

Part XII.

Normal Forms and Properties of CFLs

Chomsky Normal Form (CNF)

Definition: Let $G = (N, T, P, S)$ be a CFG.

G is in *Chomsky normal form* if every rule in P has one of these forms

- $A \rightarrow BC$, where $A, B, C \in N$;
- $A \rightarrow a$, where $A \in N, a \in T$;

Example:

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b\}$,
 $P = \{S \rightarrow CB, C \rightarrow AS, S \rightarrow AB, A \rightarrow a, B \rightarrow b\}$
 is in Chomsky normal form.

Note: $L(G) = \{a^n b^n : n \geq 1\}$

Greibach Normal Form (GNF)

Definition: Let $G = (N, T, P, S)$ be a CFG. G is in *Greibach normal form* if every rule in P is of this form

- $A \rightarrow ax$, where $A \in N$, $a \in T$, $x \in N^*$

Example:

$G = (N, T, P, S)$, where $N = \{B, S\}$, $T = \{a, b\}$,
 $P = \{S \rightarrow aSB, S \rightarrow aB, B \rightarrow b\}$
 is in Greibach normal form.

Note: $L(G) = \{a^n b^n : n \geq 1\}$

Generative Power of Normal Forms

Theorem: For every CFG G , there is an equivalent grammar G' in Chomsky normal form.

Proof: See page 348 in [Meduna: Automata and Languages]

Theorem: For every CFG G , there is an equivalent grammar G' in Greibach normal form.

Proof: See page 376 in [Meduna: Automata and Languages]

Note: Main properties of CNF and GNF:

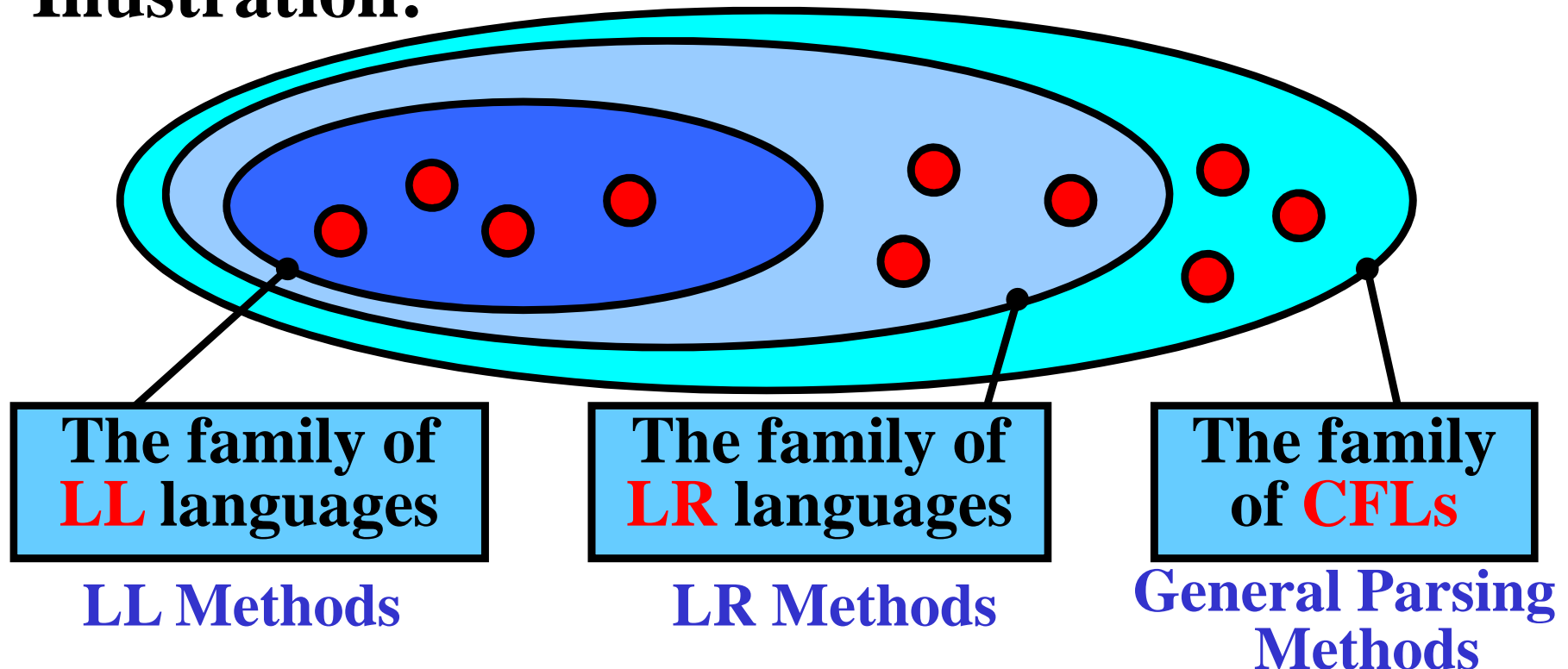
CNF: if $S \Rightarrow^n w$; $w \in T^*$ then $n = 2|w| - 1$

GNF: if $S \Rightarrow^n w$; $w \in T^*$ then $n = |w|$

General Parsing Methods

- **General Parsing methods** (GP) are applicable to all context-free languages (CFLs)

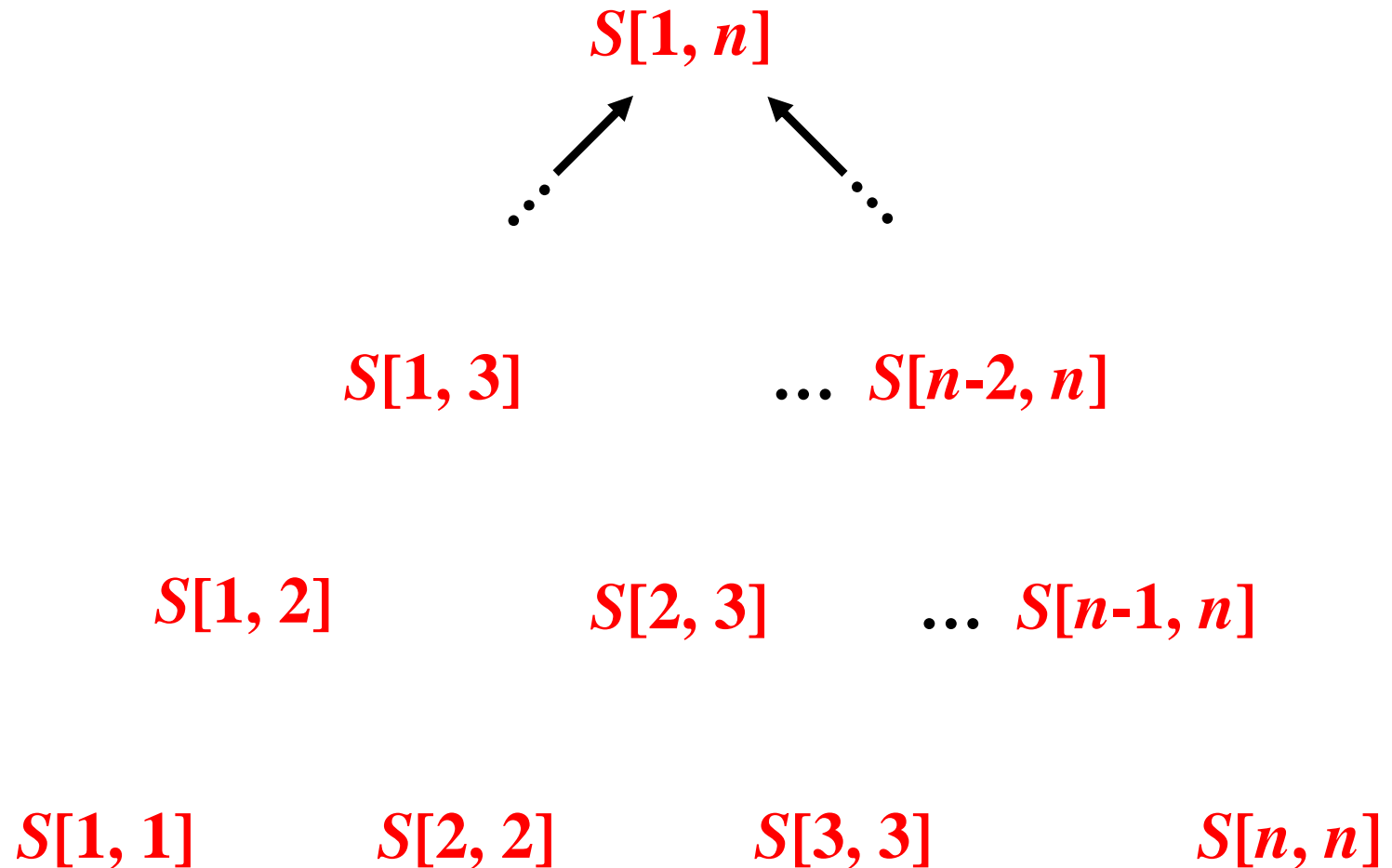
Illustration:



- **Note:** The family of **LR** languages = the family of a **deterministic CFL**

GP Based on Chomsky Normal Form

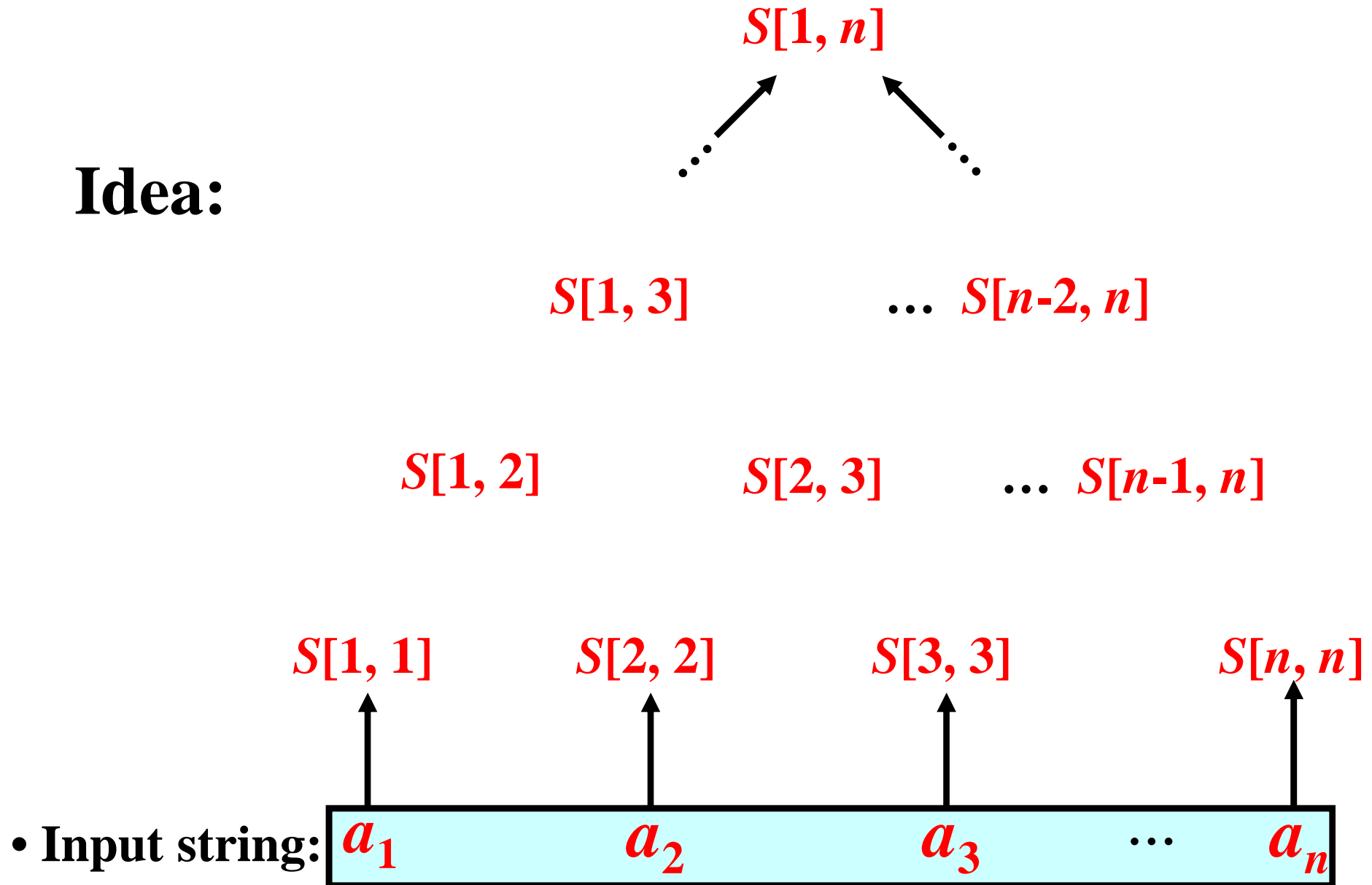
Idea:



