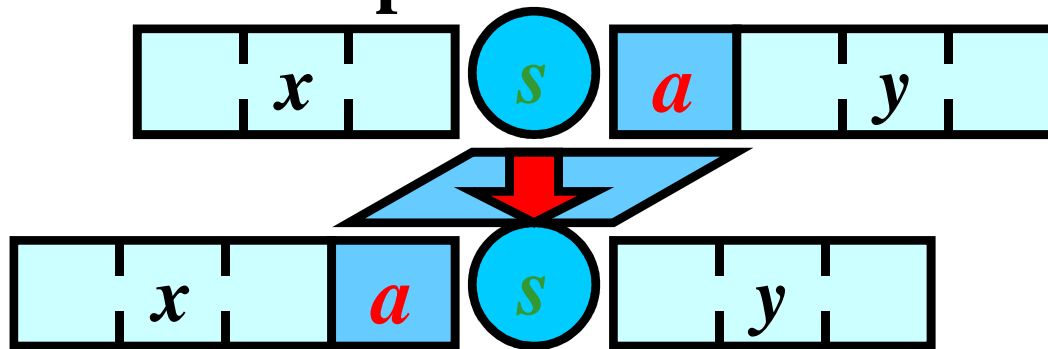
1) M contains *shift* rules that copy the input symbols onto the pushdown:



EPDAs as Models of Bottom-Up Parsers 1/2

Gist: An EPDA M underlies a bottom-up parser

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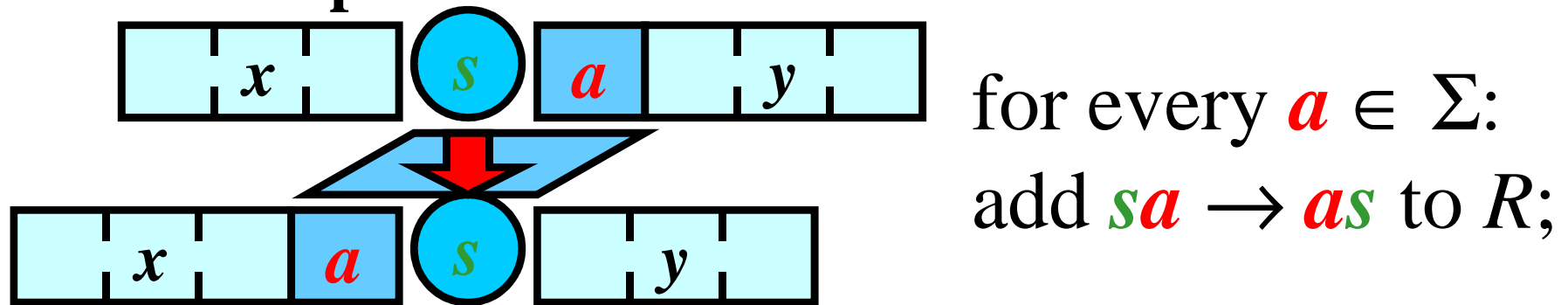


for every $a \in \Sigma$:
add $sa \rightarrow as$ to R ;

EPDAs as Models of Bottom-Up Parsers 1/2

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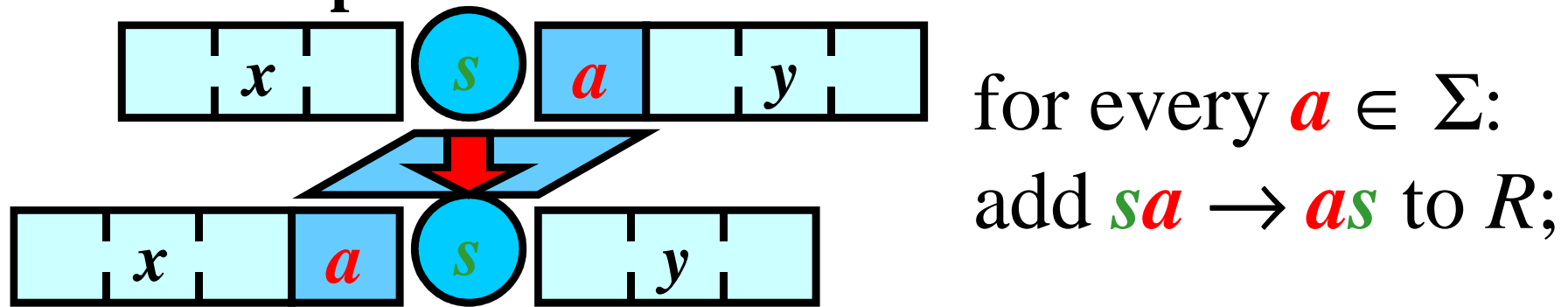
2) M contains *reduction* rules that simulate the application of a grammatical rule in reverse:



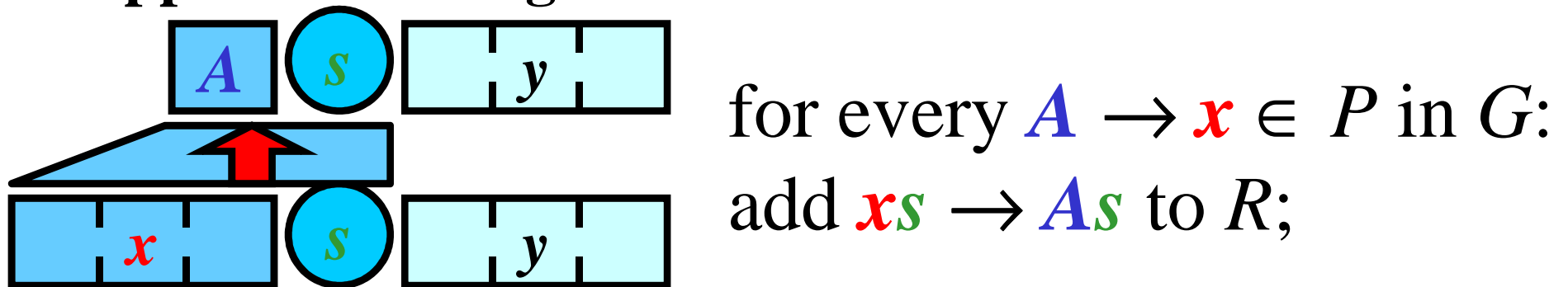
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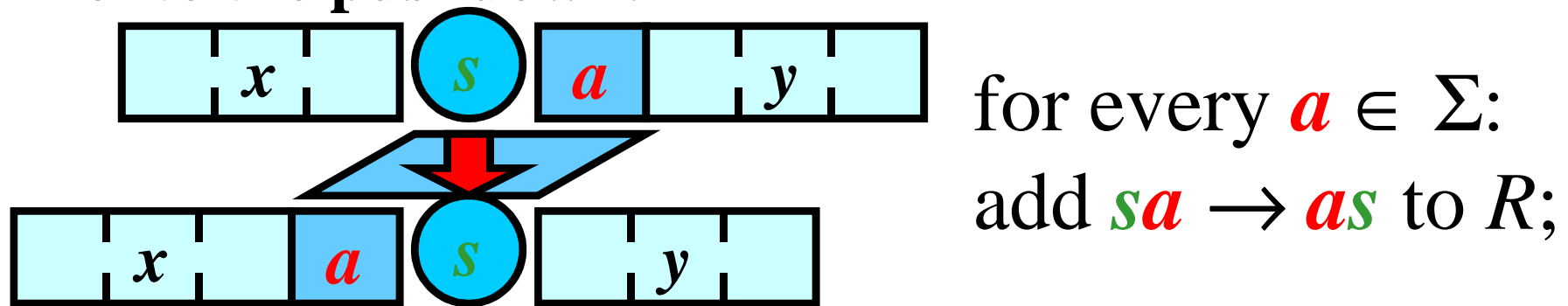
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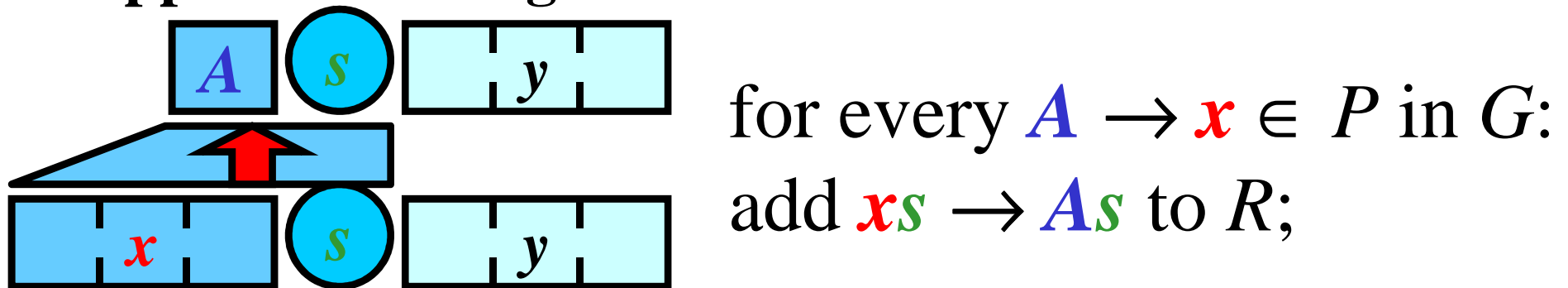
EPDAs as Models of Bottom-Up Parsers 1/2

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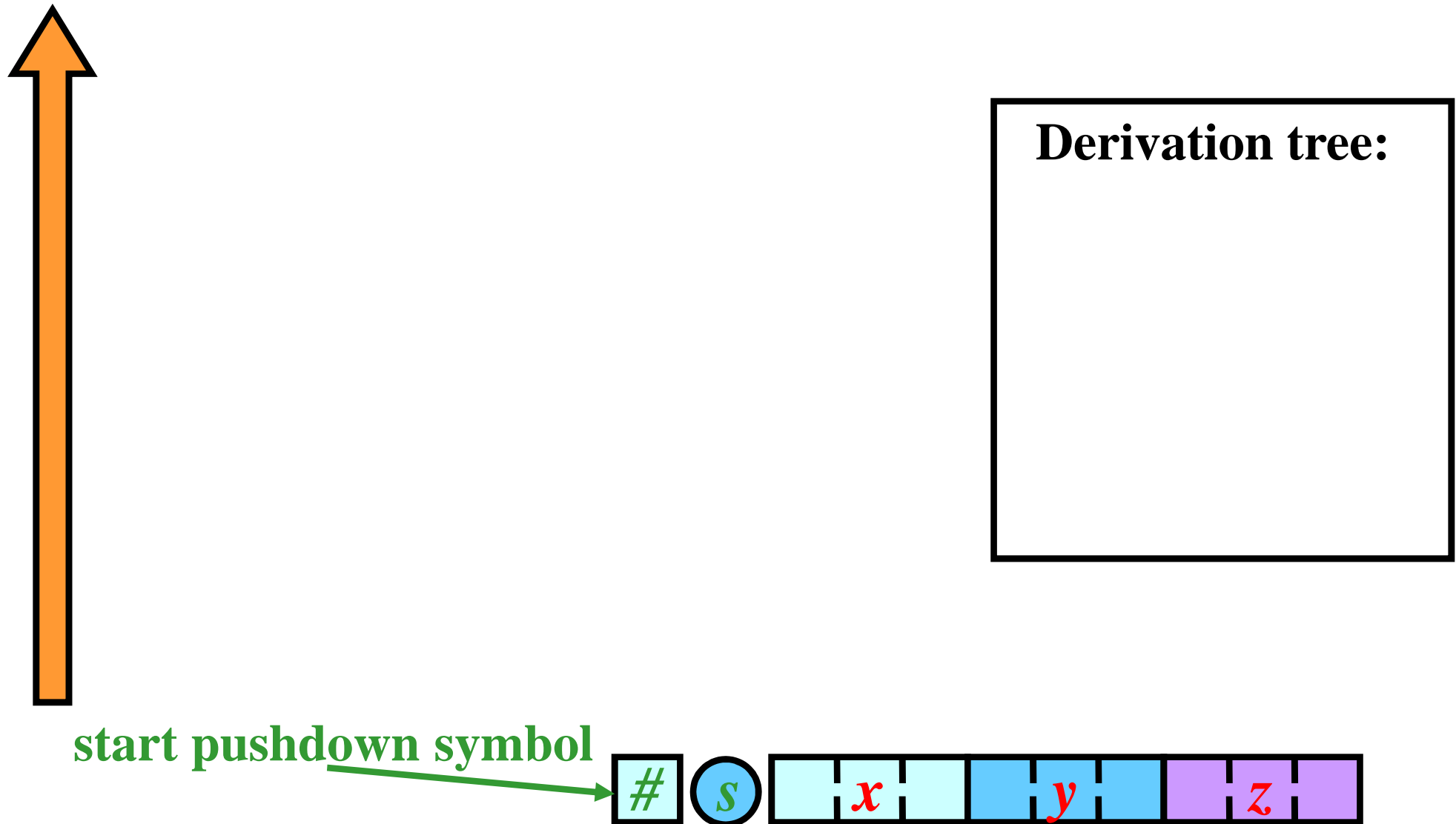
2) M contains *reduction* rules that simulate the application of a grammatical rule in reverse:



3) M also contains the rule $\#Ss \rightarrow f$ that takes M to a final state f

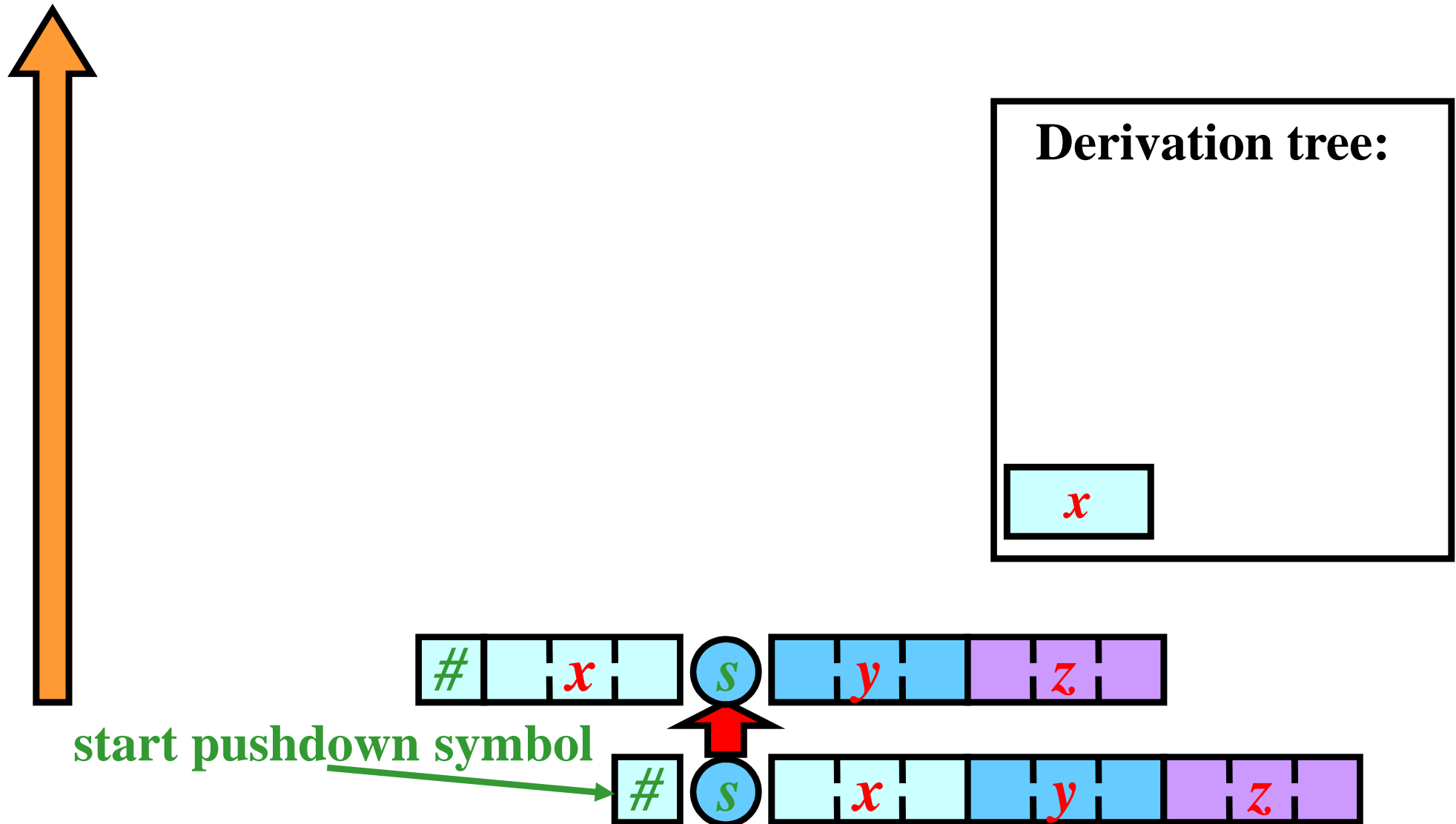
EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



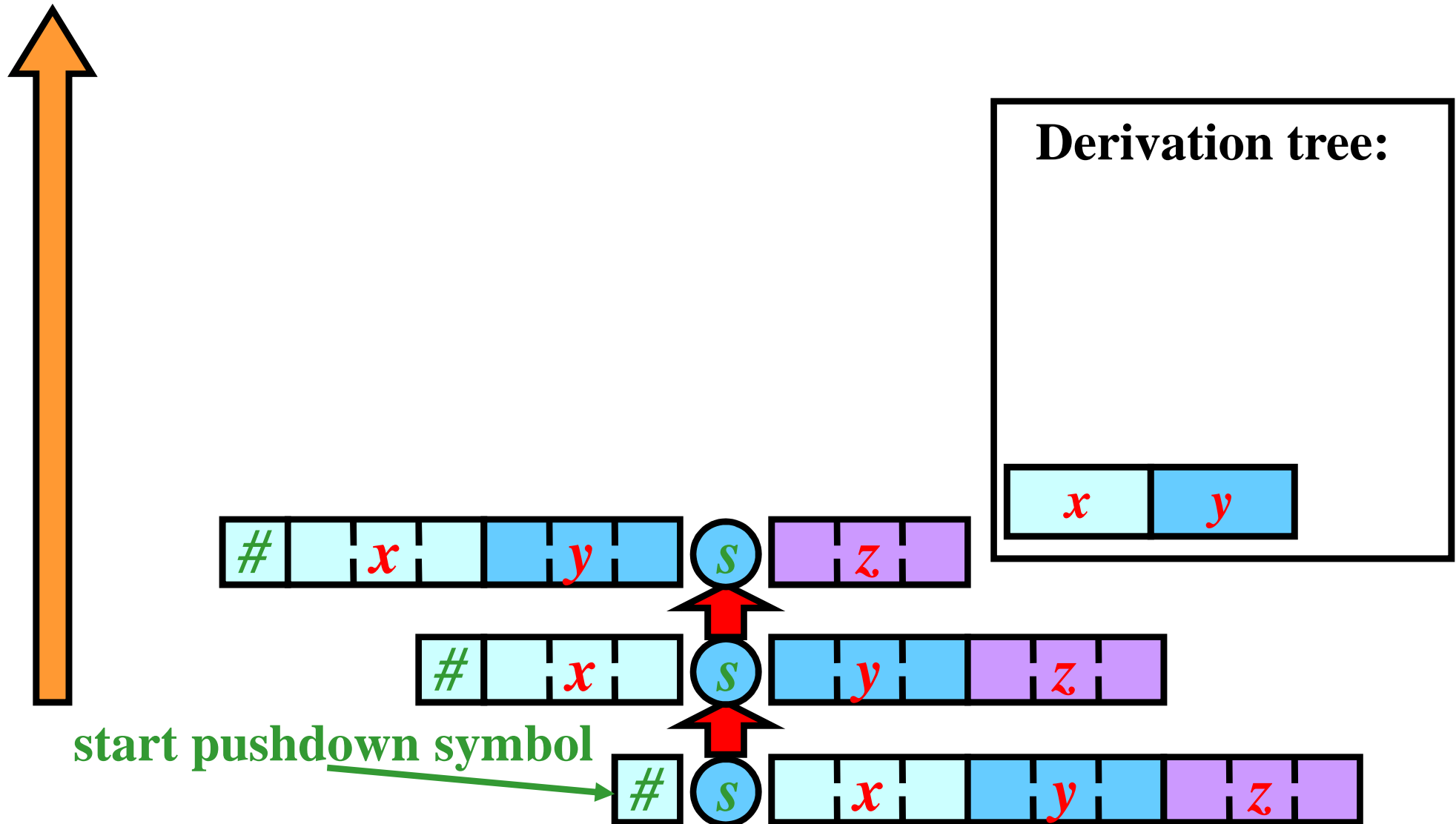
EPDAs as Models of Bottom-Up Parsers 2/2

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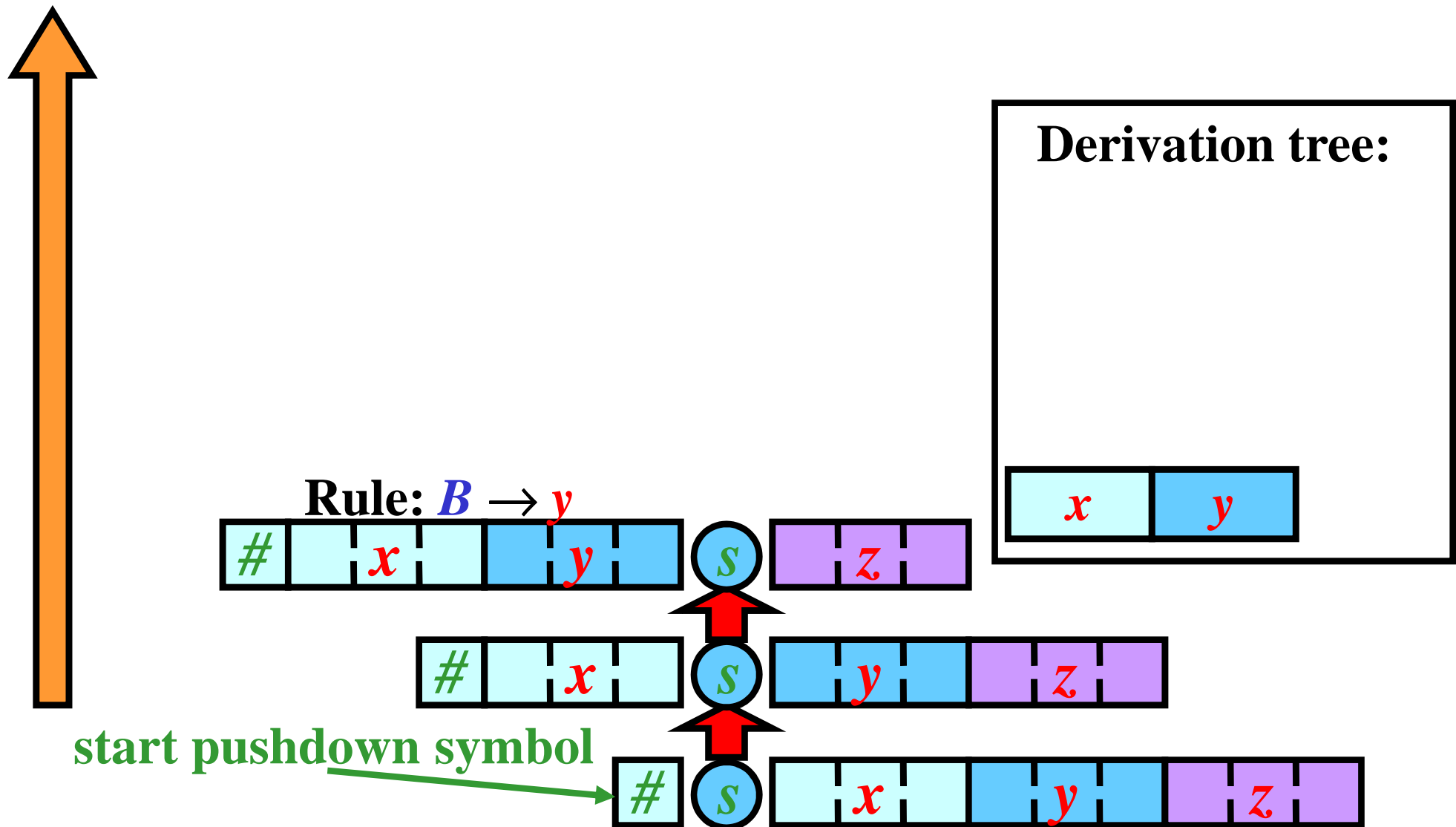
EPDAs as Models of Bottom-Up Parsers 2/2

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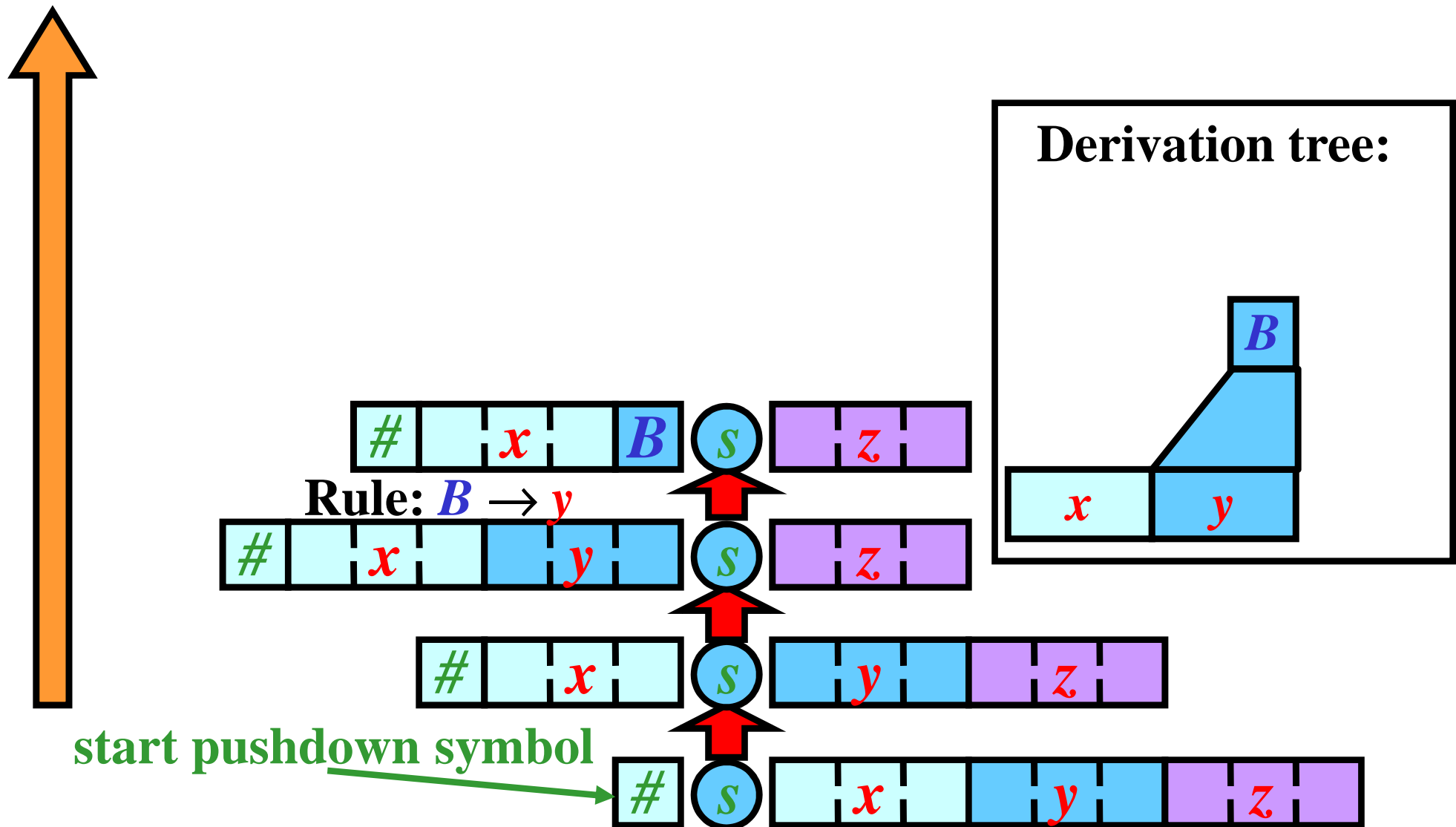
EPDAs as Models of Bottom-Up Parsers 2/2