- Input string: $a_1 \quad a_2 \quad a_3 \quad \dots \quad a_n$

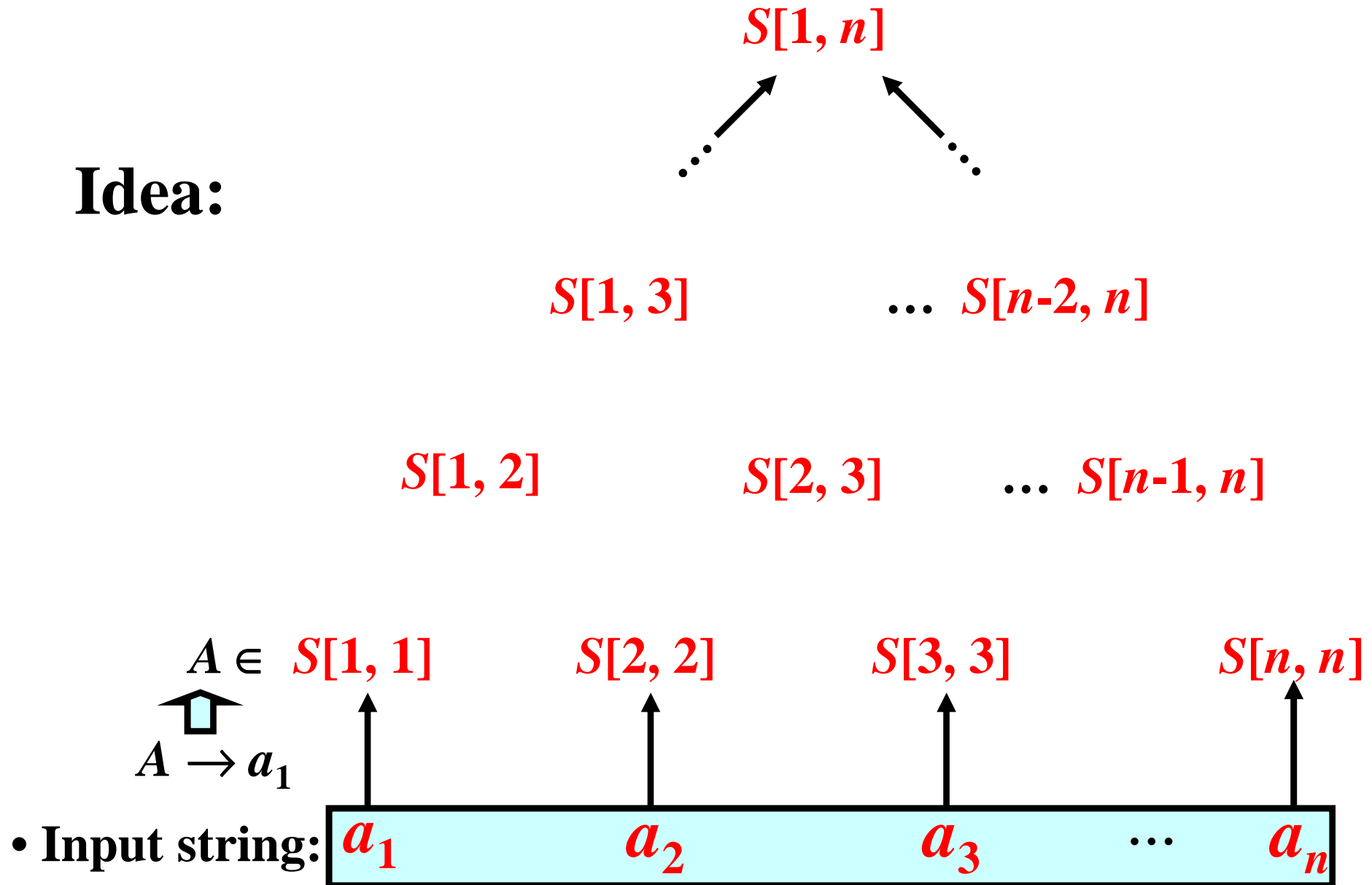
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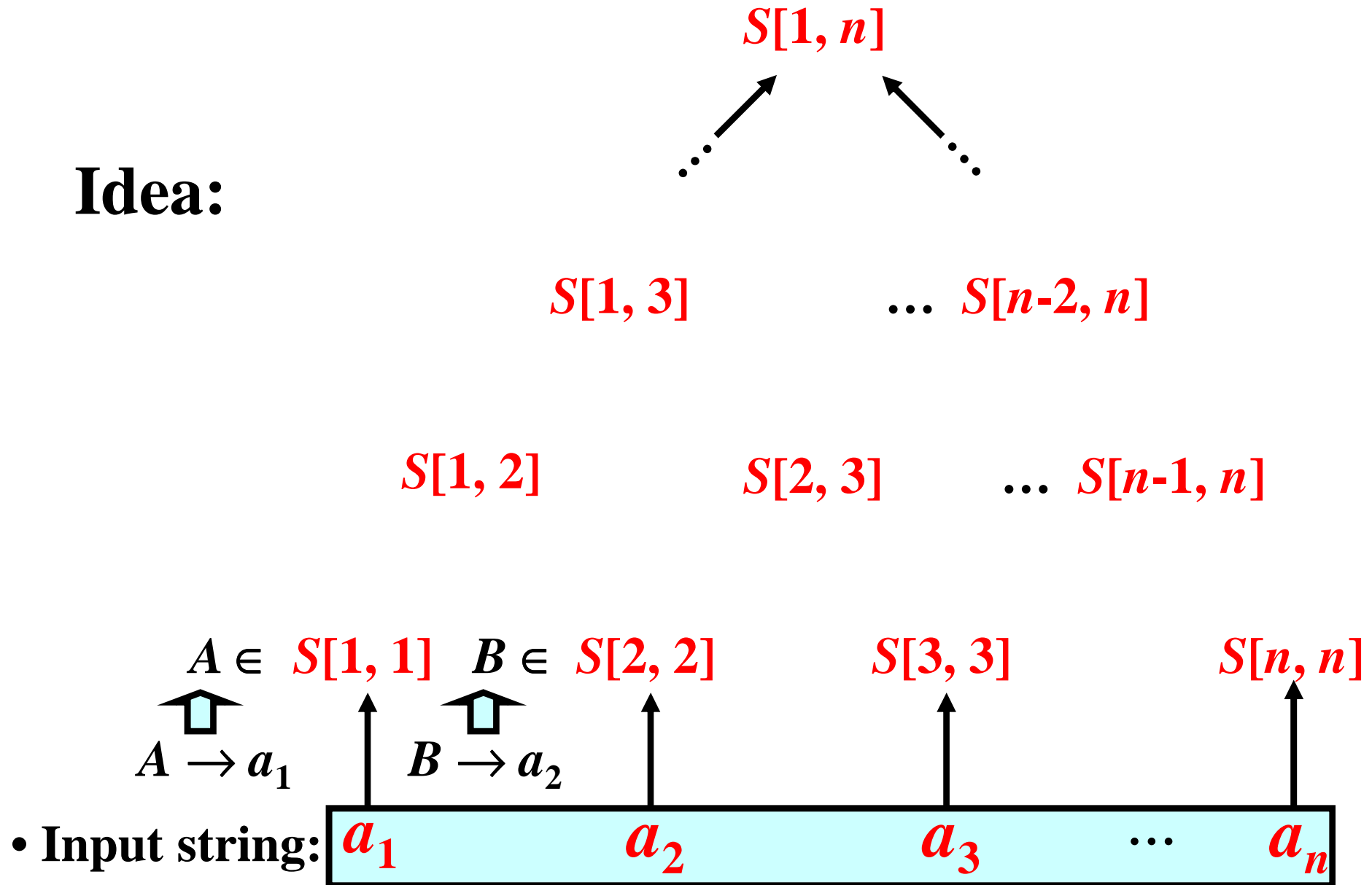
GP Based on Chomsky Normal Form

Idea:



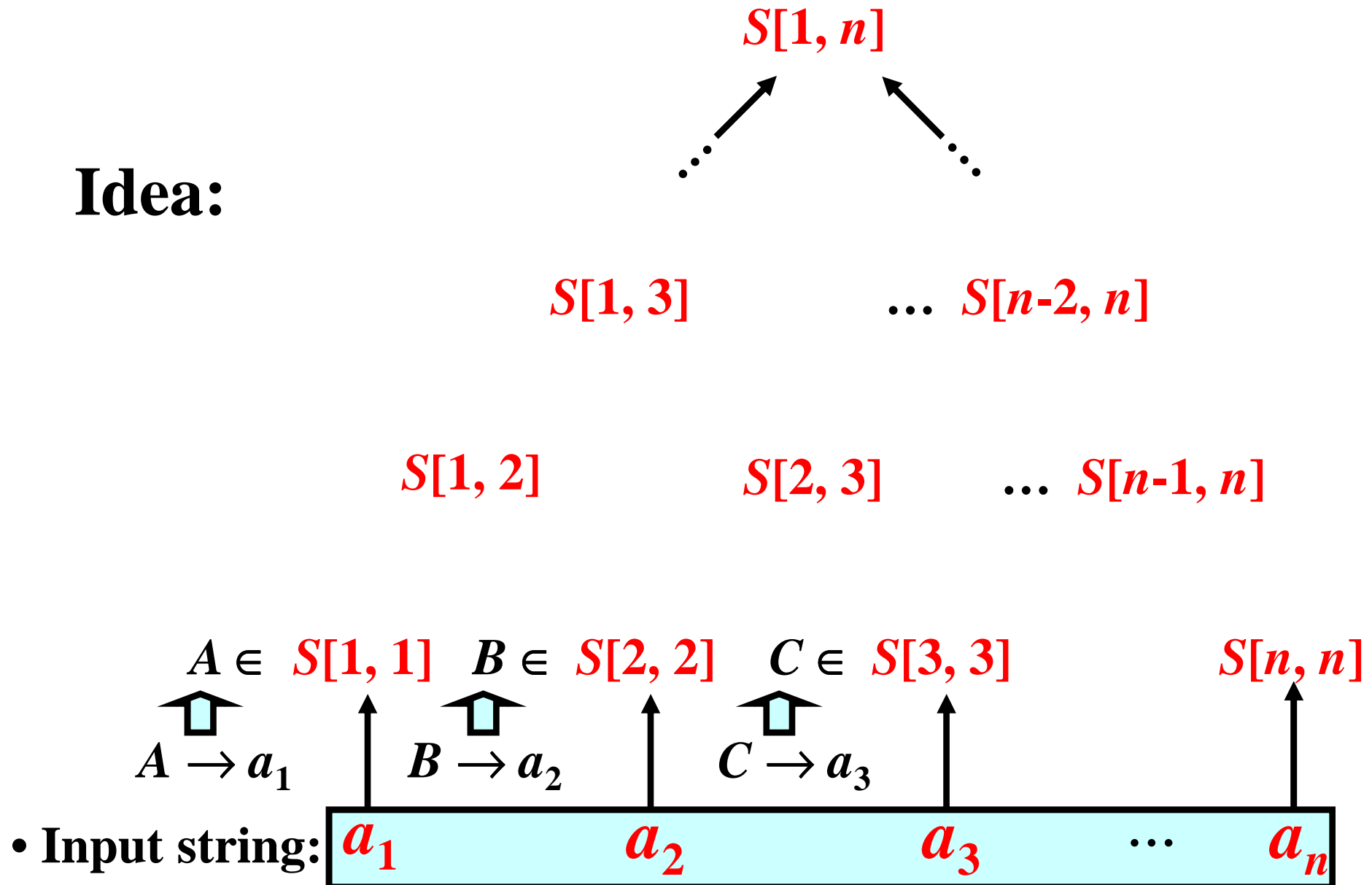
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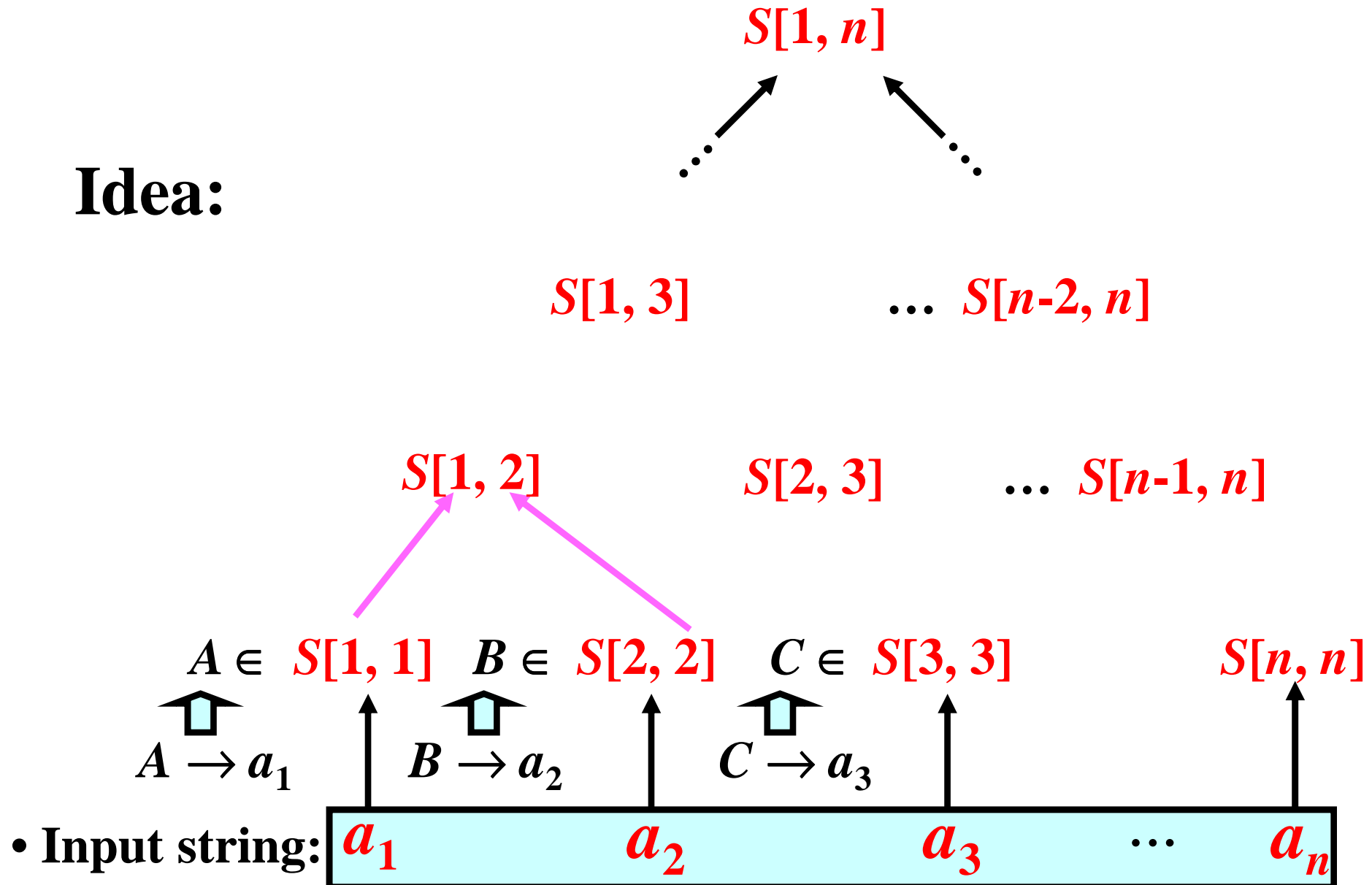
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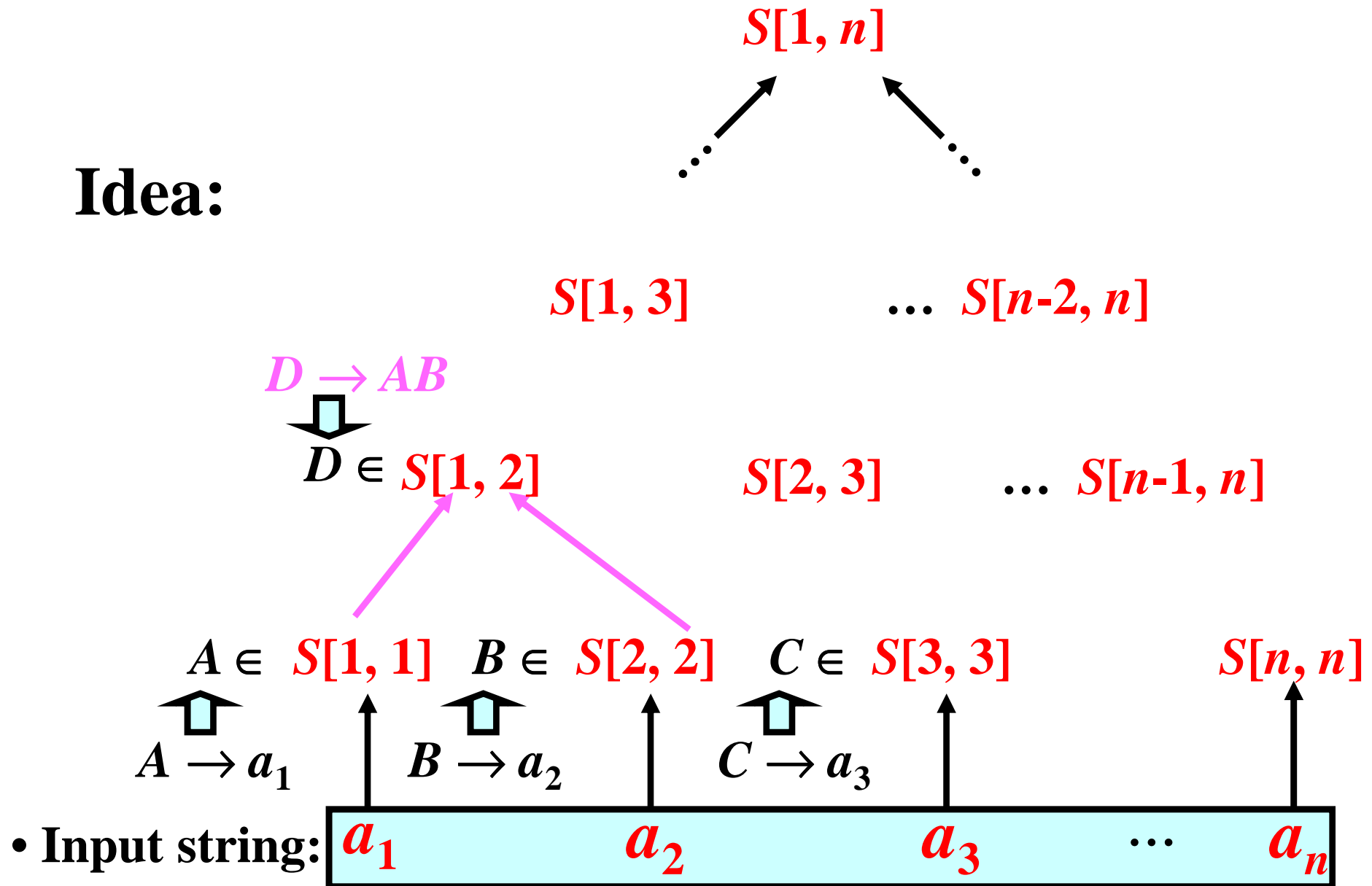
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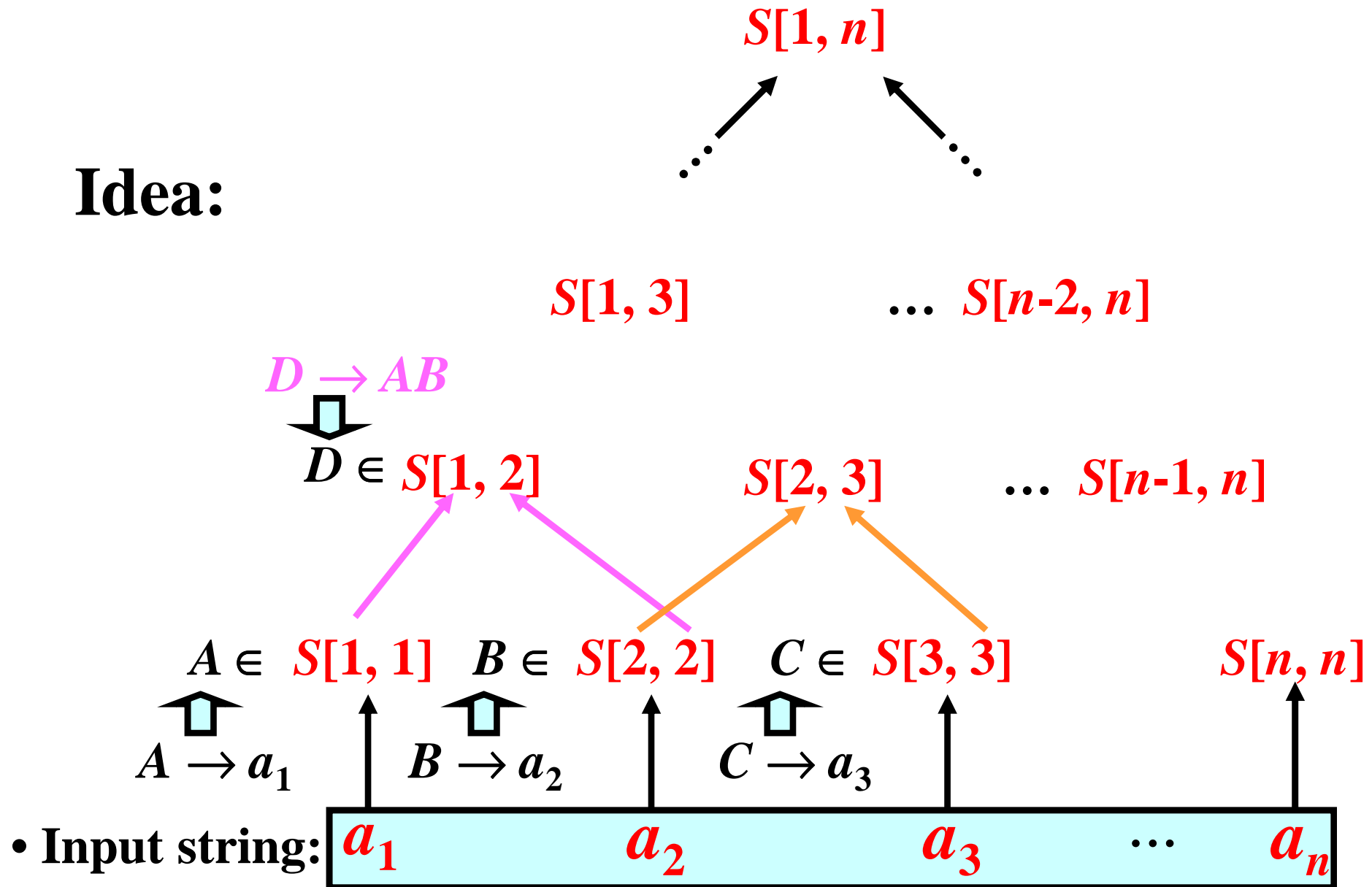
GP Based on Chomsky Normal Form