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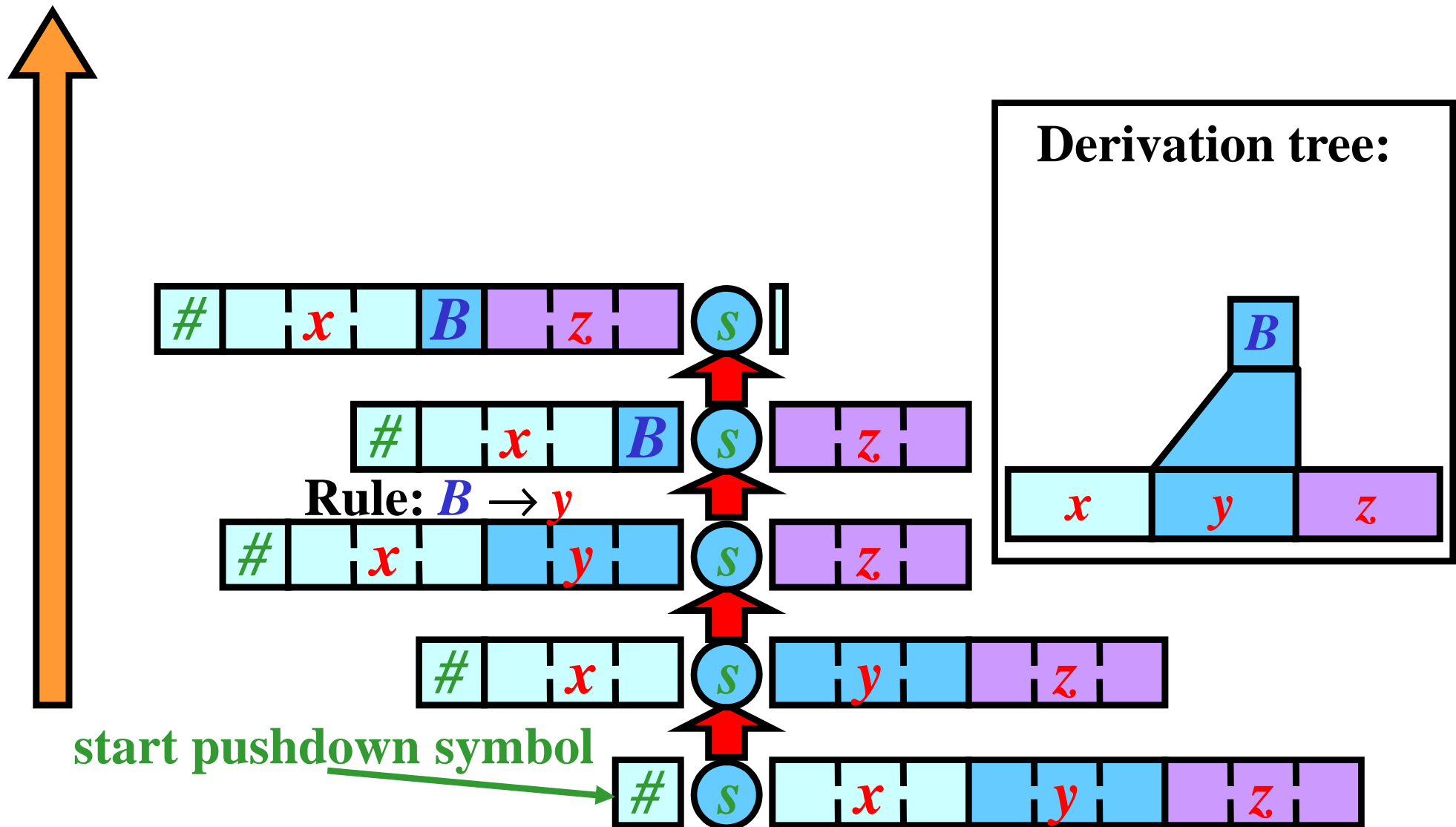
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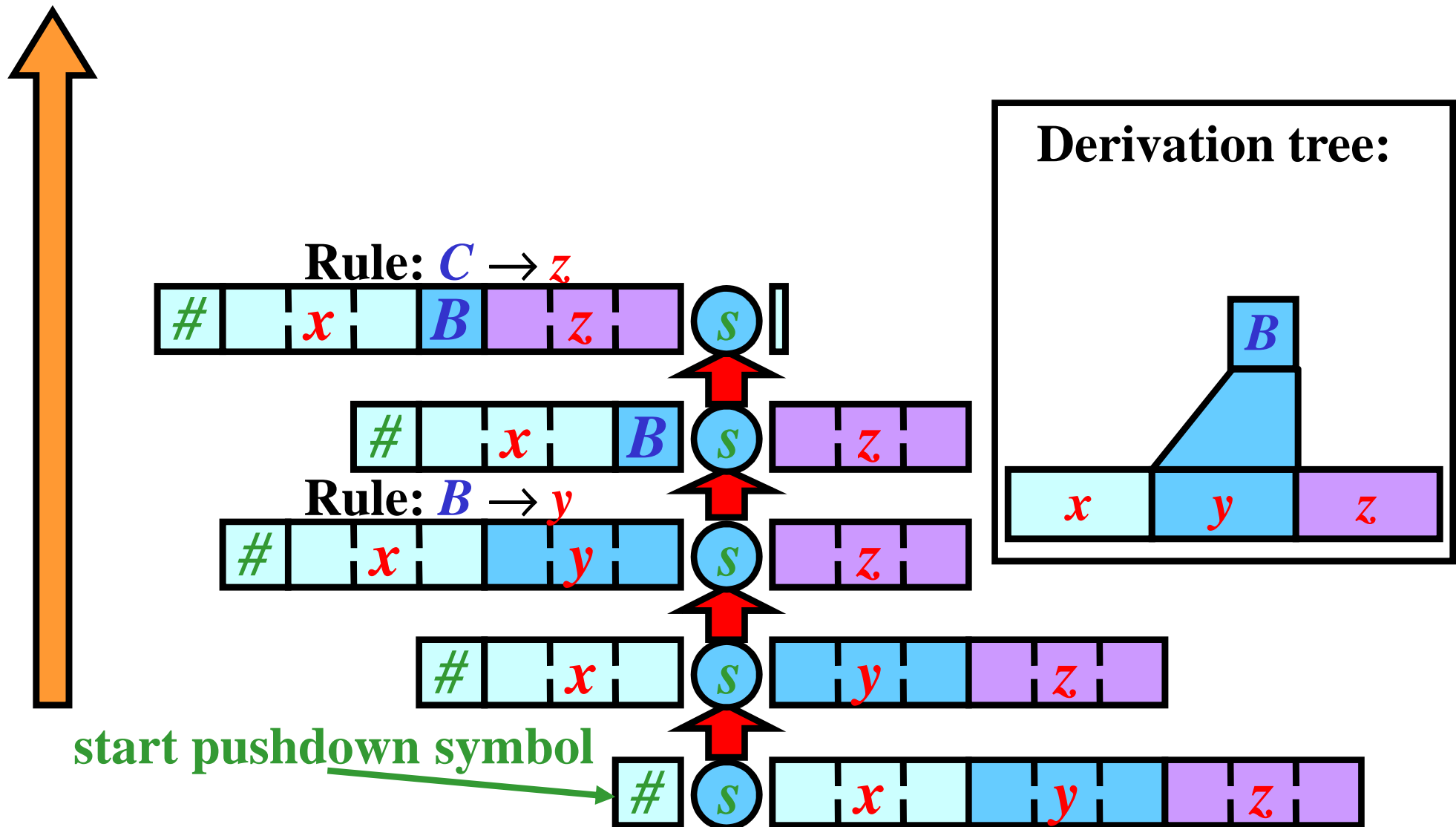
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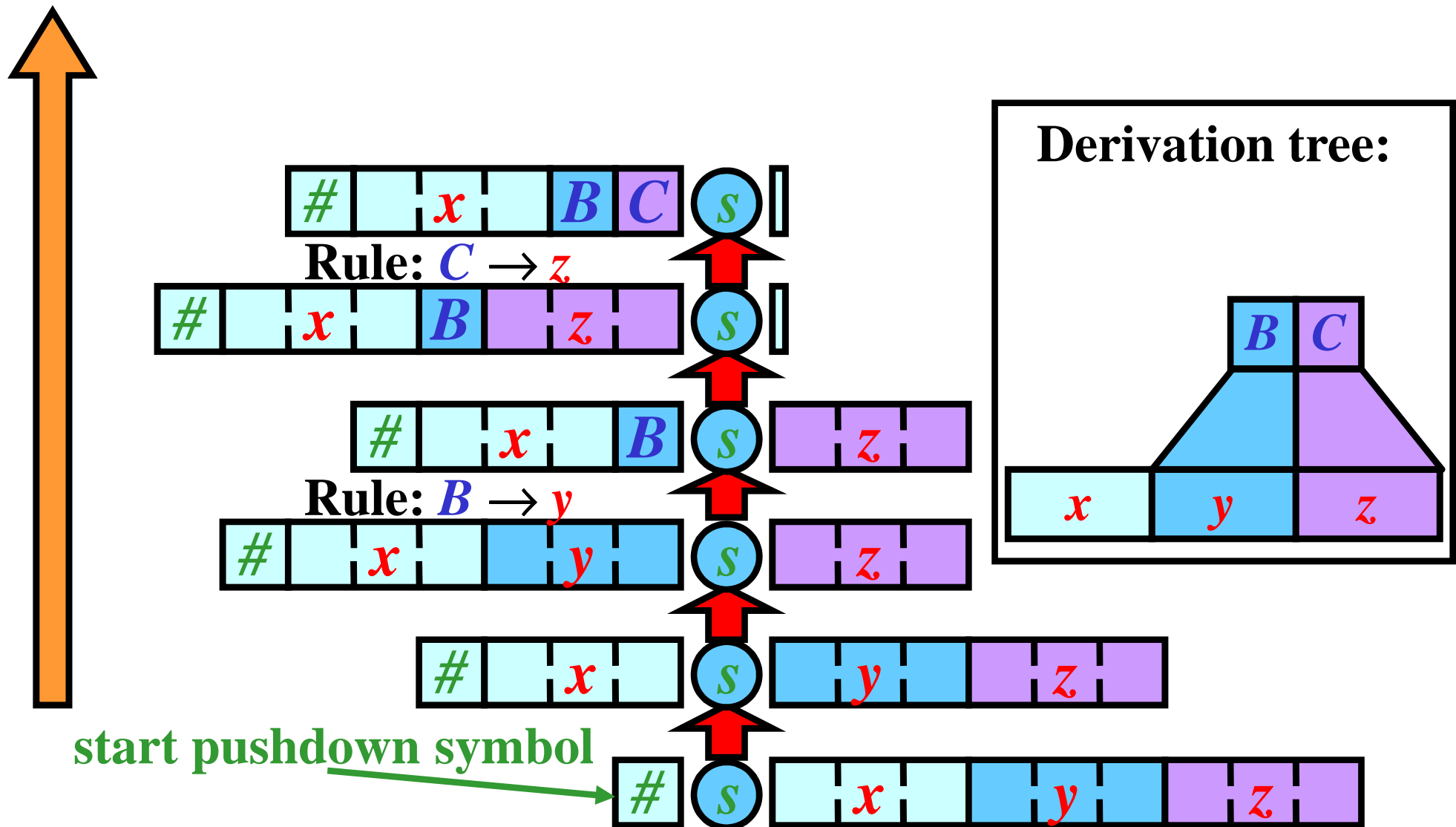
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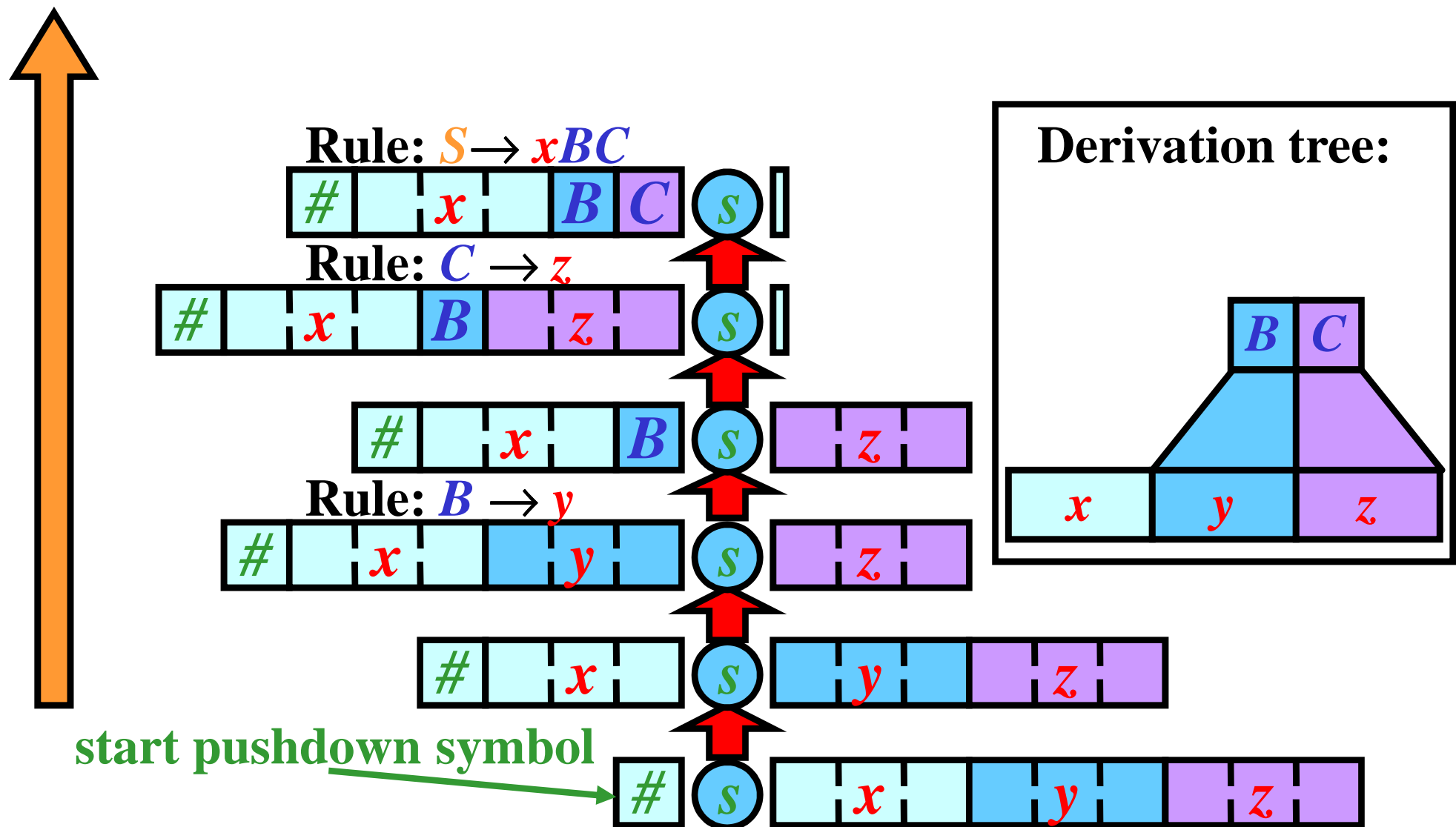
EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



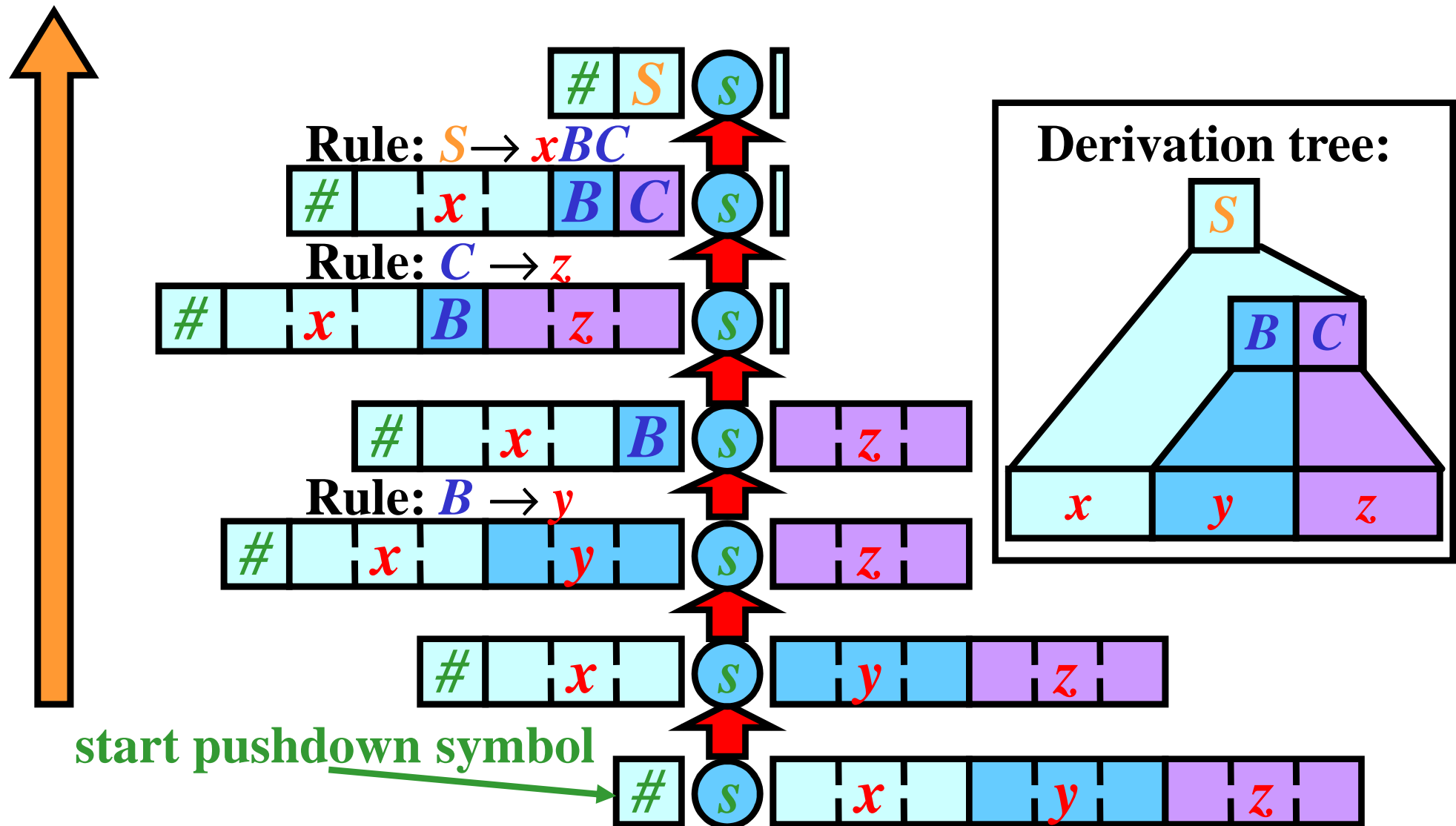
EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



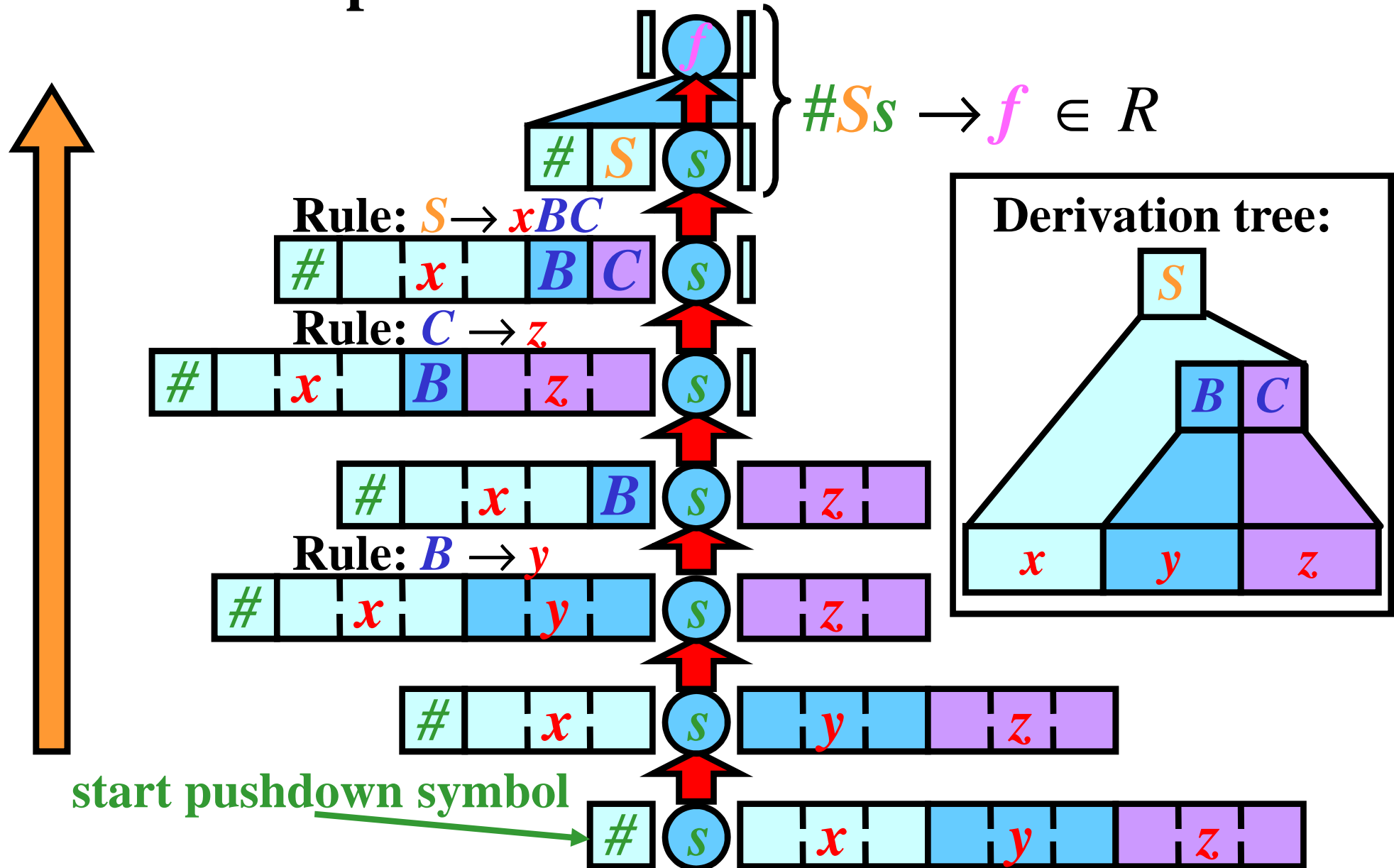
EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