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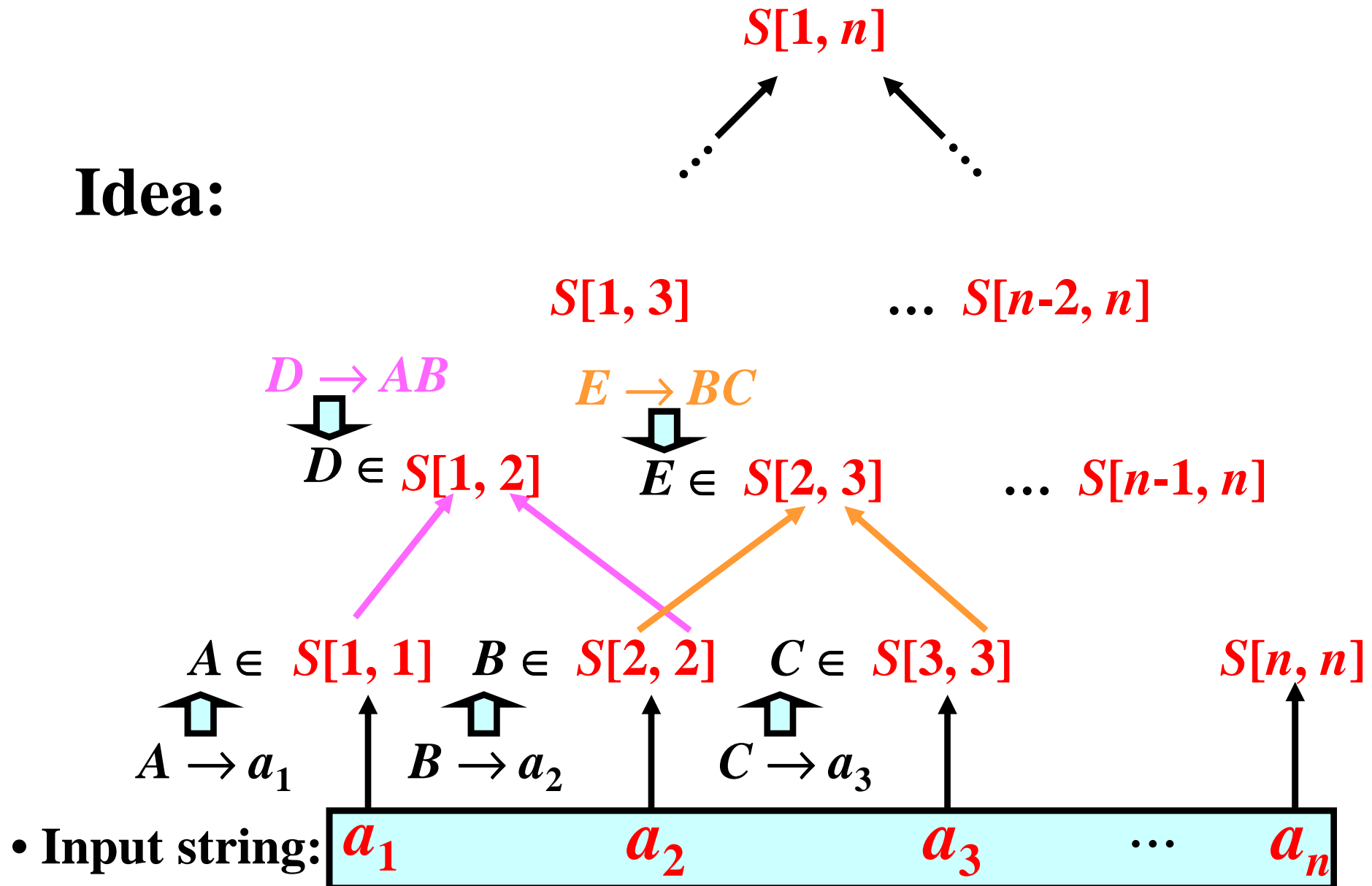
GP Based on Chomsky Normal Form

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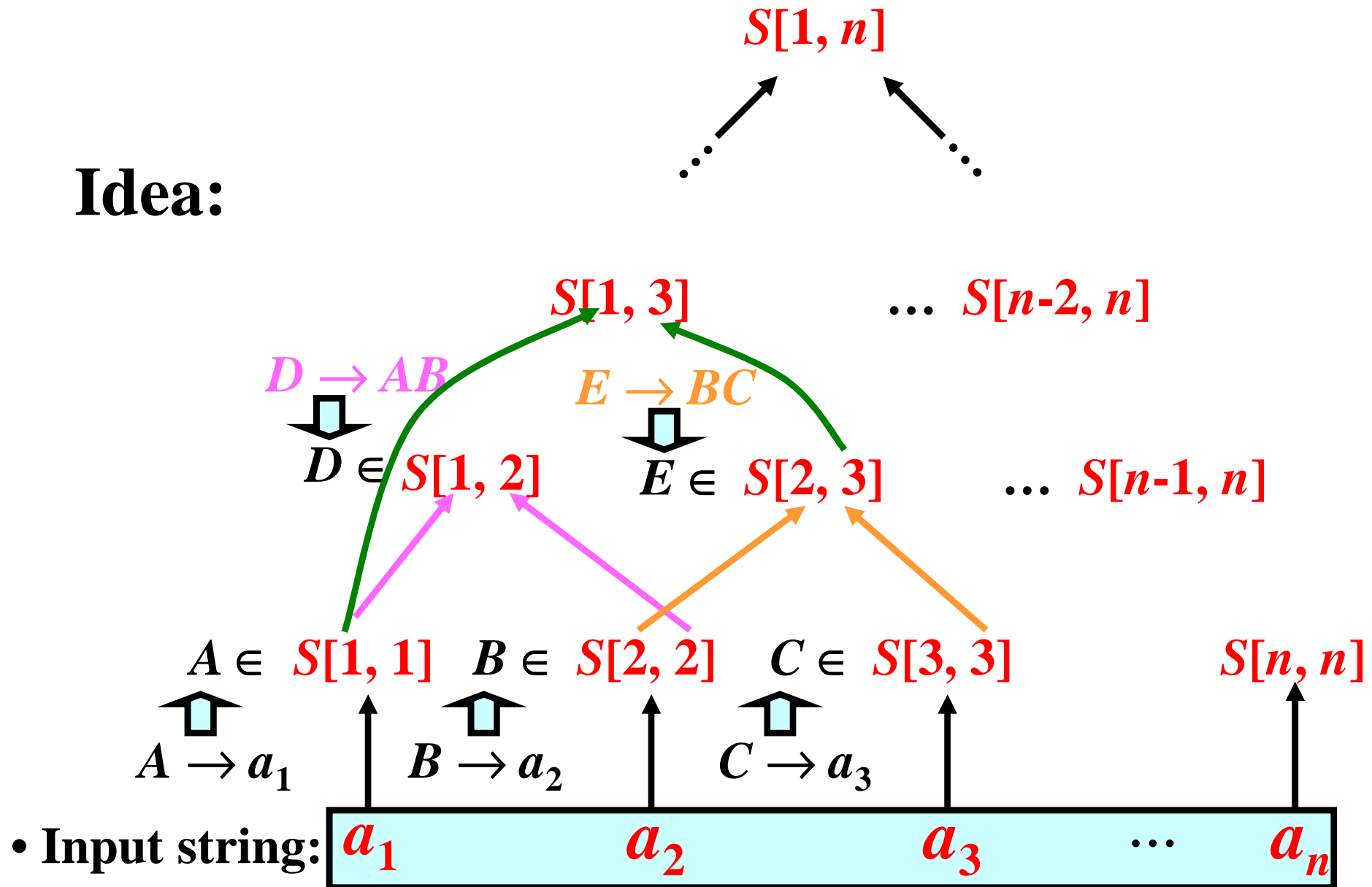
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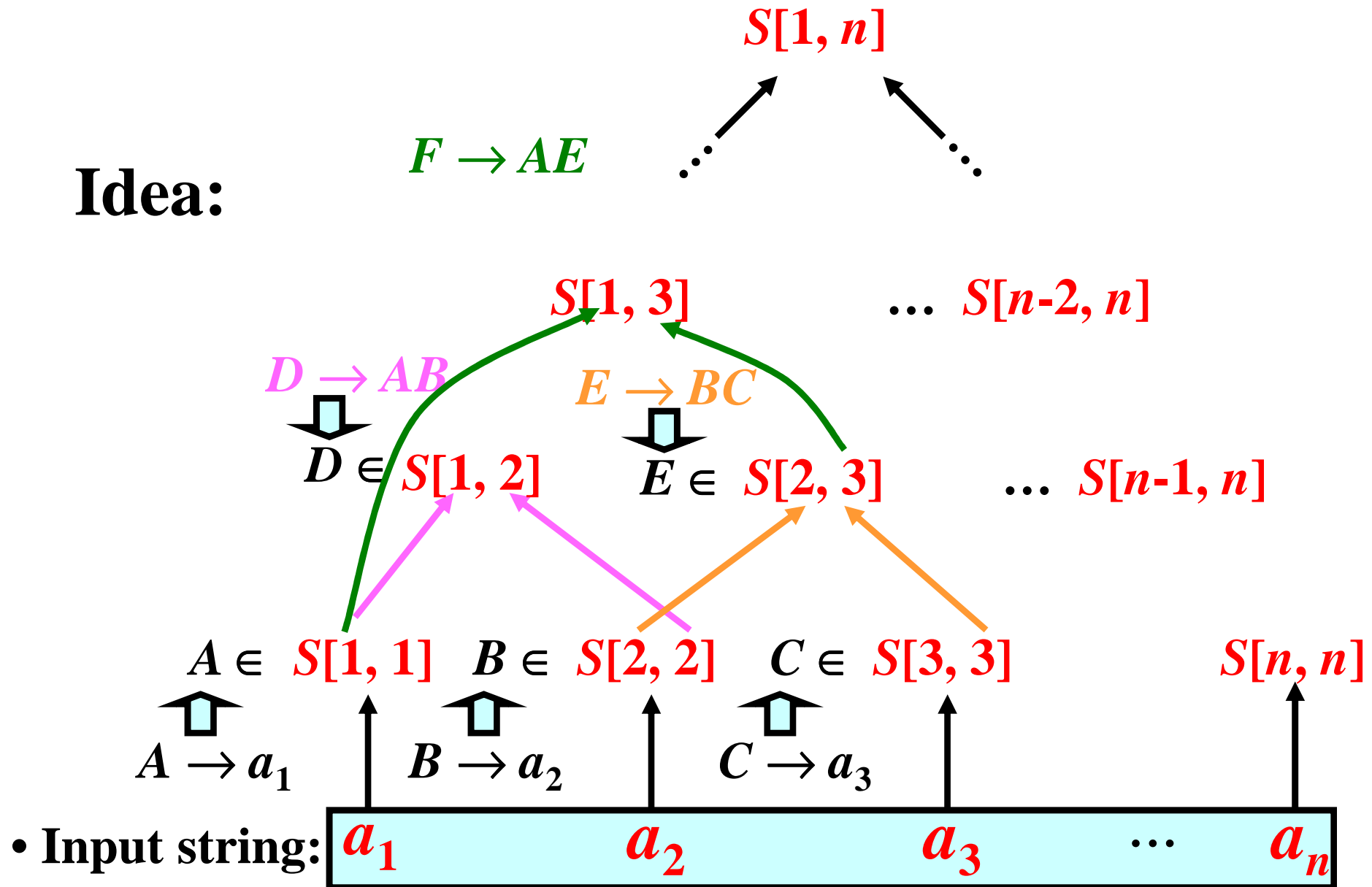
GP Based on Chomsky Normal Form