Algorithm: From CFG to EPDA

- **Input:** CFG $G = (N, T, P, S)$
 - **Output:** EPDA $M = (Q, \Sigma, \Gamma, R, s, \#, F)$; $L(G) = L(M)_f$
-
- **Method:**
 - $Q := \{s, f\};$
 - $\Sigma := T;$
 - $\Gamma := N \cup T \cup \{\#\};$
 - Construction of R :
 - for every $a \in \Sigma$, add $sa \rightarrow as$ to R ;
 - for every $A \rightarrow x \in P$, add $xs \rightarrow As$ to R ;
 - add $\#Ss \rightarrow f$ to R ;
 - $F := \{f\};$

From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

$$N = \{S\}, T = \{ (,) \}, P = \{ S \rightarrow (S), S \rightarrow () \}$$

Objective: An EPDA M such that $L(G) = L(M)_f$

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$ where:

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$$“(” \in T \quad “)” \in T$$



$$R = \{s(\rightarrow (s, s) \rightarrow)s,$$

From CFG to EPDA: Example 1/2

- $G = (N, T, P, \mathbf{S})$, where:

$$N = \{\textcolor{blue}{S}\}, T = \{(\textcolor{red}{,} \textcolor{red}{})\}, P = \{\textcolor{blue}{S} \rightarrow (\textcolor{blue}{S}), \textcolor{blue}{S} \rightarrow (\textcolor{red}{})\}$$

Objective: An EPDA M such that $L(G) = L(M)_f$

$$M = (Q, \Sigma, \Gamma, R, s, \#, F) \text{ where:}$$

$$Q = \{\textcolor{green}{s}, \textcolor{violet}{f}\}; \Sigma = T = \{(\textcolor{red}{,}, \textcolor{red}{})\}; \Gamma = N \cup T \cup \{\textcolor{green}{\#}\} = \{\textcolor{blue}{S}, (\textcolor{red}{,}), \textcolor{green}{\#}\}$$

$$\begin{array}{ccc}
 \text{“(”} \in T & \text{“)”} \in T & \text{S} \rightarrow \text{(S)} \in P \\
 \Downarrow & \Downarrow & \swarrow \searrow \\
 R = \{ \text{S(} \rightarrow \text{(S, } & \text{S)} \rightarrow \text{)S, } & \text{(S)S} \rightarrow \text{SS,} \\
 \end{array}$$

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Diagram illustrating the mapping from CFG rules to EPDA rules:

- CFG rules: $(\, \in T$, $\,) \in T$, $S \rightarrow (S) \in P$, $S \rightarrow (\,) \in P$
- EPDA rules: $s(\rightarrow (s, s) \rightarrow)s$ (shift rules), $(S)s \rightarrow Ss$, $(\,)s \rightarrow Ss$ (reduction rules), $\#Ss \rightarrow f$

From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

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Diagram illustrating the derivation of the reduction rules from the CFG rules:

- From $(\in T$ and $) \in T$, we derive the shift rules $s(\rightarrow (s, s) \rightarrow)s$.
- From $S \rightarrow (S) \in P$ and $S \rightarrow () \in P$, we derive the reduction rules $(S)s \rightarrow Ss$ and $()s \rightarrow Ss$.

$$F = \{f\}$$

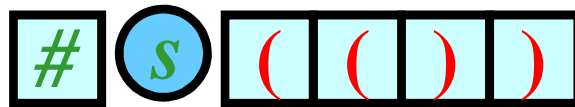
From CFG to EPDA: Example 2/2

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$R = \{ s(\rightarrow (s, s) \rightarrow)s, (S)s \rightarrow Ss, ()s \rightarrow Ss, \#Ss \rightarrow f \}$

Question: $(()) \in L(M)_f$?



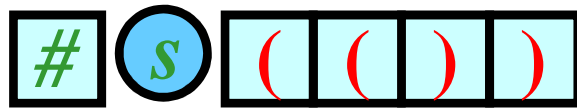
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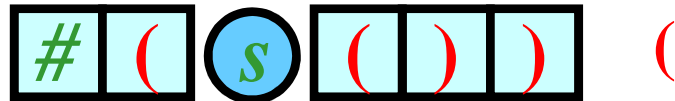
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Rule: $s(\rightarrow (s$



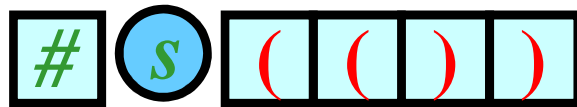
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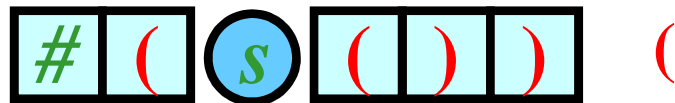
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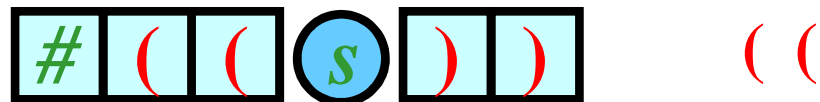
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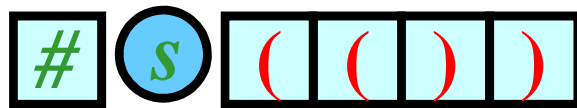
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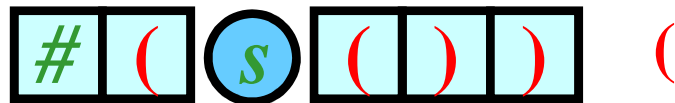
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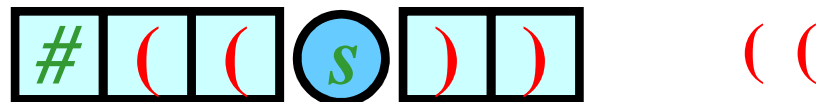
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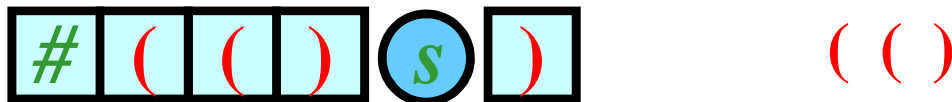
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Rule: $s(\rightarrow (s$



Rule: $s) \rightarrow)s$



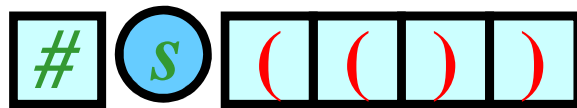
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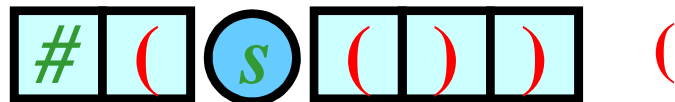
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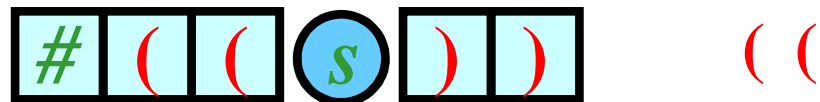
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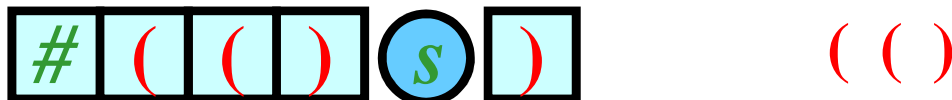
Rule: $s(\rightarrow (s$



Rule: $s(\rightarrow (s$



Rule: $s) \rightarrow)s$



Rule: $()s \rightarrow S$

After applying the rule $()s \rightarrow S$, the head has moved to the right and written a blue S on the tape. The tape now contains $\#$, $($, S , s , $)$, and $($. The head is now over the s . A blue S with an arrow pointing down to the $($ is shown to the right of the tape.

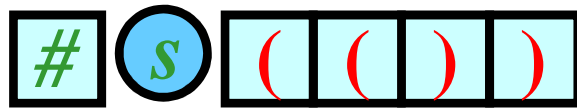
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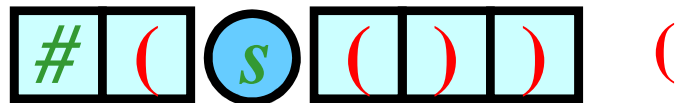
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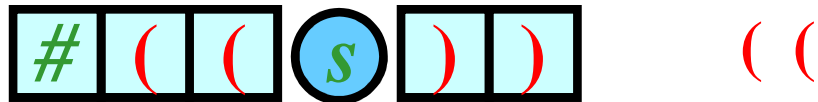
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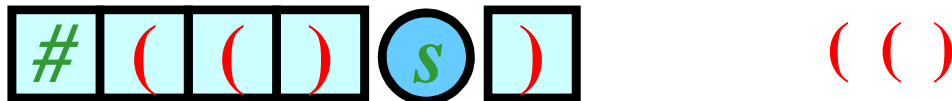
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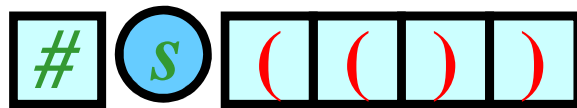
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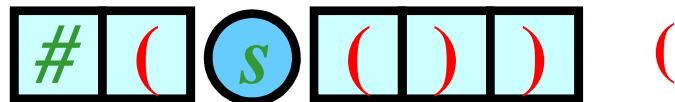
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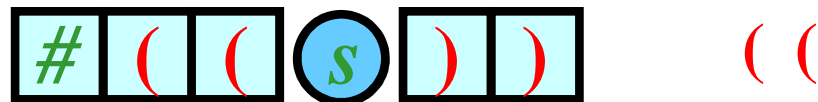
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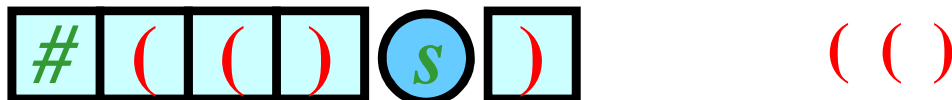
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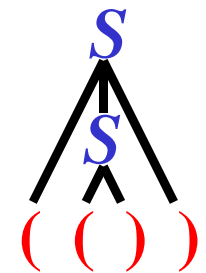
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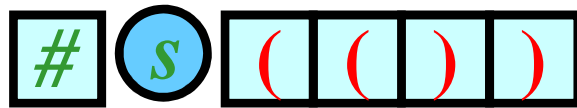
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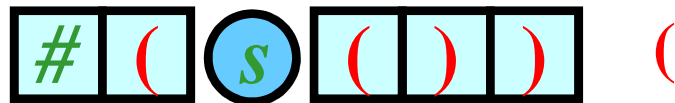
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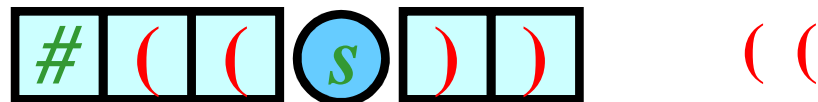
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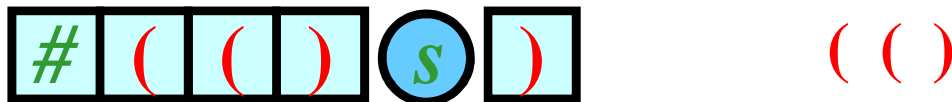
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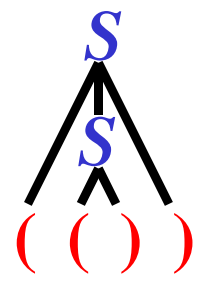
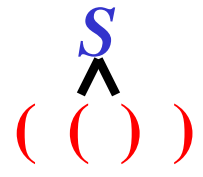
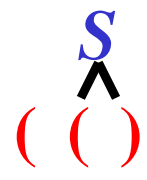
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Rule: $(S) \rightarrow S$



Rule: $\#Ss \rightarrow f$



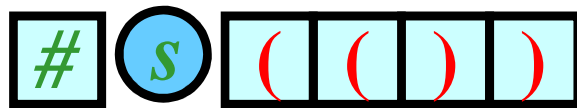
From CFG to EPDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$, where:

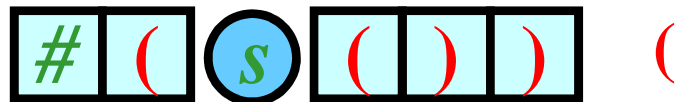
$Q = \{s, f\}$, $\Sigma = T = \{ (,) \}$, $\Gamma = \{ (,), S, \# \}$, $F = \{f\}$

$R = \{ s(\rightarrow (s, s) \rightarrow)s, (S)s \rightarrow Ss, ()s \rightarrow Ss, \#Ss \rightarrow f \}$

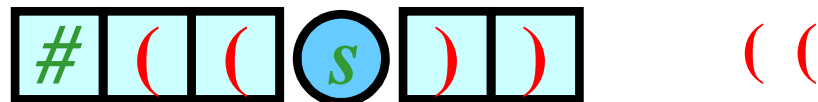
Question: $(()) \in L(M)_f?$



Rule: $s(\rightarrow (s$



Rule: $s(\rightarrow (s$



Rule: $s) \rightarrow)s$



Rule: $()s \rightarrow S$



Rule: $s) \rightarrow)s$



Rule: $(S) \rightarrow S$



Rule: $\#Ss \rightarrow f$

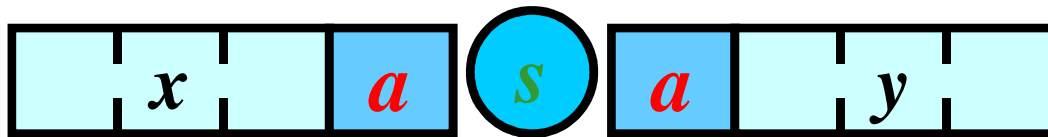


Answer: YES

PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA M underlies a top-down parser

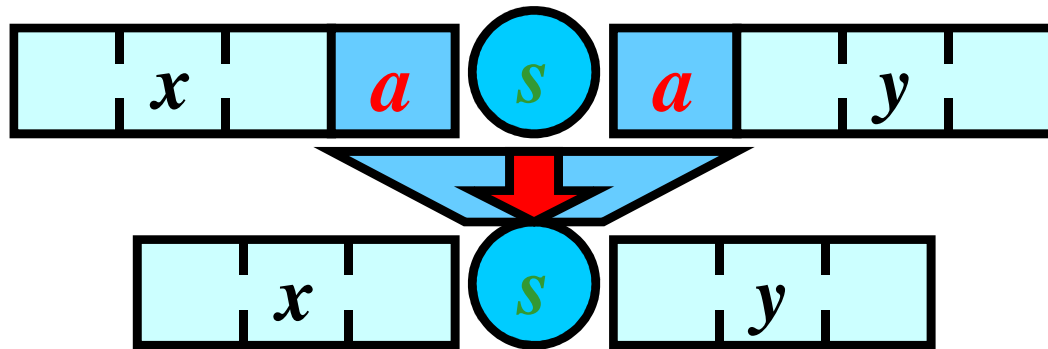
1) M contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:



PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA M underlies a top-down parser

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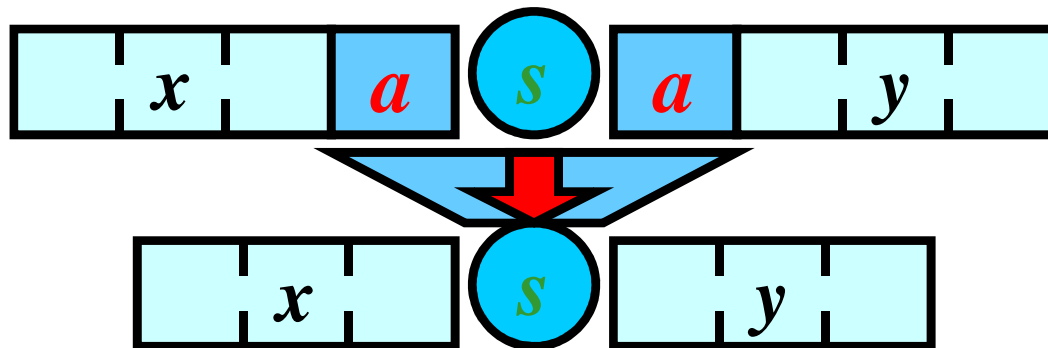
for every $a \in \Sigma$:

add $asa \rightarrow s$ to R ;

PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA M underlies a top-down parser

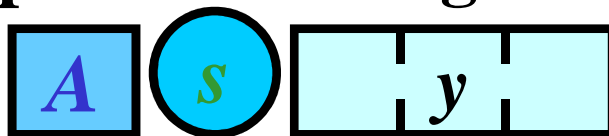
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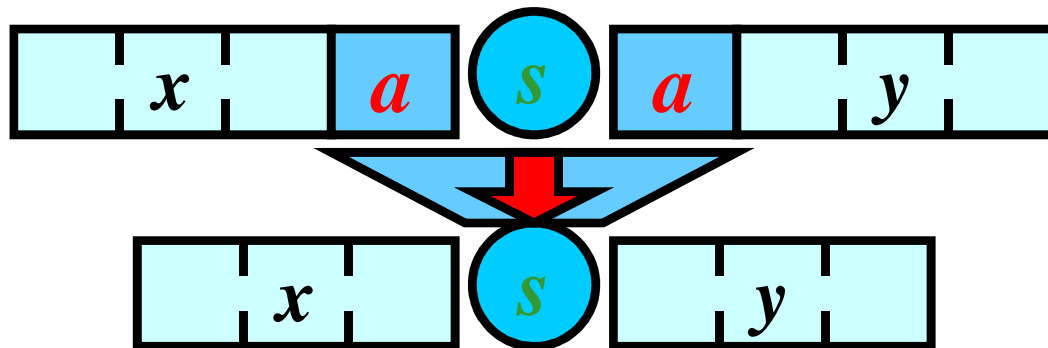
- 2) M contains *expansion* rules that simulate the application of a grammatical rule:



PDAs as Models of Top-Down Parsers 1/2

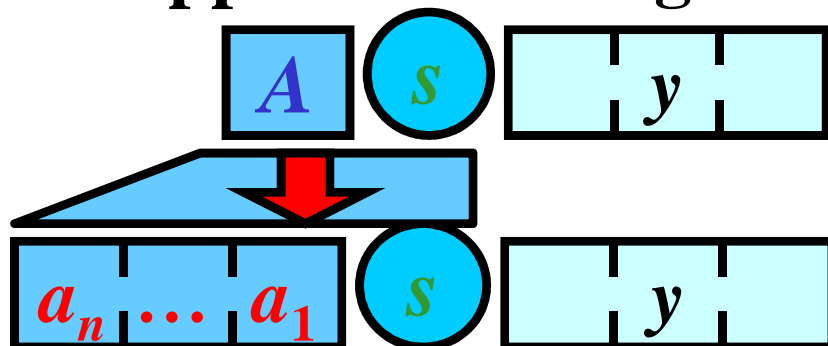
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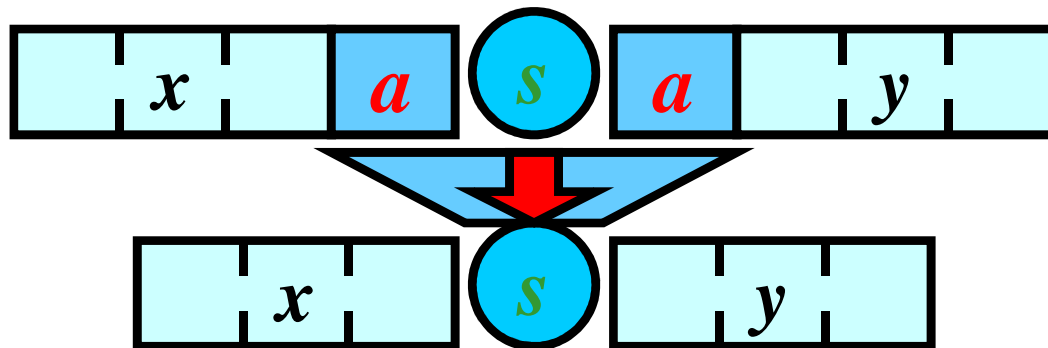
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PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA M underlies a top-down parser

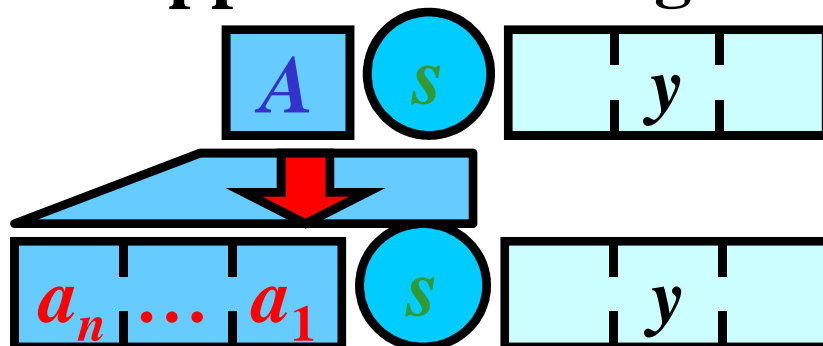
1) M contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:



for every $a \in \Sigma$:

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2) M contains *expansion* rules that simulate the application of a grammatical rule:



for every $A \rightarrow a_1 \dots a_n \in P$ in G ,

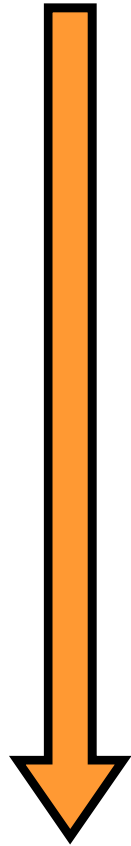
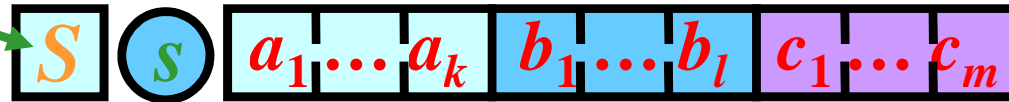
add $As \rightarrow \underbrace{a_n \dots a_1}_{} s$ to R ;

$= \text{reversal}(a_1 \dots a_n)$

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

start pushdown symbol

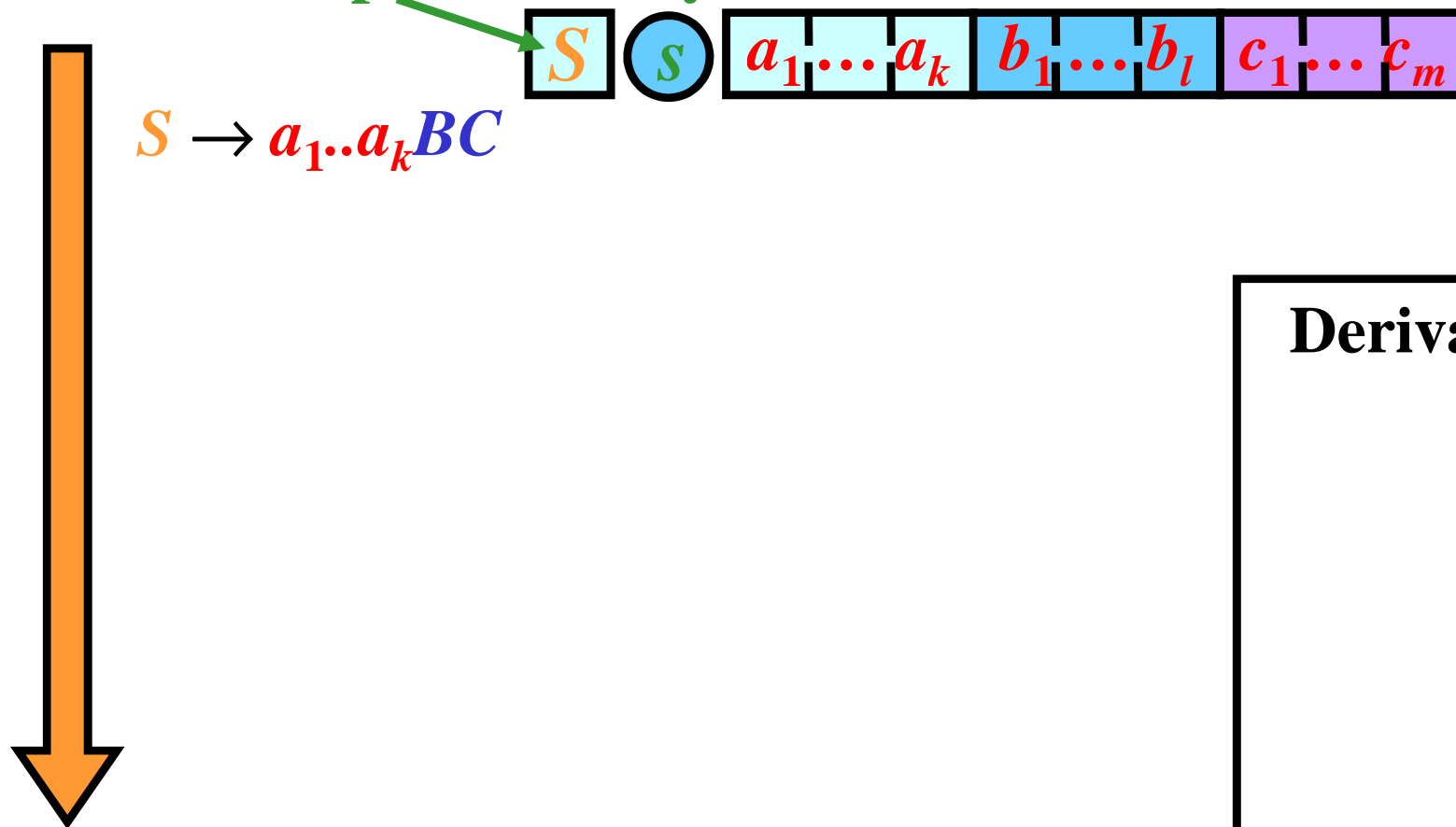


Derivation tree:

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

start pushdown symbol

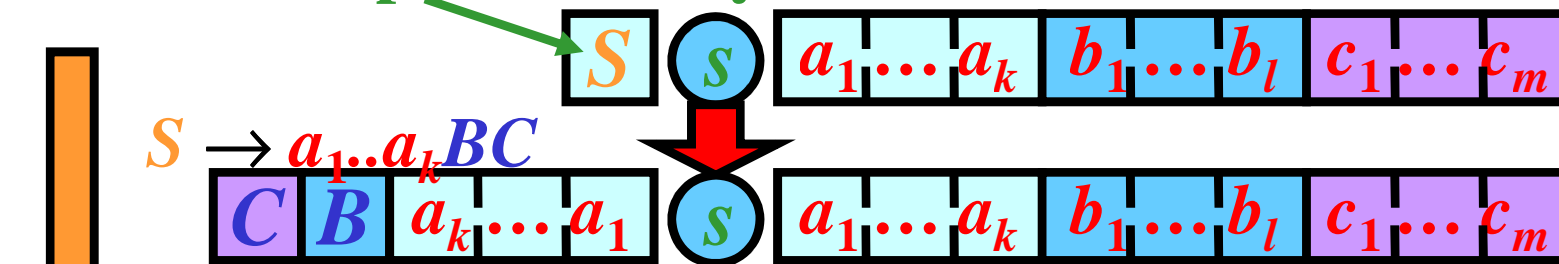


Derivation tree:

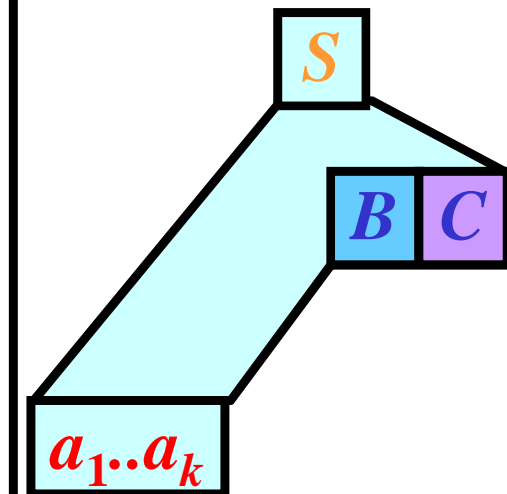
PDAs as Models of Top-Down Parsers 2/2

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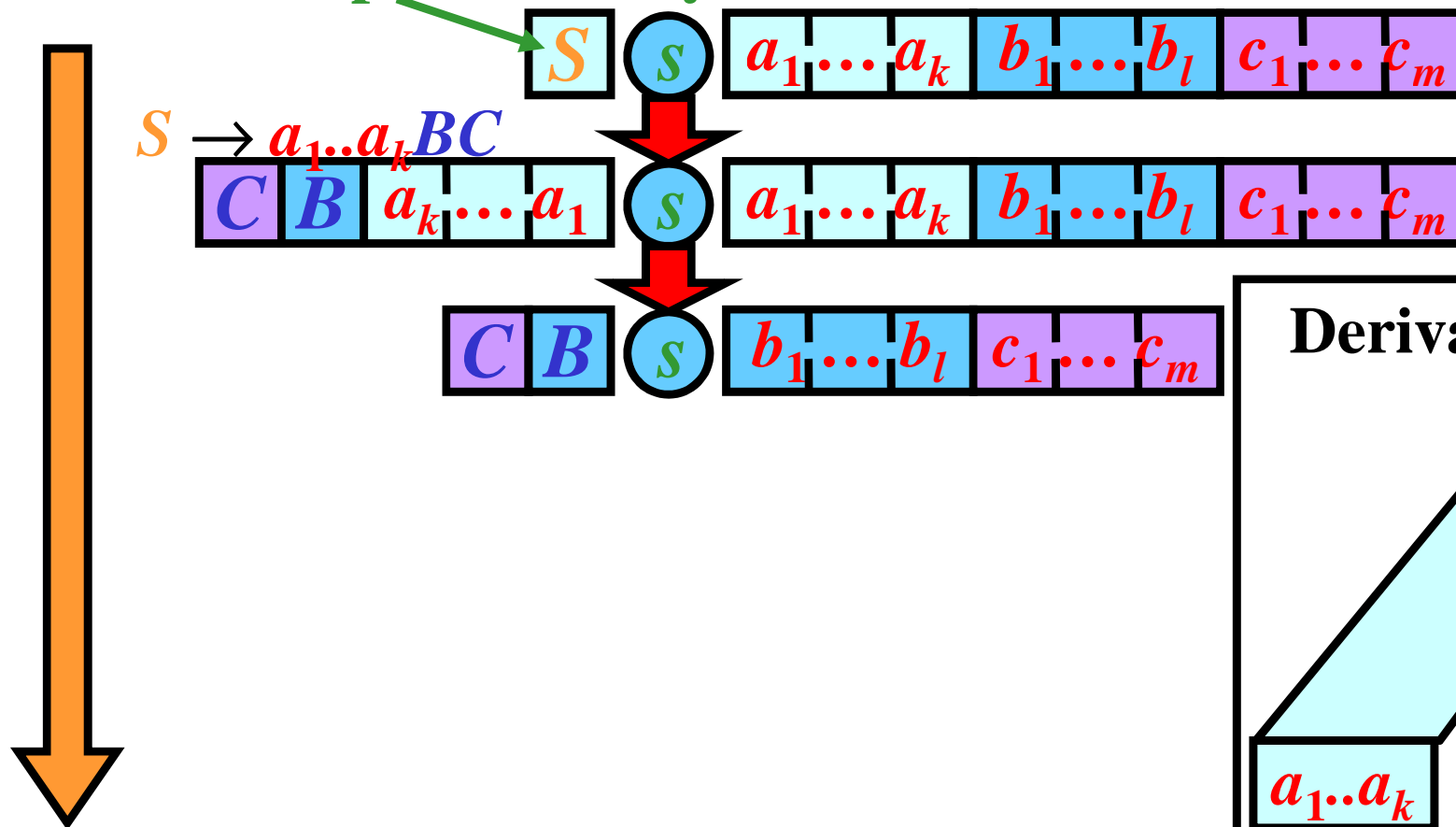
Derivation tree:



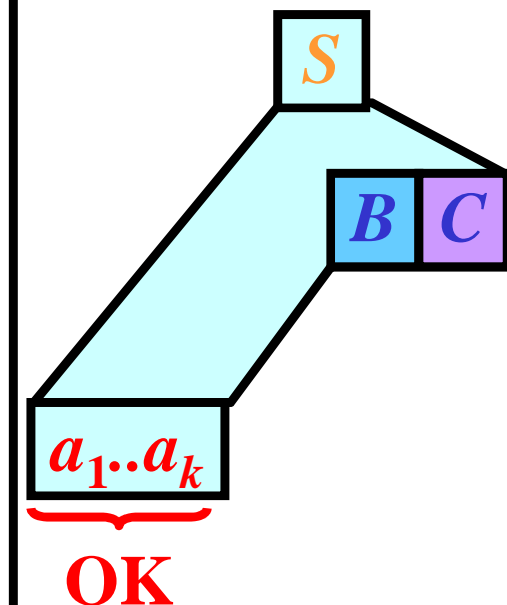
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