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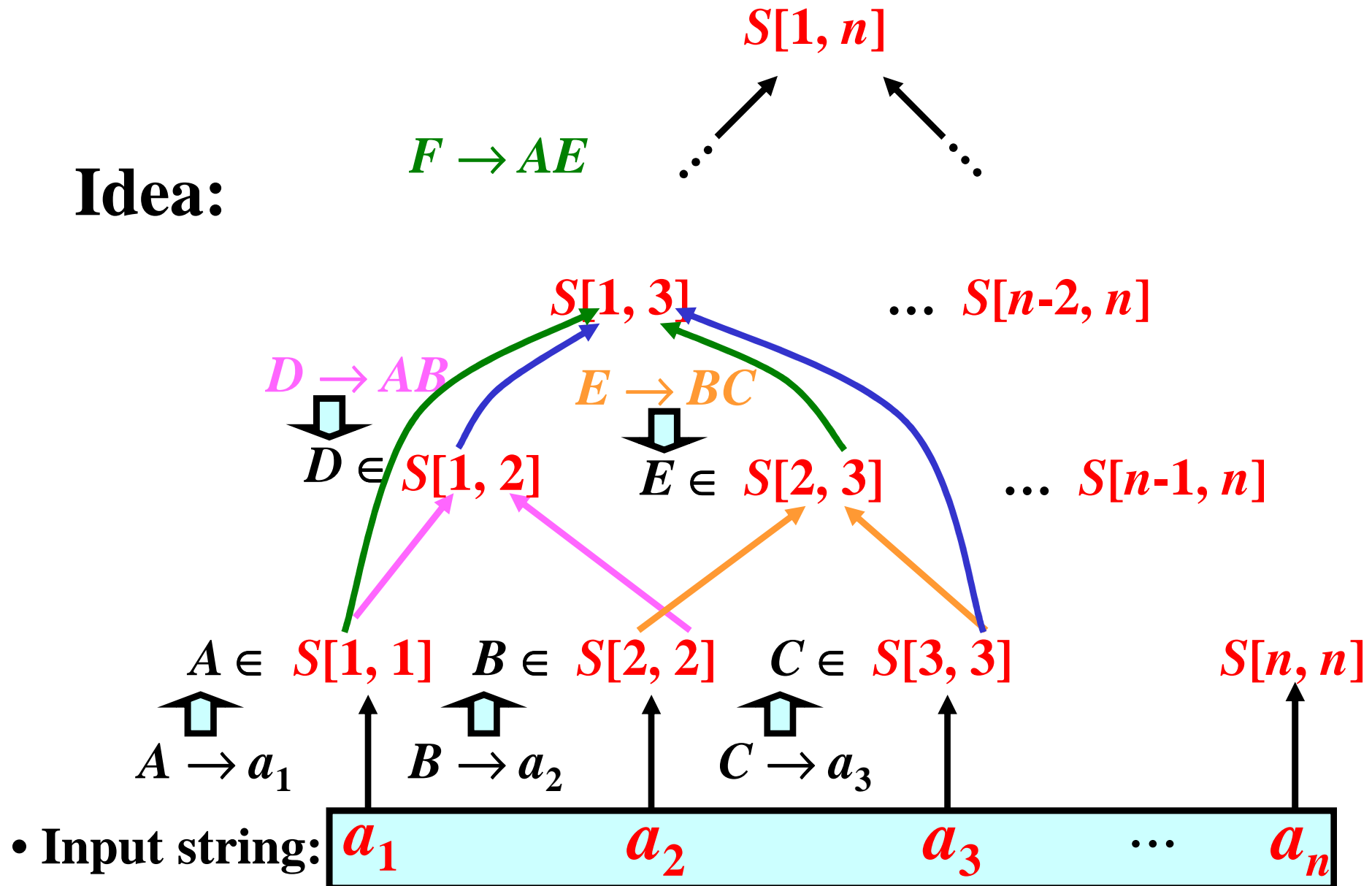
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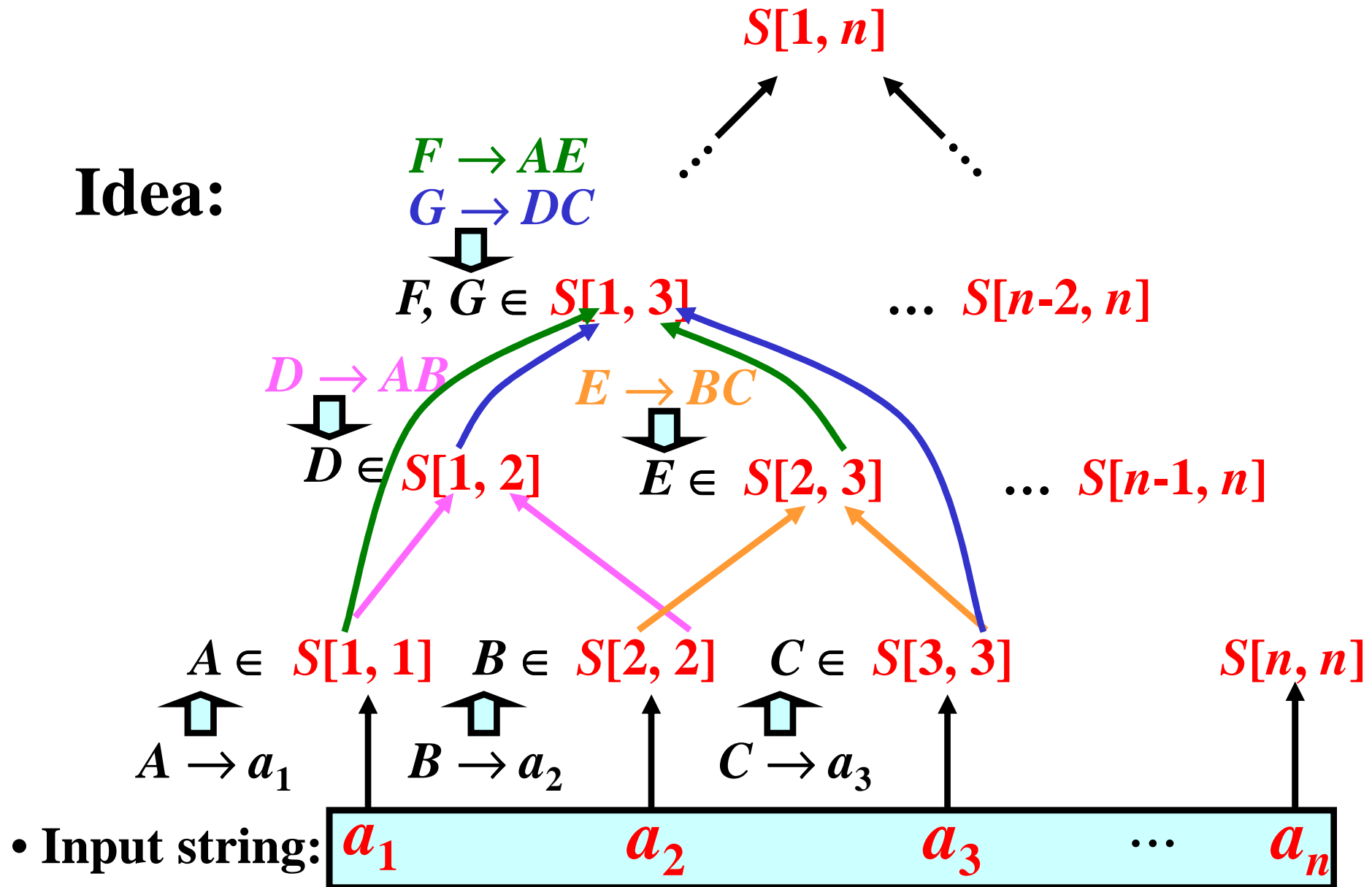
GP Based on Chomsky Normal Form

Idea:



GP Based on Chomsky Normal Form

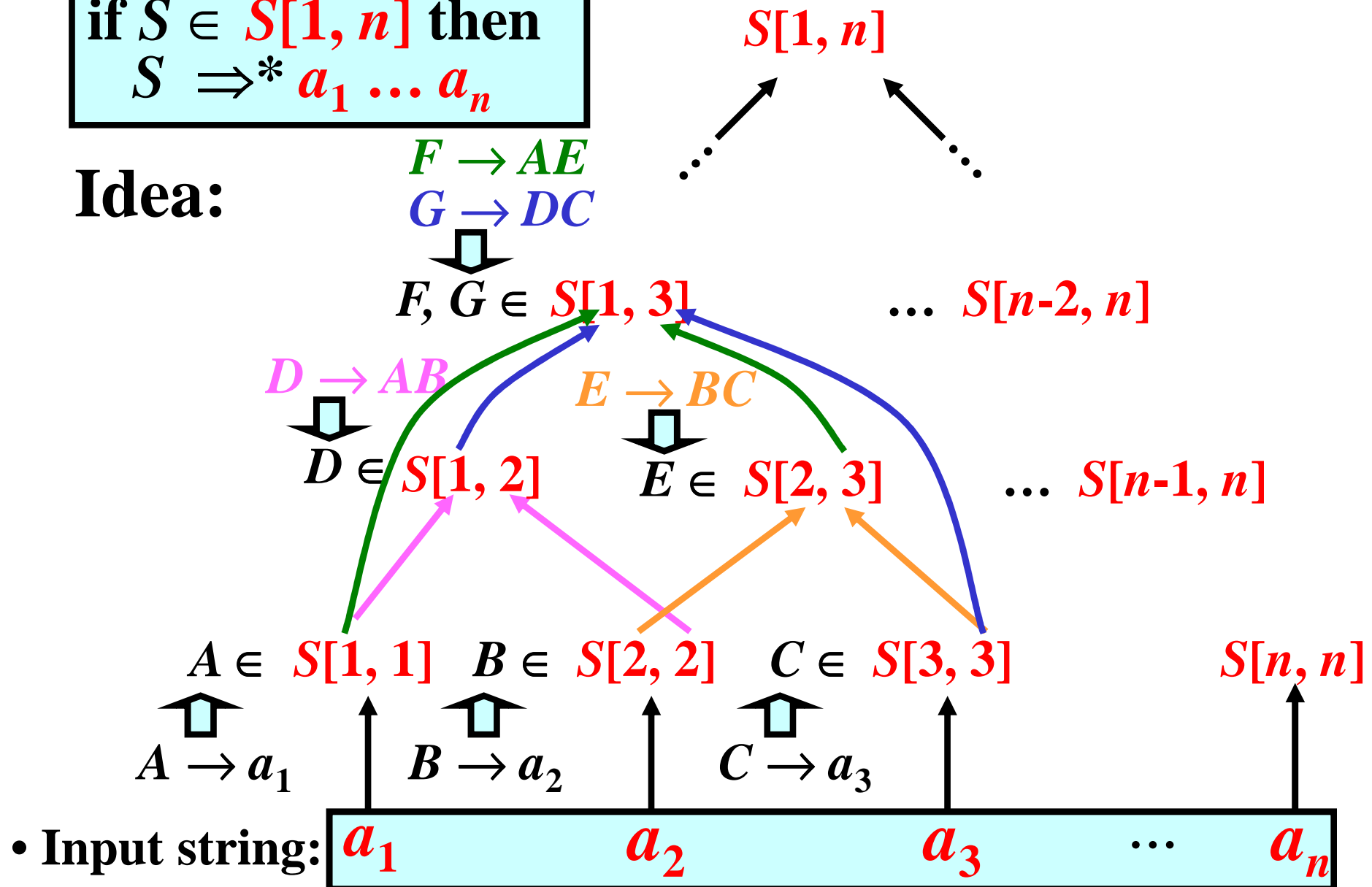
Idea:



GP Based on Chomsky Normal Form

if $S \in S[1, n]$ then
 $S \Rightarrow^* a_1 \dots a_n$

Idea:



Algorithm: GP Based on CNF

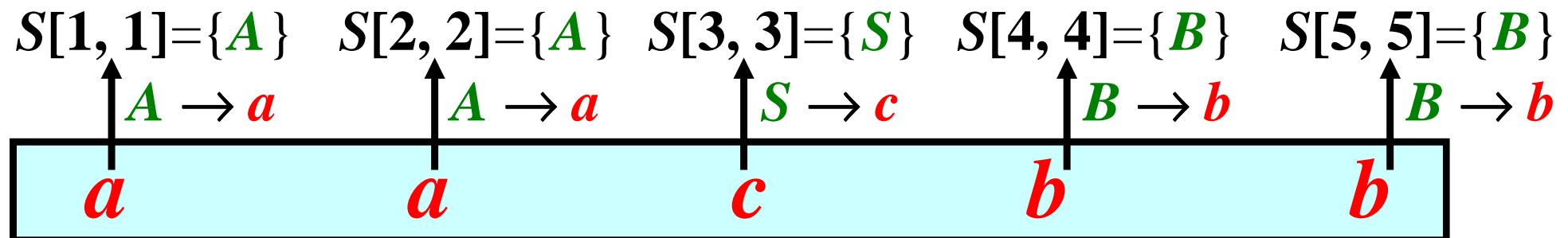
- **Input:** $G = (N, T, P, S)$ in CNF, $w = a_1 \dots a_n$
 - **Output:** **YES** if $w \in L(G)$
NO if $w \notin L(G)$
-

- **Method:**
- for each $a_i, i = 1, \dots, n$ do
 - $S[i, i] := \{A : A \rightarrow a_i \in P\}$
- Apply the following rule until no $S[i, k]$ can be changed:
 - if $A \rightarrow BC \in P, B \in S[i, j], C \in S[j+1, k]$,
 where $1 \leq i \leq j < k \leq n$ then add A to $S[i, k]$
- if $S \in S[1, n]$ then write ('**YES**')
 else write ('**NO**')

GP Based on CNF: Example 1/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b\}$,
 $P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$

Question: $aacbb \in L(G)$?



GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b\}$,
 $P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$

Question: $aacbb \in L(G)$?

$S[1, 2]$

$S[2, 3]$

$S[3, 4]$

$S[4, 5]$

$S[1, 1] = \{A\}$ $S[2, 2] = \{A\}$ $S[3, 3] = \{S\}$ $S[4, 4] = \{B\}$ $S[5, 5] = \{B\}$

a

a

c

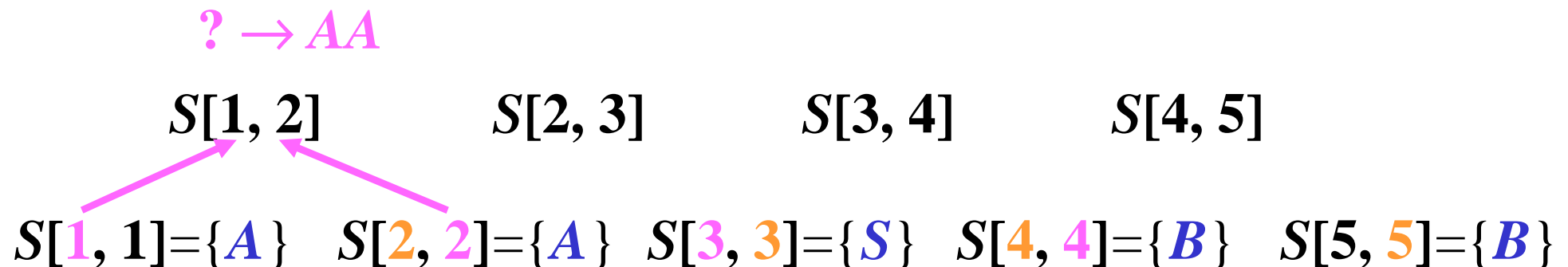
b

b

GP Based on CNF: Example 2/5

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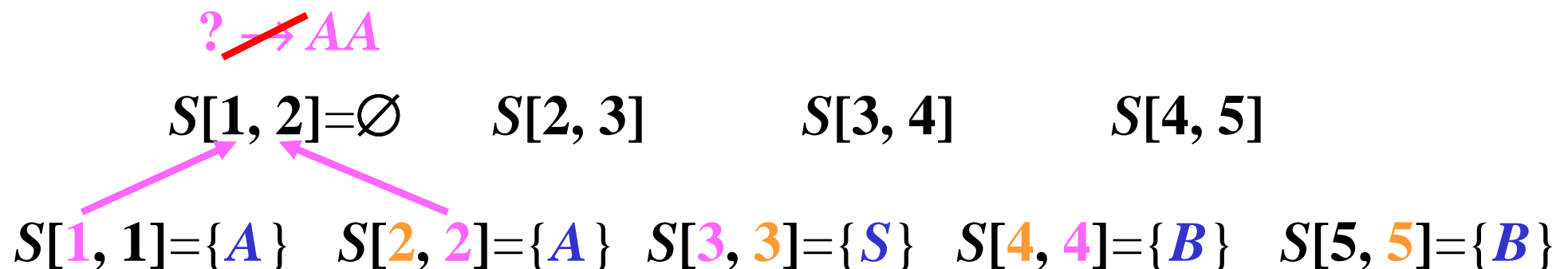
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*a**a**c**b**b*

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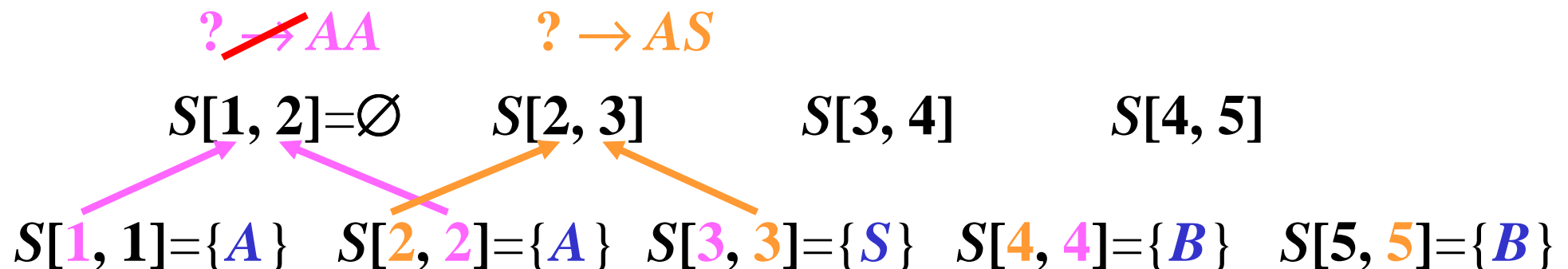
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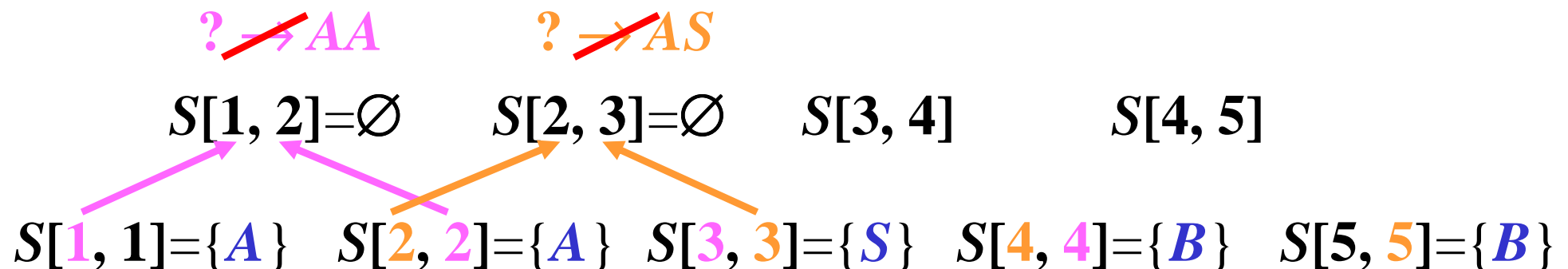
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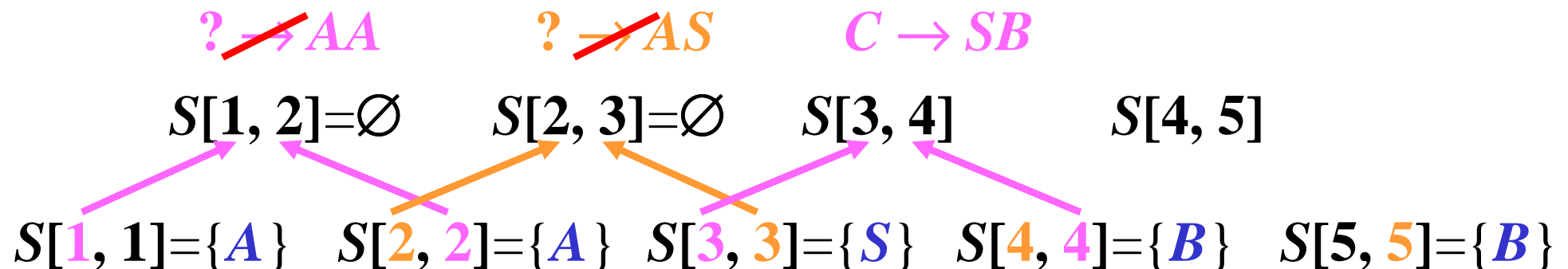
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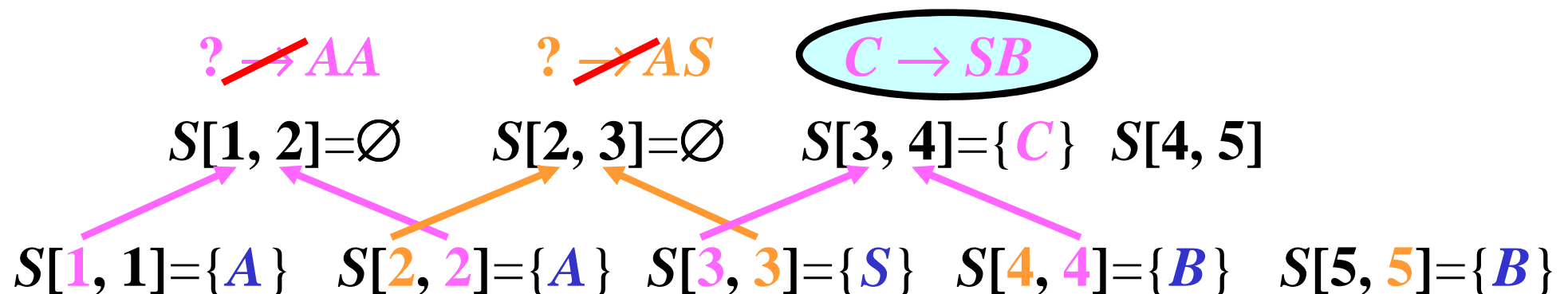
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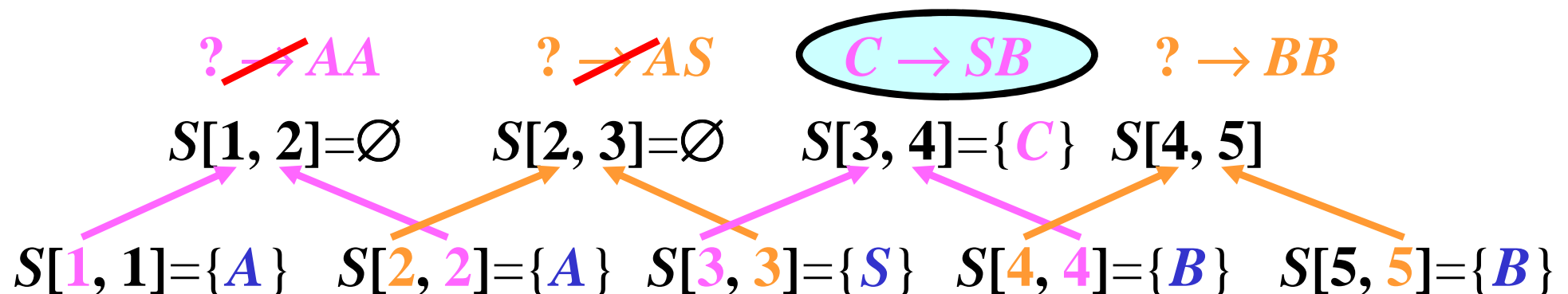
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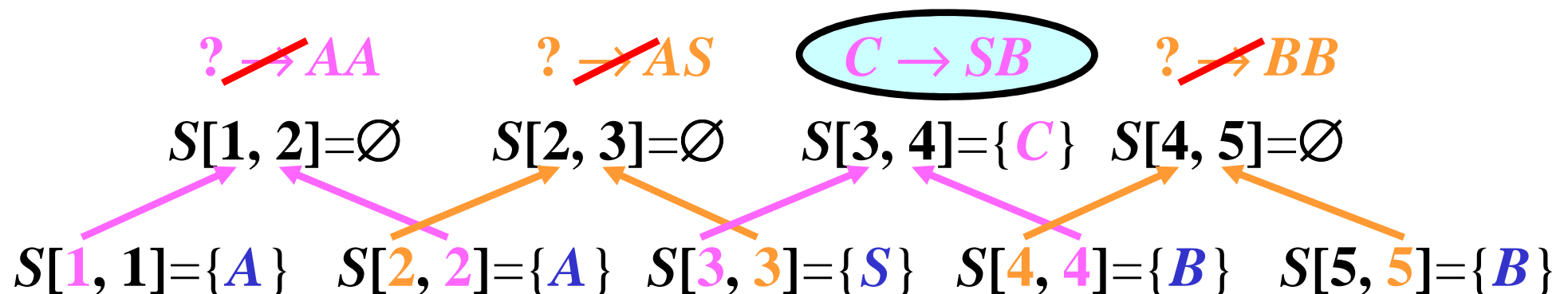
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 a a c b b

GP Based on CNF: Example 3/5

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Question: $aacbb \in L(G)$?