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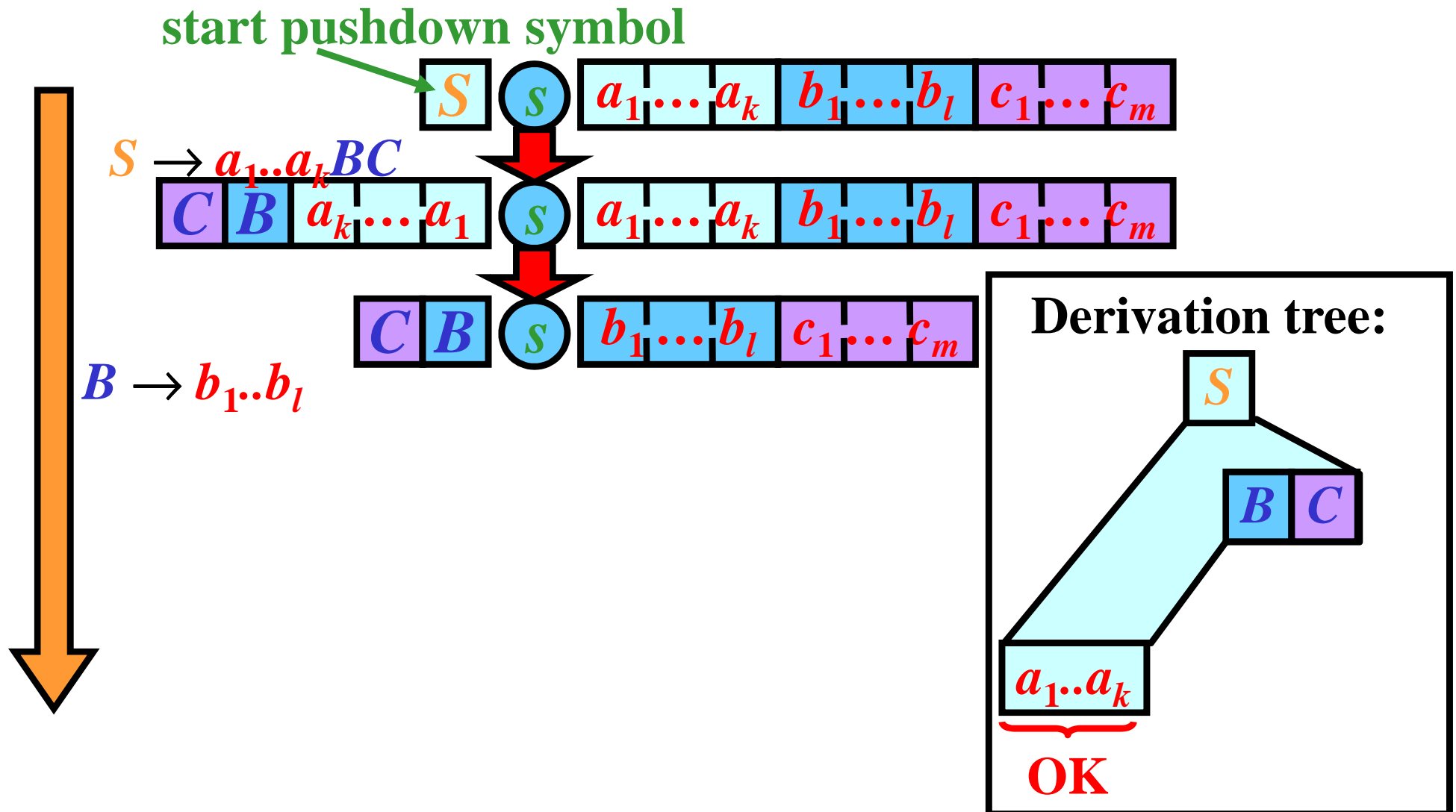


Derivation tree:



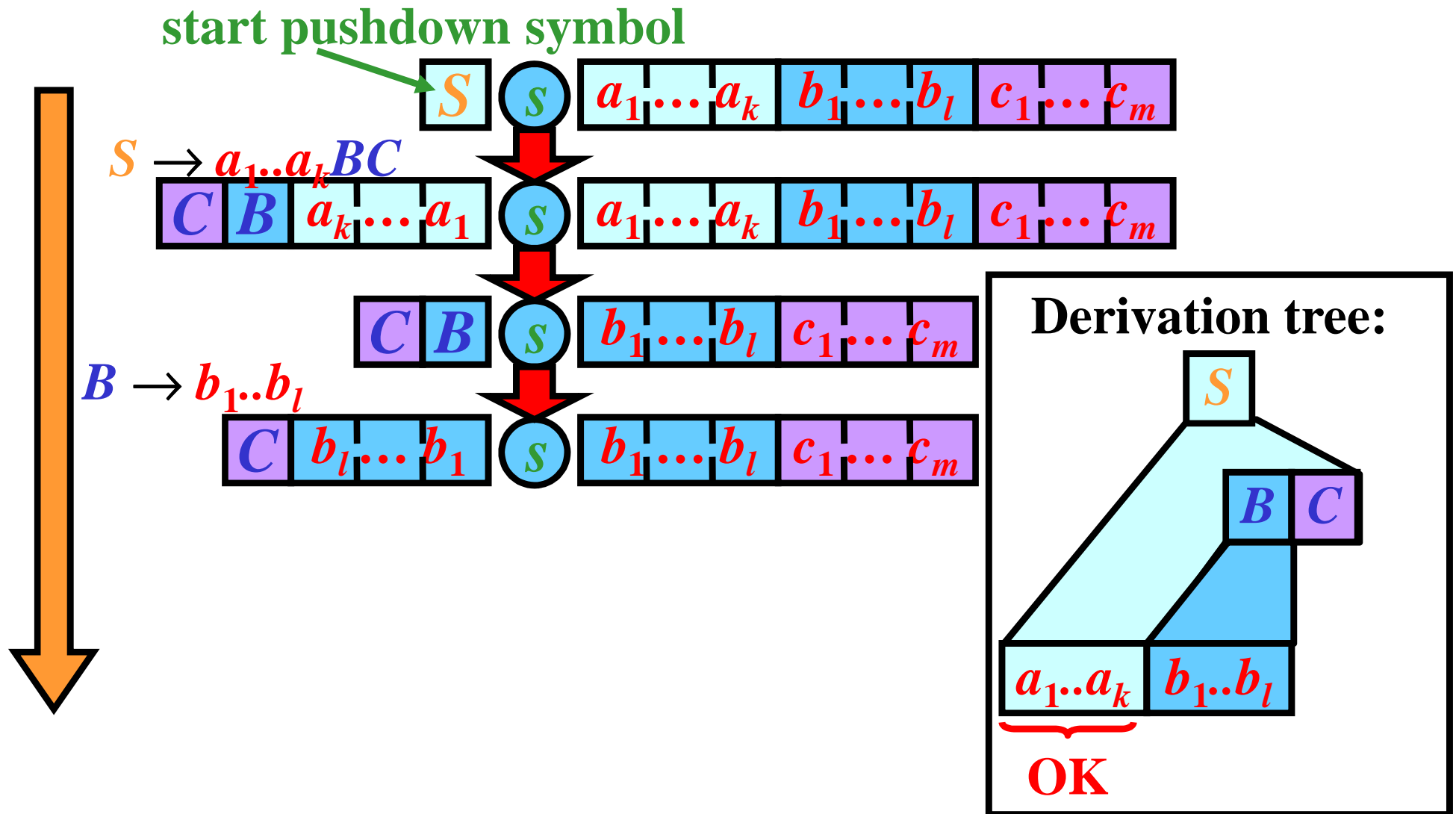
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



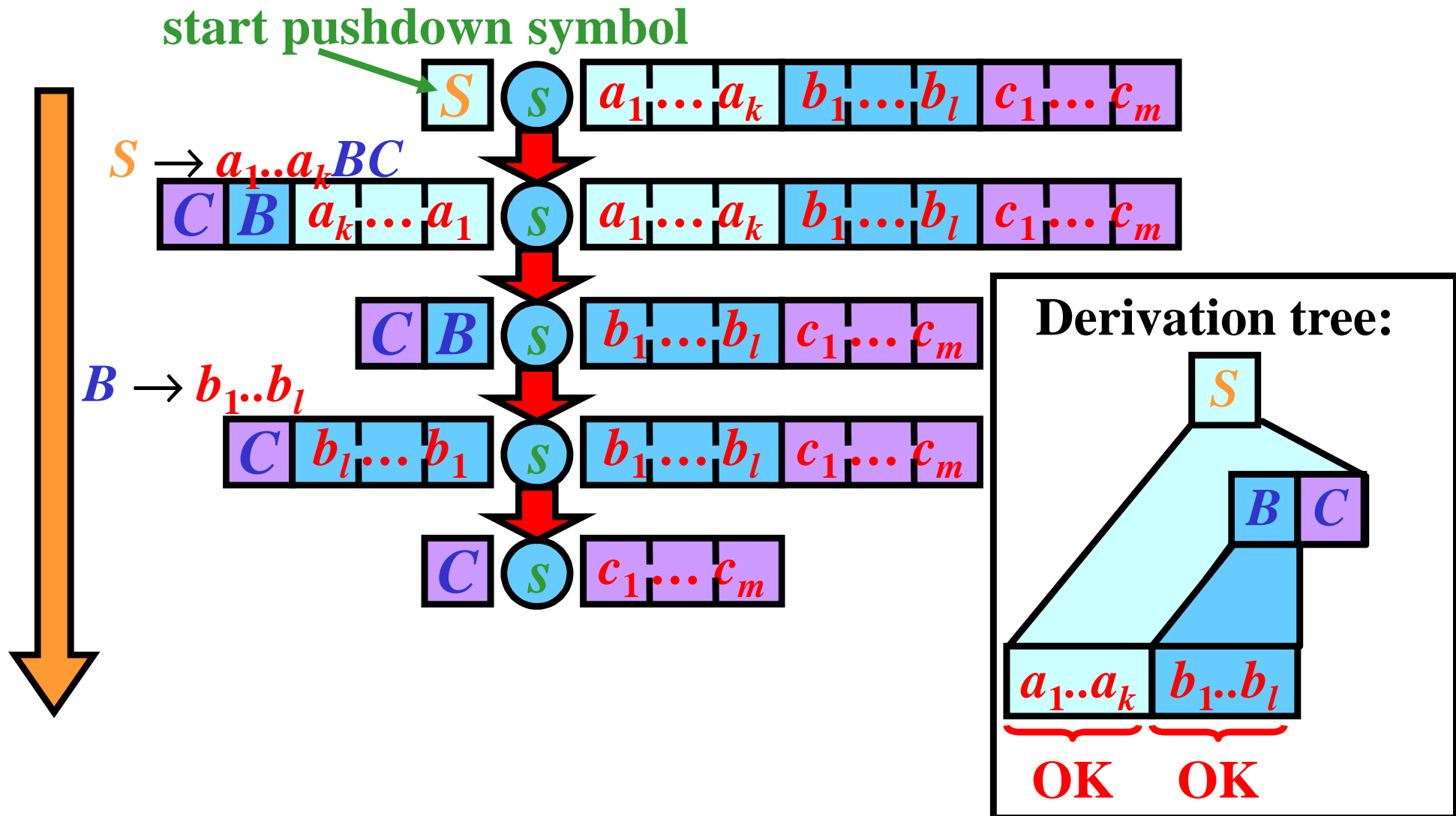
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



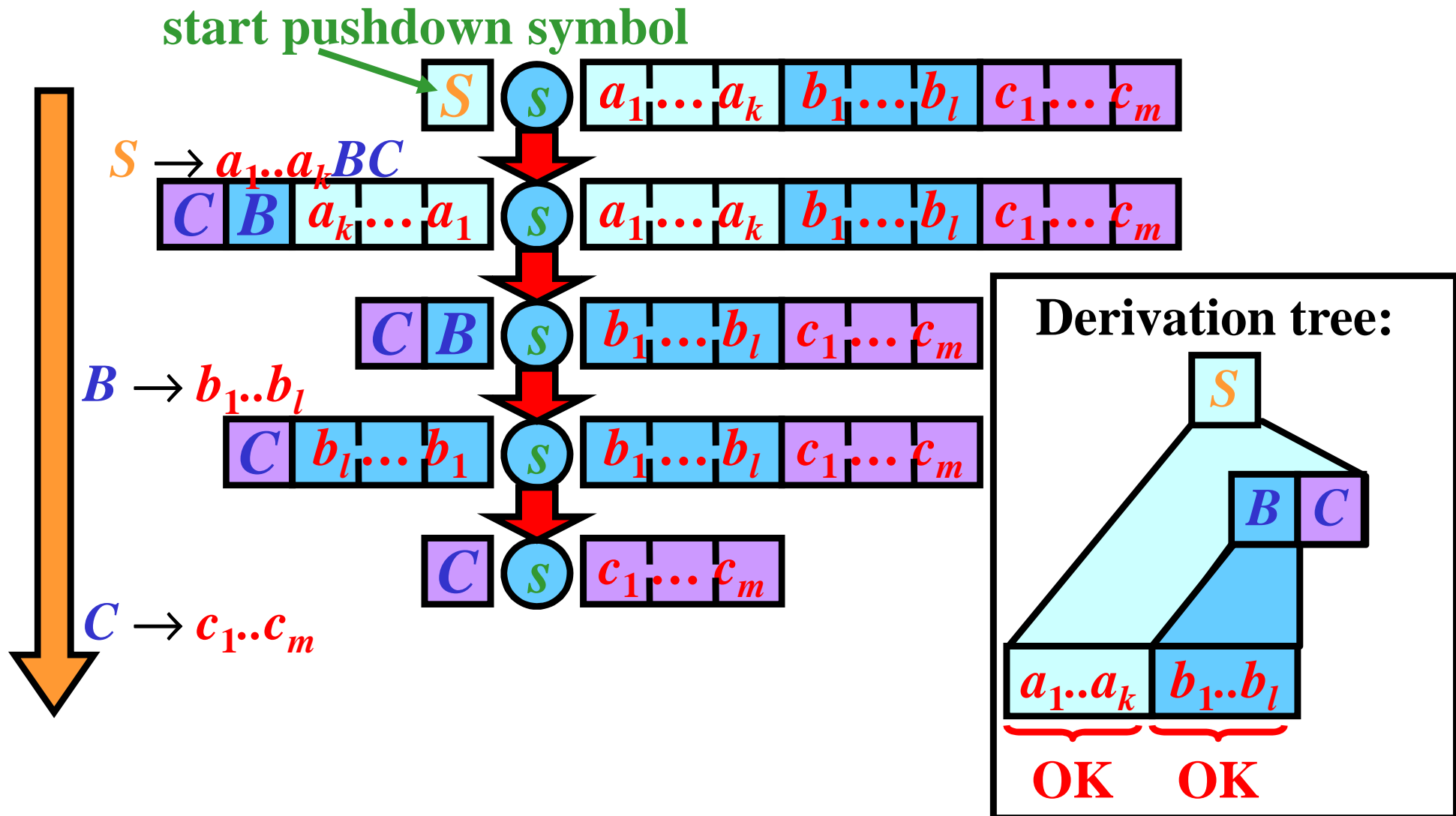
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



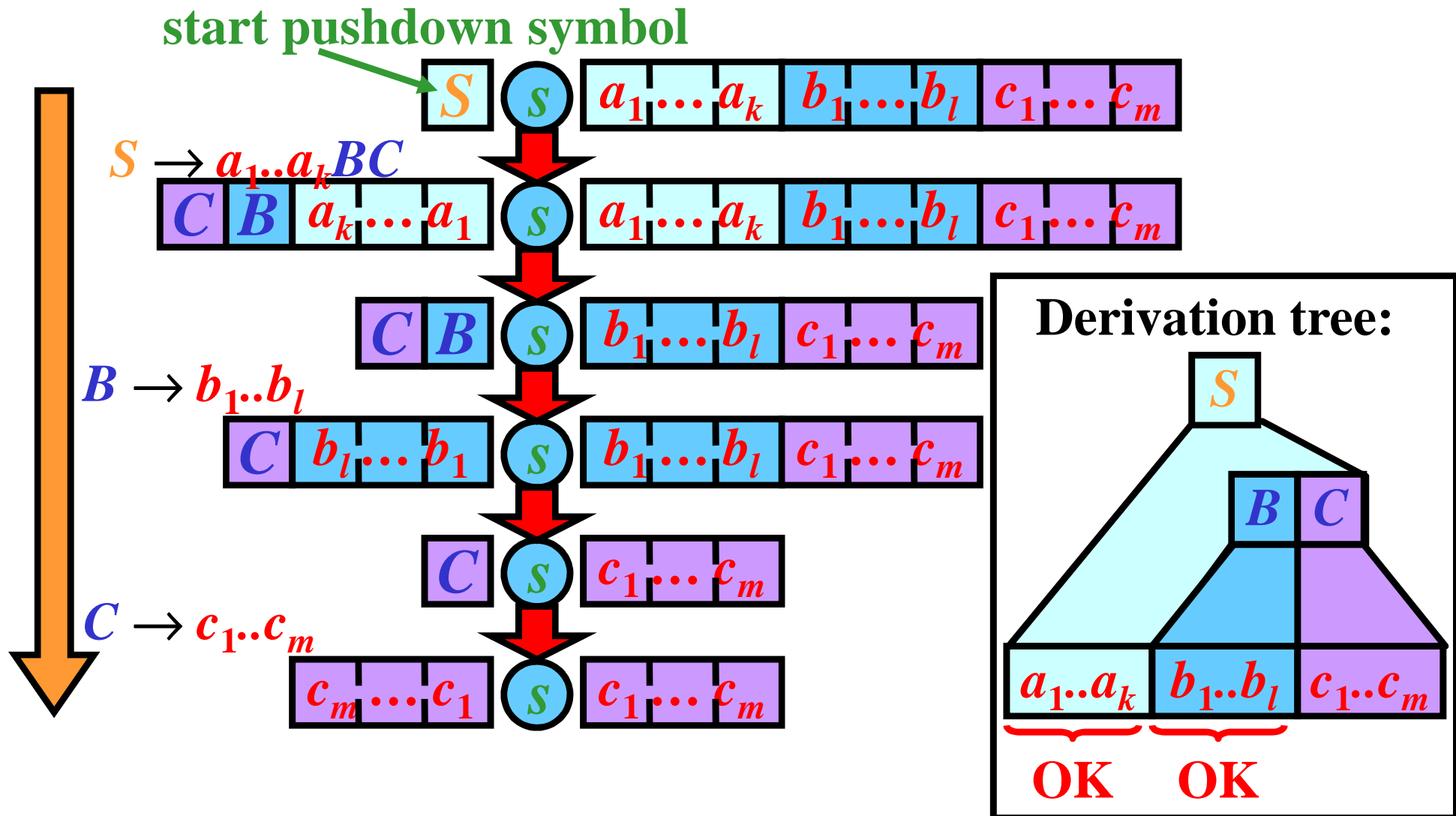
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