$S[1, 3]$

$S[2, 4]$

$S[3, 5]$

$S[1, 2] = \emptyset$

$S[2, 3] = \emptyset$

$S[3, 4] = \{C\}$

$S[4, 5] = \emptyset$

$S[1, 1] = \{A\}$

$S[2, 2] = \{A\}$

$S[3, 3] = \{S\}$

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a

a

c

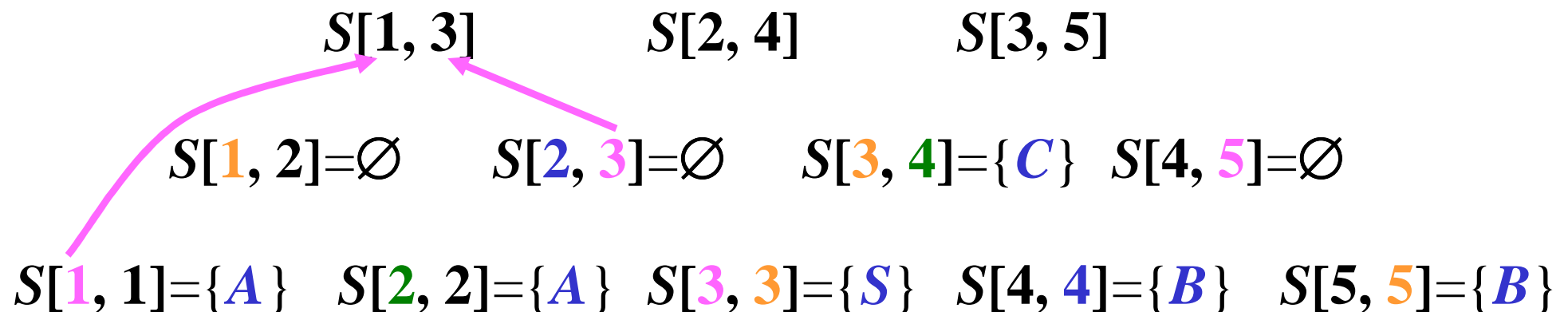
b

b

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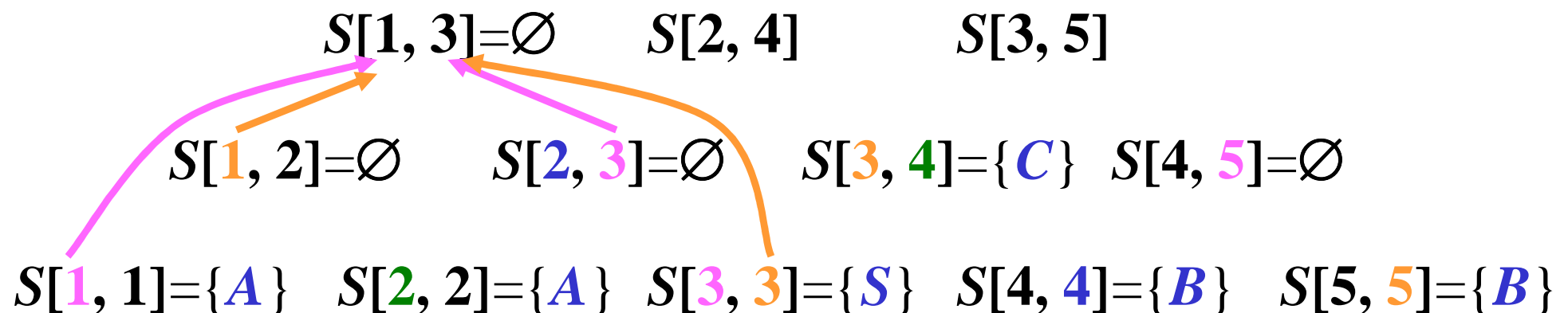
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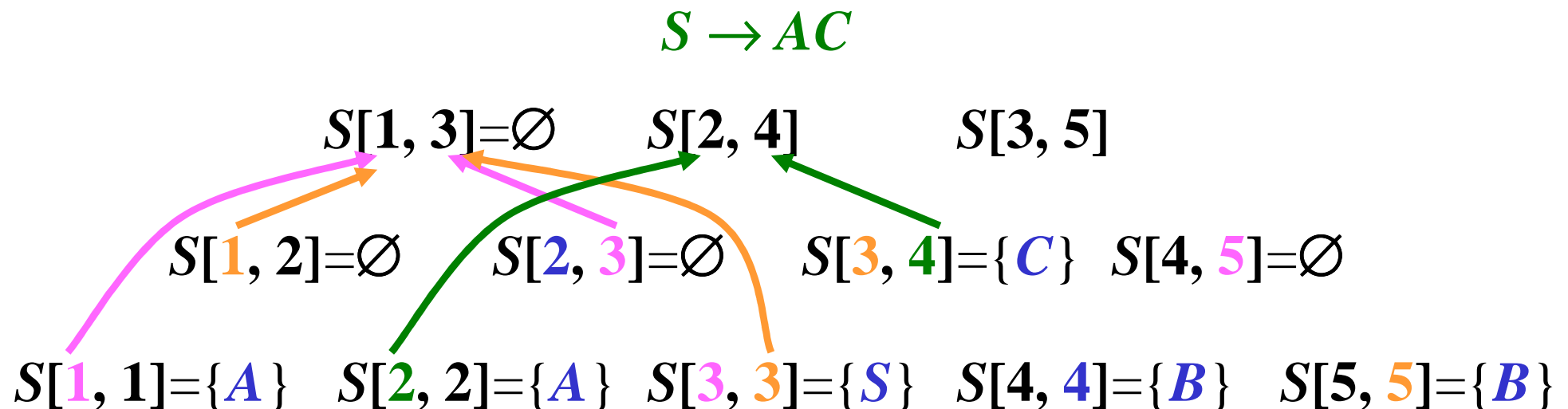
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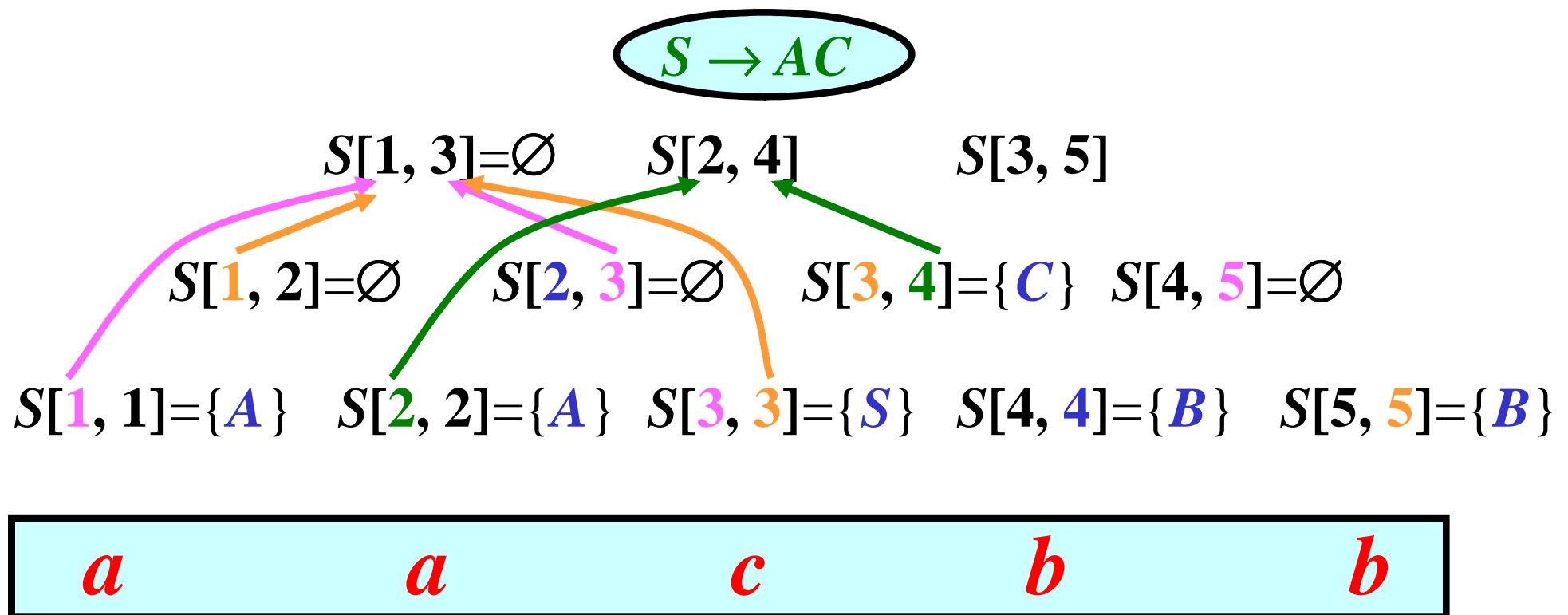
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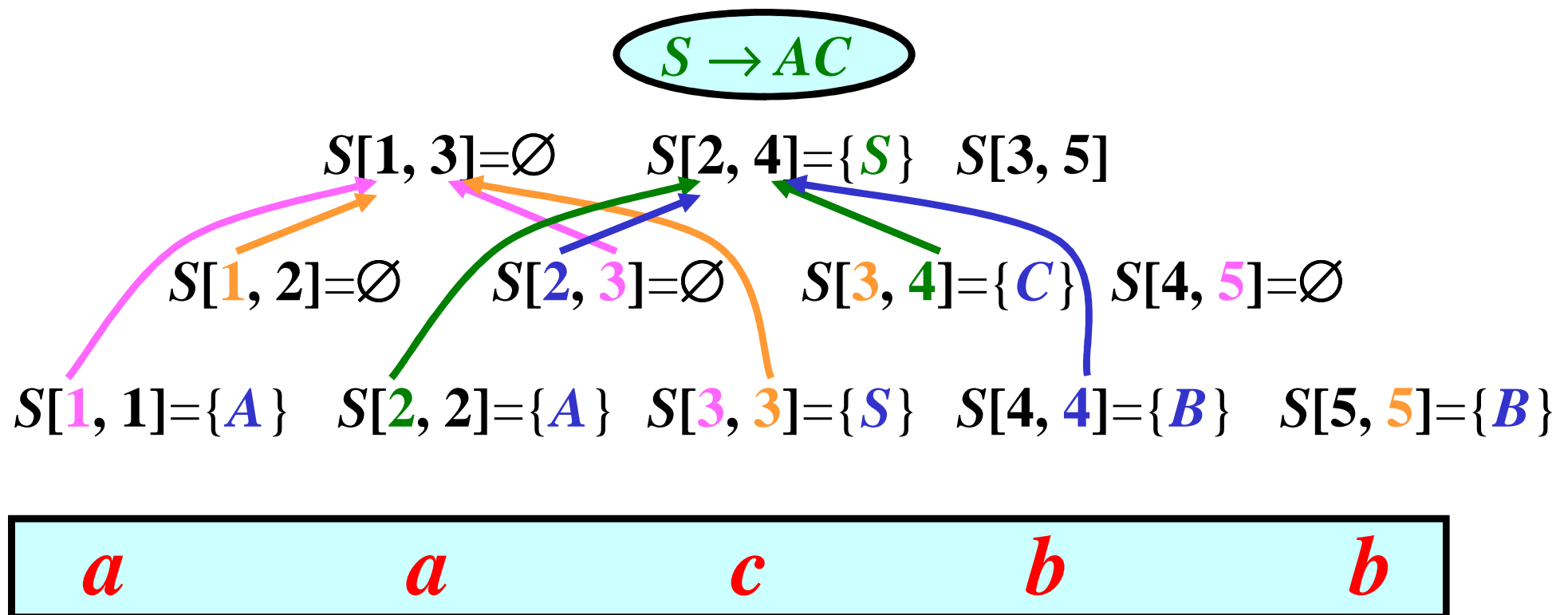
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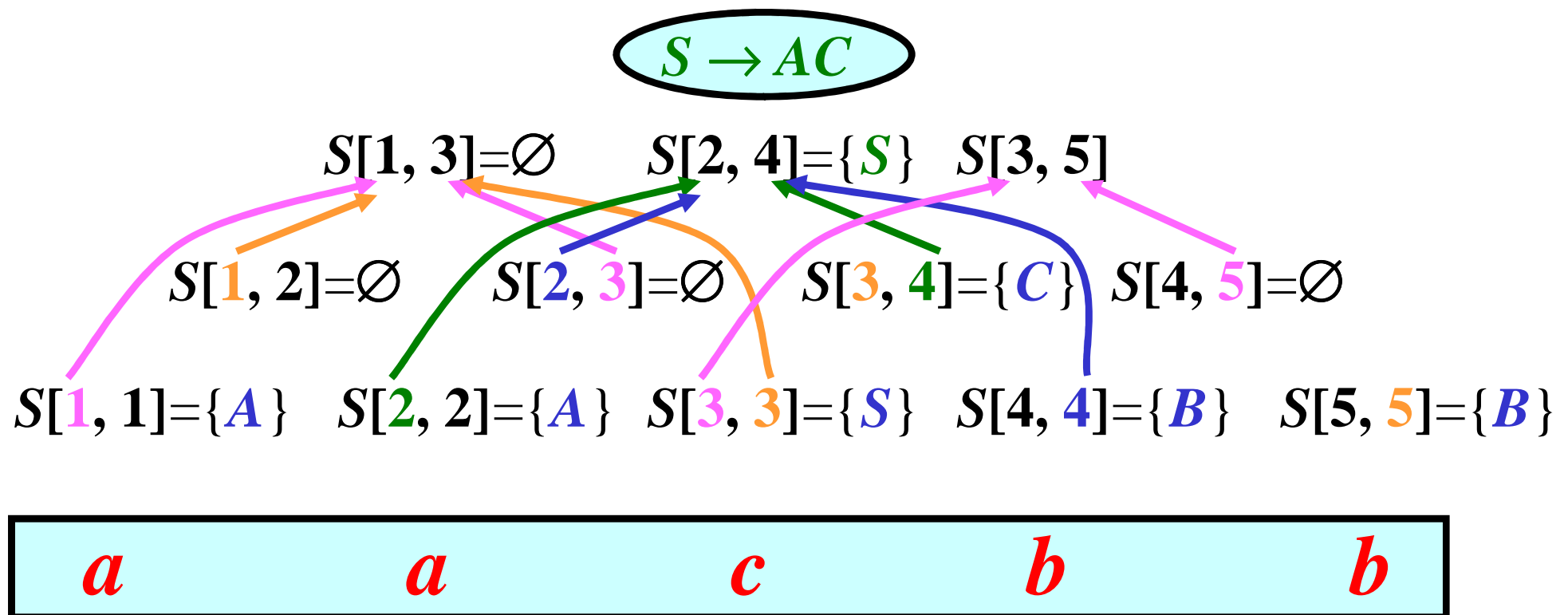
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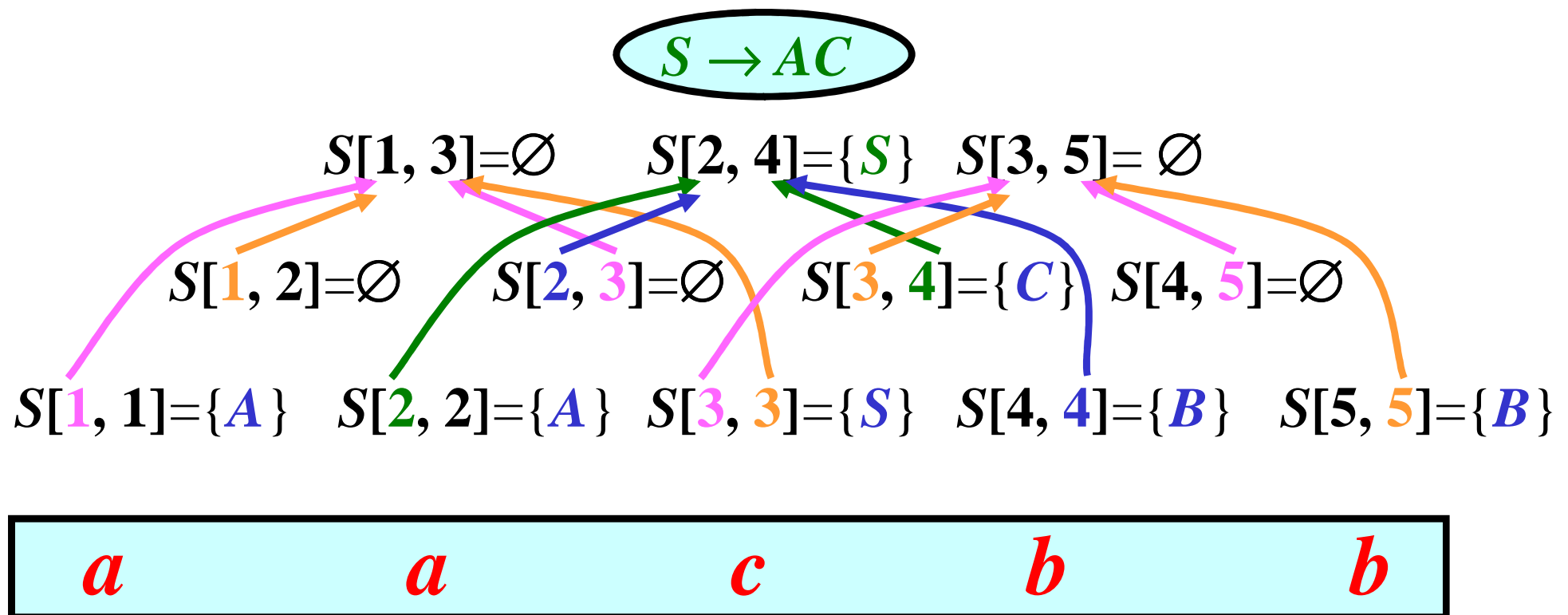
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GP Based on CNF: Example 4/5