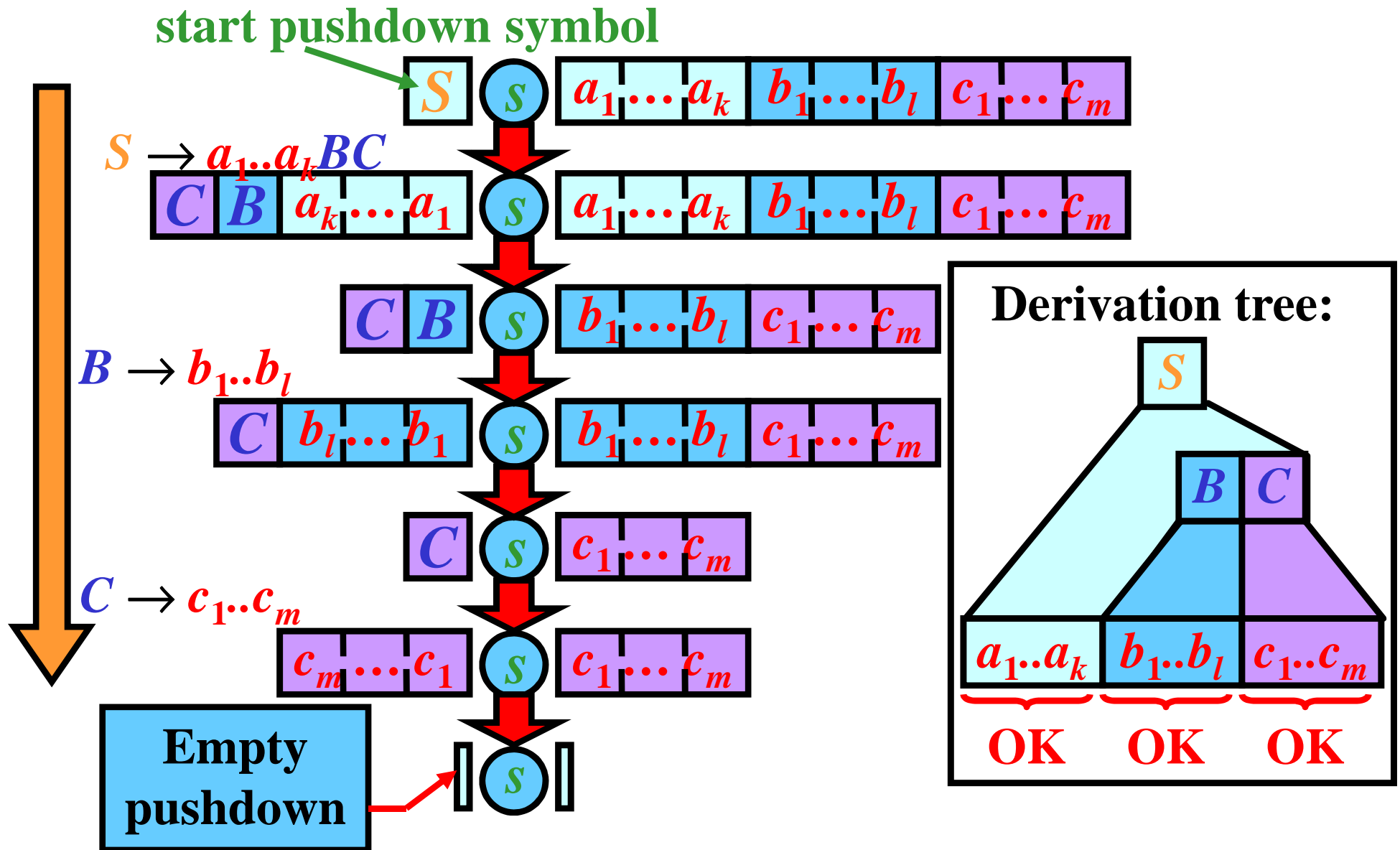
PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



Algorithm: From CFG to PDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** PDA $M = (Q, \Sigma, \Gamma, R, s, S, F)$; $L(G) = L(M)_\varepsilon$

- **Method:**
 - $Q := \{s\};$
 - $\Sigma := T;$
 - $\Gamma := N \cup T;$
 - Construction of R :
 - for every $a \in \Sigma$, add $asa \rightarrow s$ to R ;
 - for every $A \rightarrow x \in P$, add $As \rightarrow ys$ to R ,
where $y = \text{reversal}(x)$;
 - $F := \emptyset;$

From CFG to PDA: Example 1/2

- $G = (N, T, P, S)$, where:

$$N = \{S\}, T = \{ (,) \}, P = \{ S \rightarrow (S), S \rightarrow () \}$$

Objective: An PDA M such that $L(G) = L(M)_\varepsilon$

$M = (Q, \Sigma, \Gamma, R, s, S, F)$ where:

From CFG to PDA: Example 1/2

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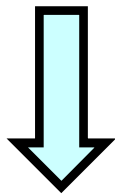
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$$“(” \in T$$



$$R = \{ (s(\rightarrow s,$$

From CFG to PDA: Example 1/2

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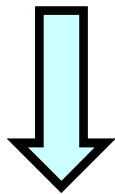
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$$“(” \in T \quad “)” \in T$$



$$R = \{ (s(\rightarrow s,)s) \rightarrow s,$$

From CFG to PDA: Example 1/2

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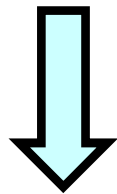
$$N = \{S\}, T = \{(\, , \,)\}, P = \{S \rightarrow (S), S \rightarrow (\,)\}$$

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rev



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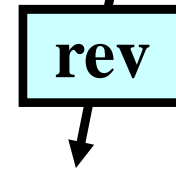
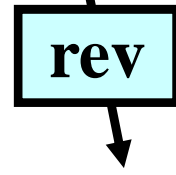
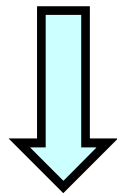
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$$“(” \in T \quad “)” \in T \quad S \rightarrow (S) \in P \quad S \rightarrow (\,) \in P$$



$$R = \{ (s(\rightarrow s, \,)s) \rightarrow s, \, Ss \rightarrow)S(s, \, Ss \rightarrow)(s \}$$

From CFG to PDA: Example 1/2

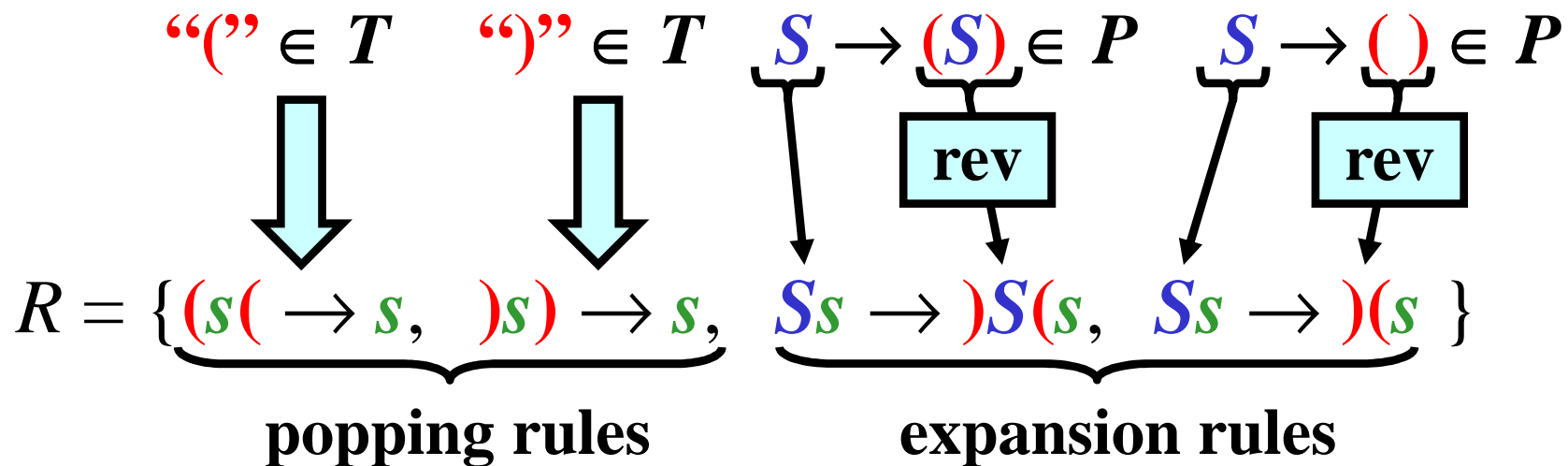
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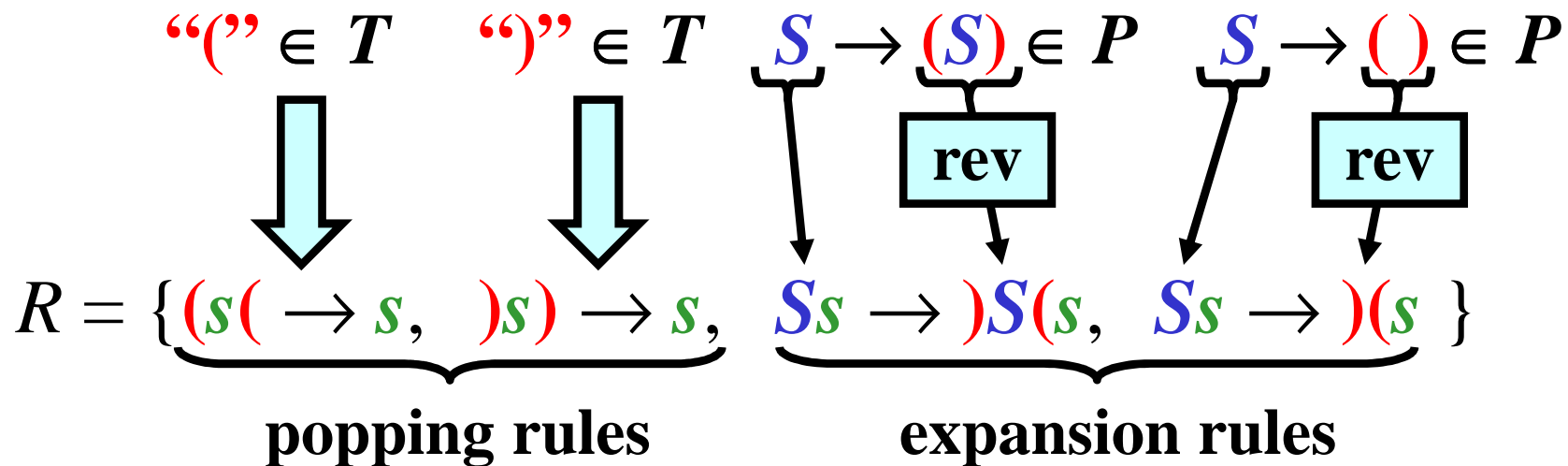
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$$F = \emptyset$$

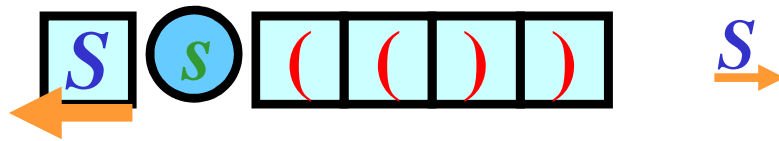
From CFG to PDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, \mathbf{s}, \mathbf{S}, F)$, where:

$Q = \{\mathbf{s}\}$, $\Sigma = T = \{(\mathbf{, , \mathbf{)}}$, $\Gamma = \{(\mathbf{, , \mathbf{S}}\}$, $F = \emptyset$

$P = \{(\mathbf{s(} \rightarrow \mathbf{s, \mathbf{)s} \rightarrow \mathbf{s, \mathbf{Ss} \rightarrow \mathbf{)S(s, \mathbf{Ss} \rightarrow \mathbf{)(s} \}$

Question: $((\mathbf{))} \in L(M)_\epsilon$?



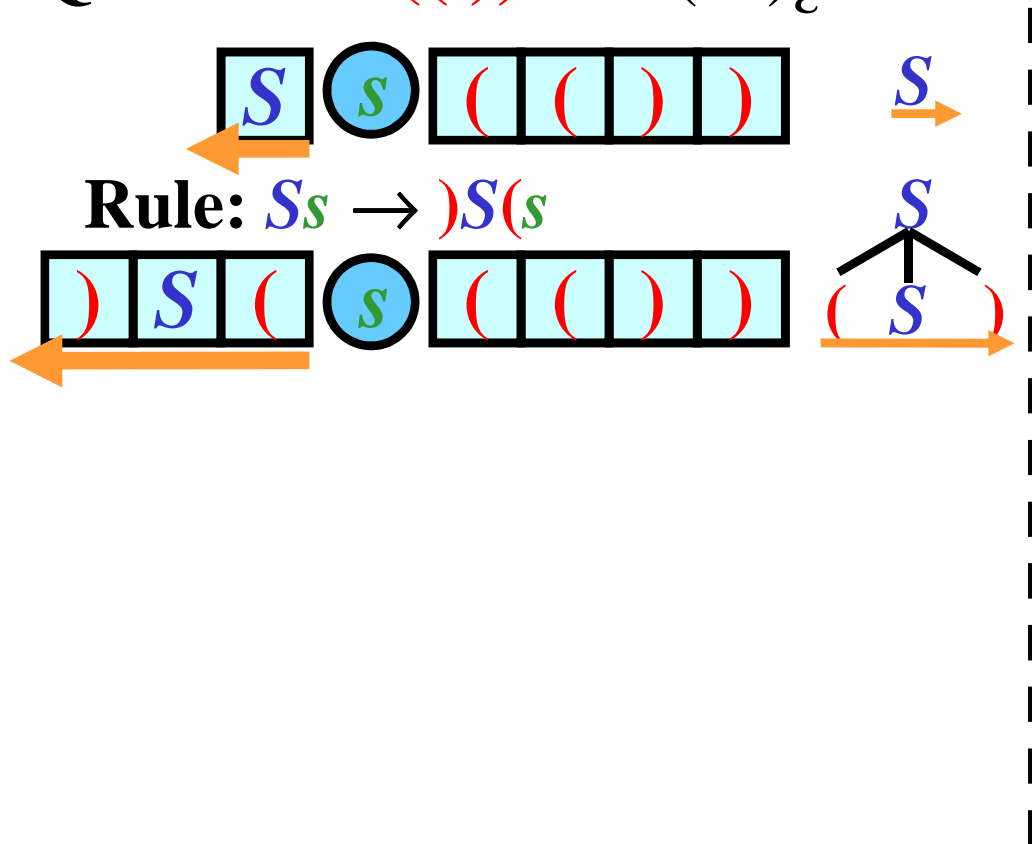
From CFG to PDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, S, F)$, where:

$Q = \{s\}$, $\Sigma = T = \{ (,) \}$, $\Gamma = \{ (,), S \}$, $F = \emptyset$

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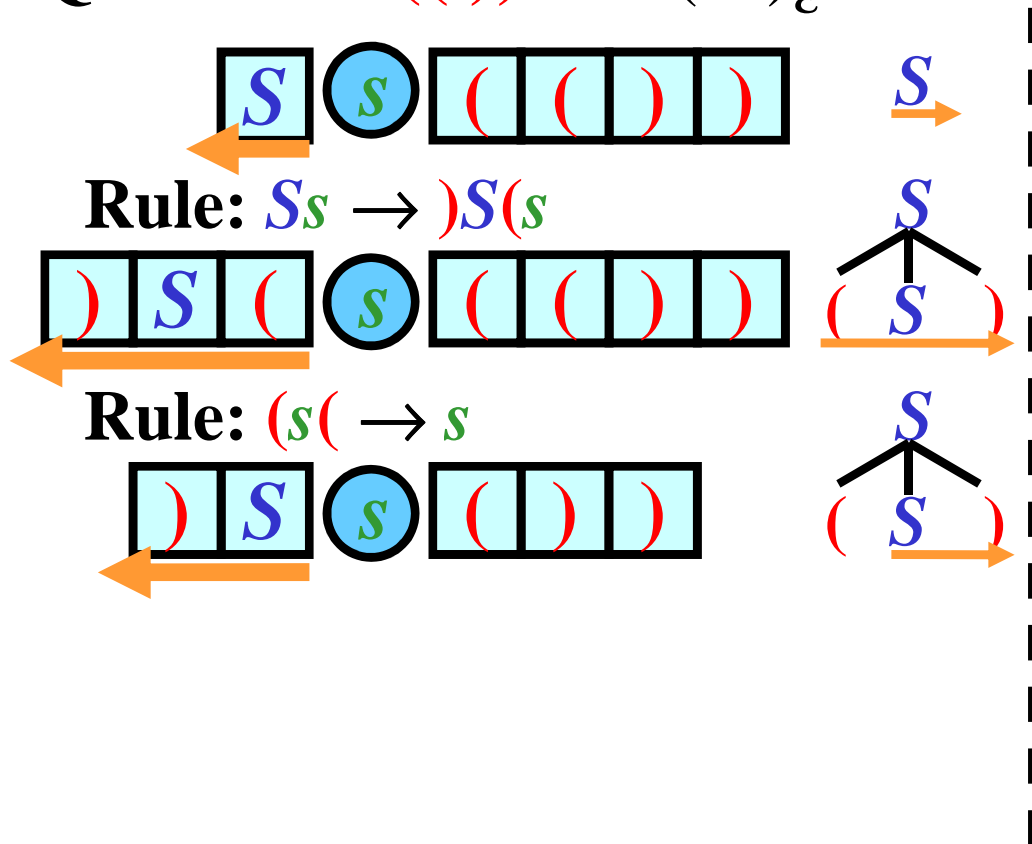
From CFG to PDA: Example 2/2

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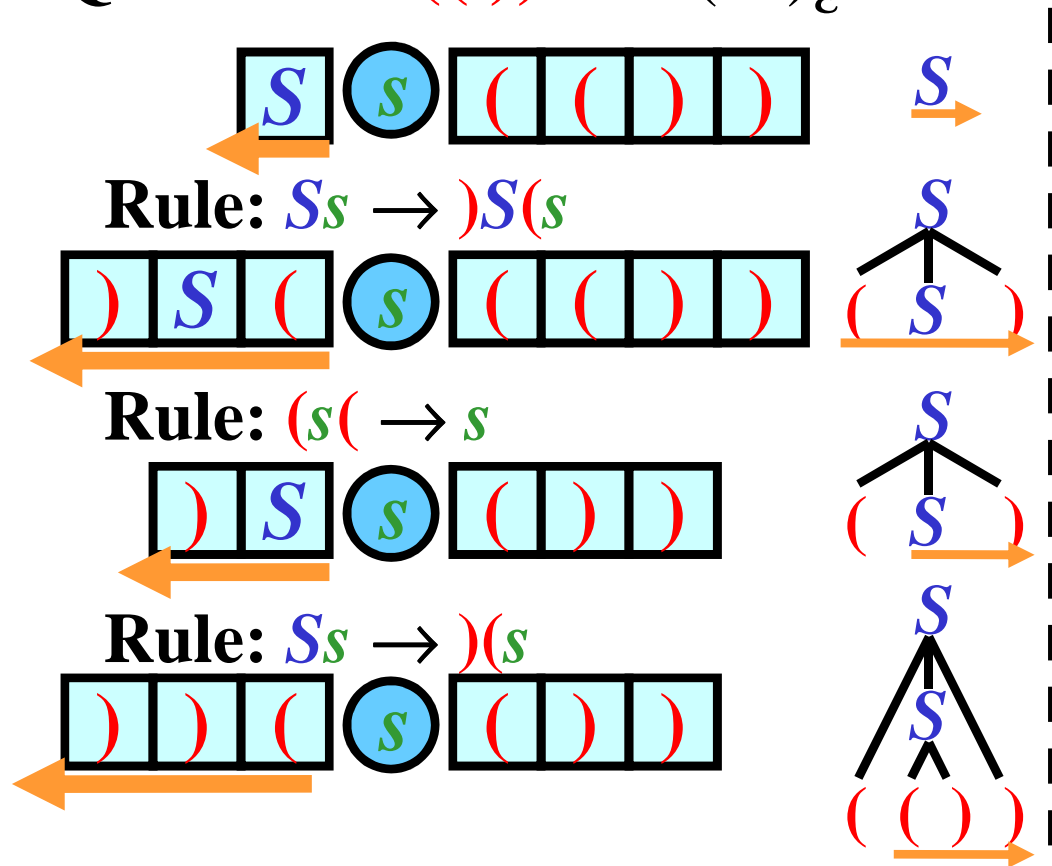
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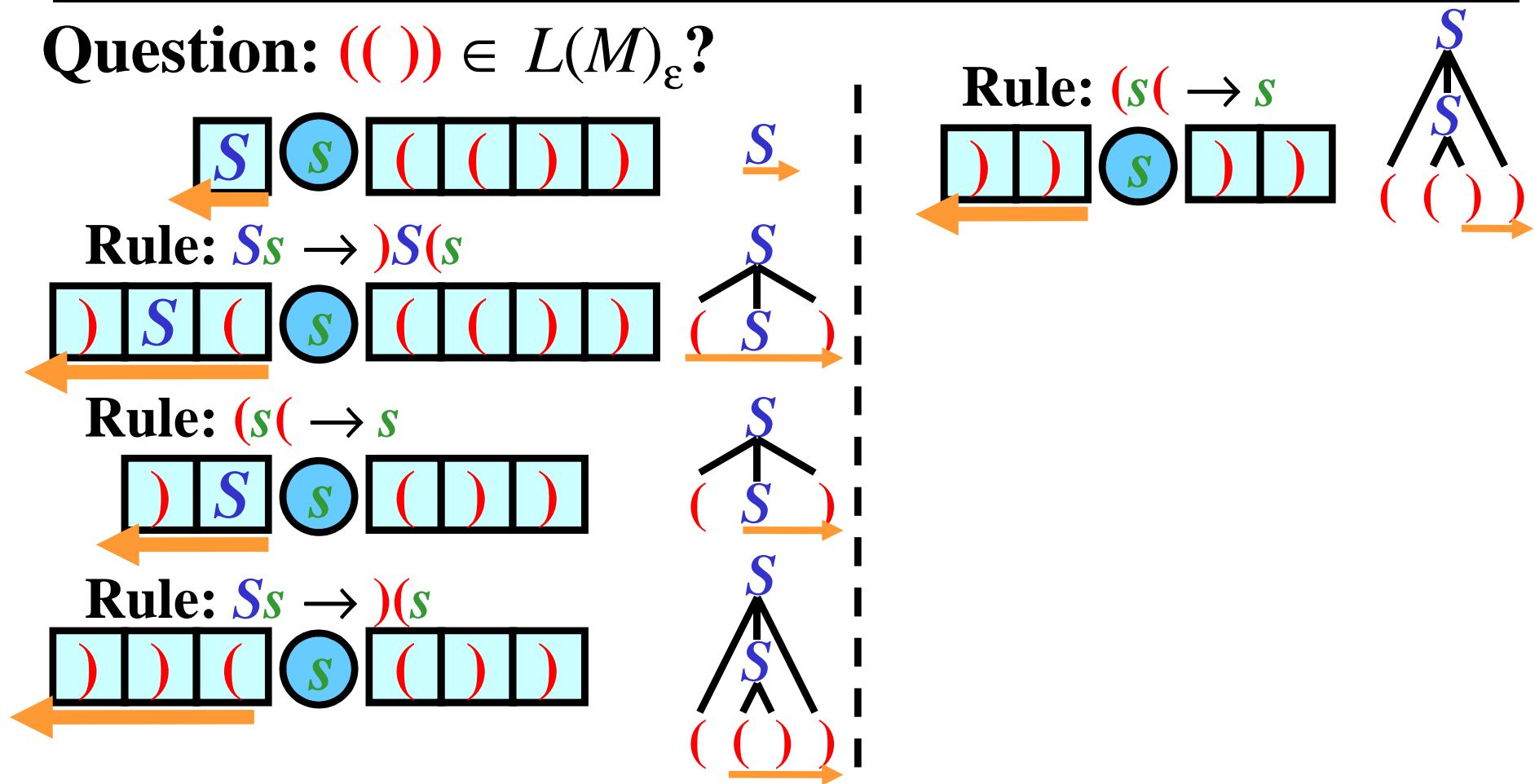
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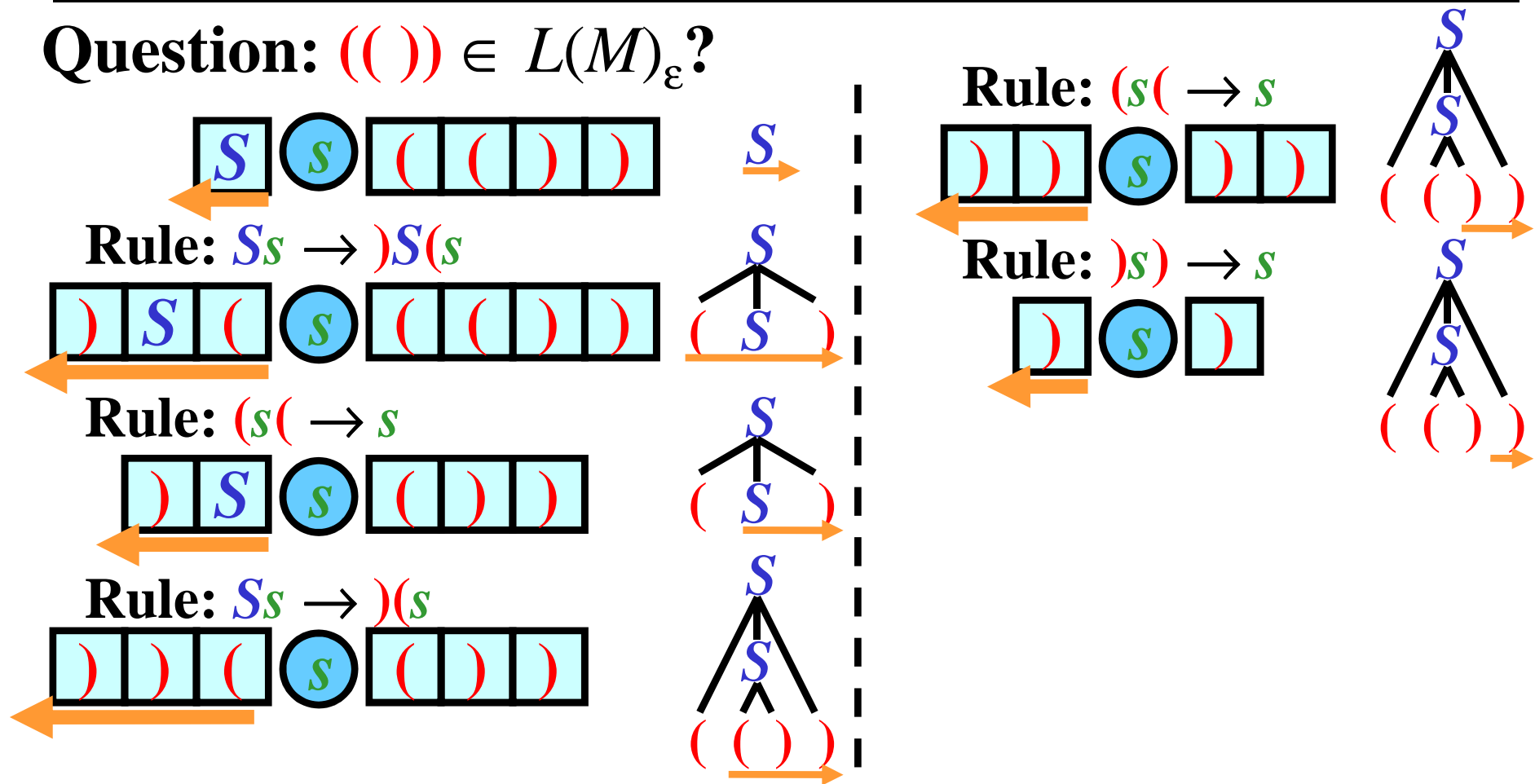
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Question: $((\,)) \in L(M)_\varepsilon$?



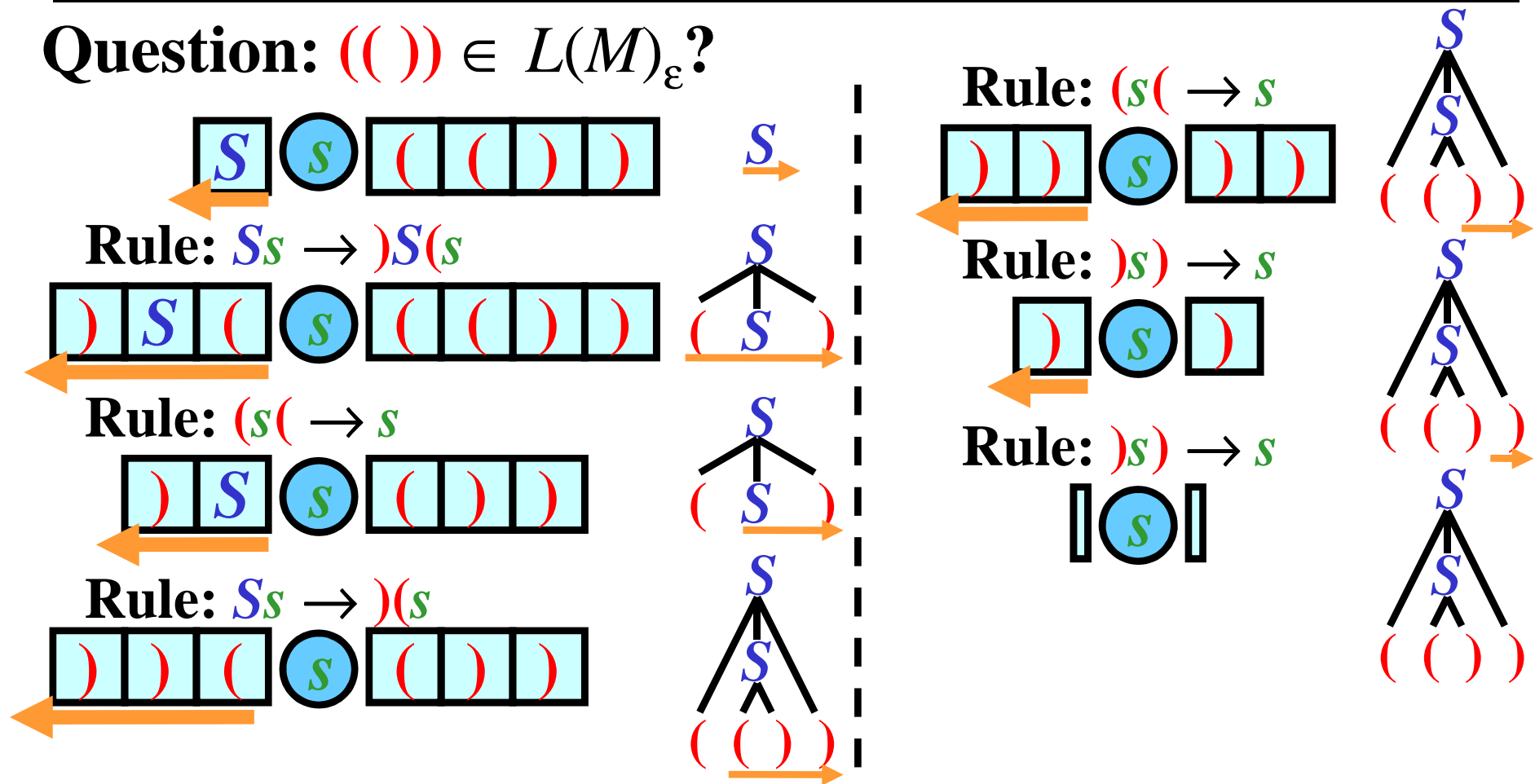
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Models for Context-free Languages

Theorem: For every CFG G , there is a PDA M such that $L(G) = L(M)_\varepsilon$.

Proof: See the previous algorithm.

Theorem: For every PDA M , there is a CFG G such that $L(M)_\varepsilon = L(G)$.

Proof: See page 486 in [Meduna: Automata and Languages]

Conclusion: The fundamental models for context-free languages are

1) **Context-free grammars** 2) **Pushdown automata**