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Question: $aacbb \in L(G)$?

$S[1, 4]$

$S[2, 5]$

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$S[1, 2] = \emptyset$ $S[2, 3] = \emptyset$ $S[3, 4] = \{C\}$ $S[4, 5] = \emptyset$

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a

a

c

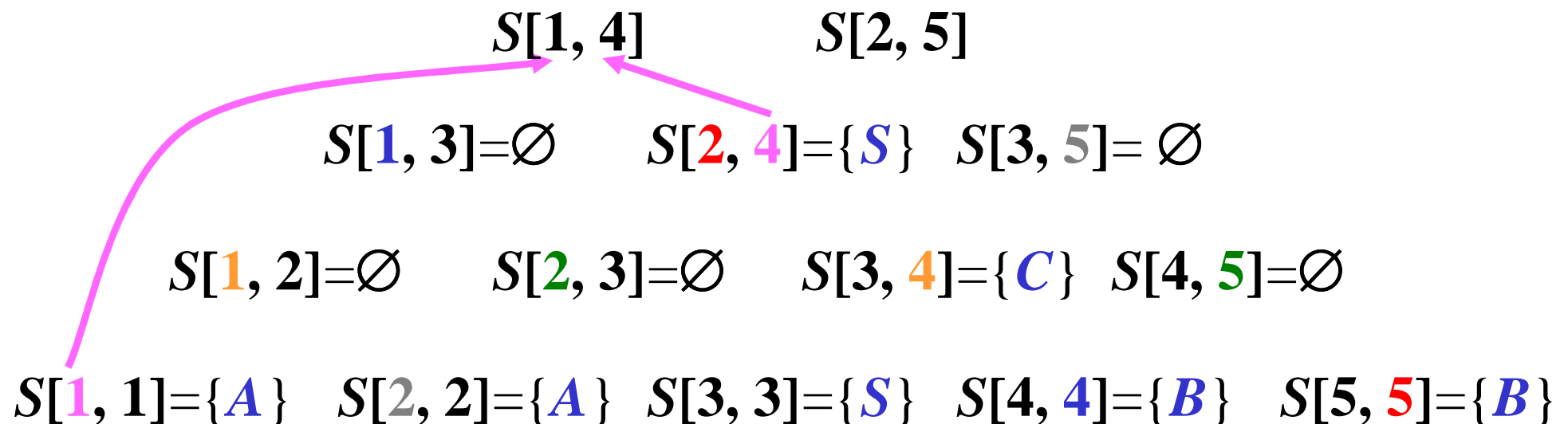
b

b

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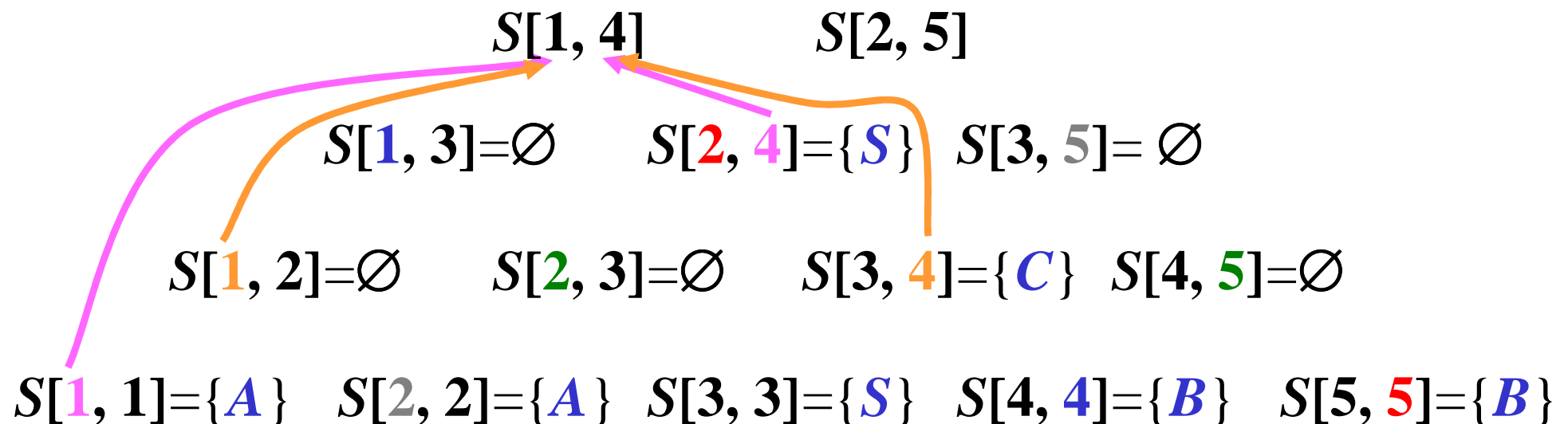
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 a a c b b

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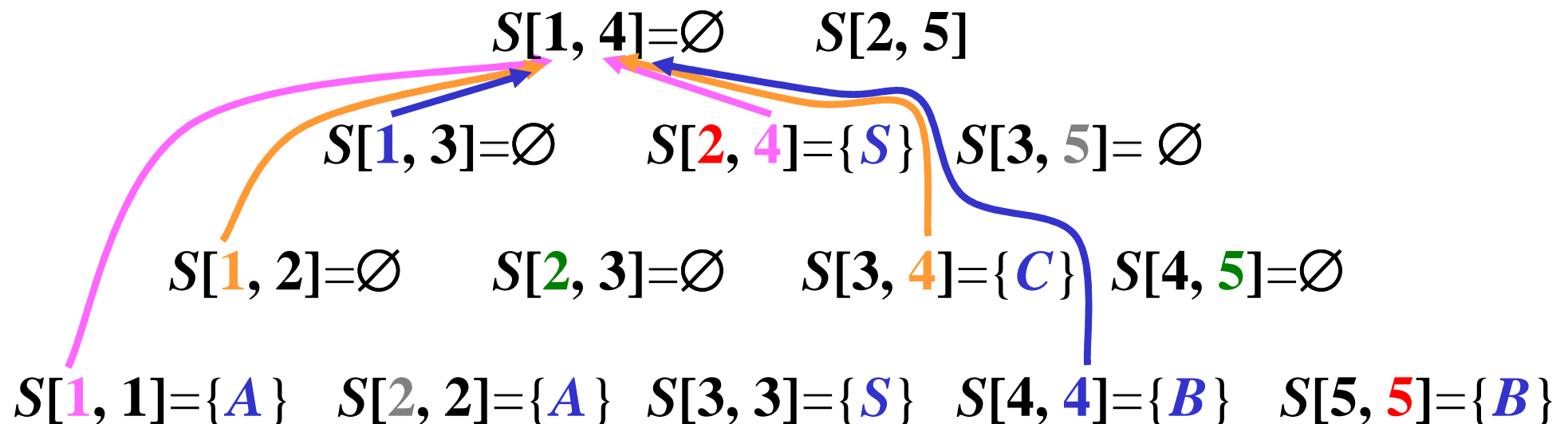
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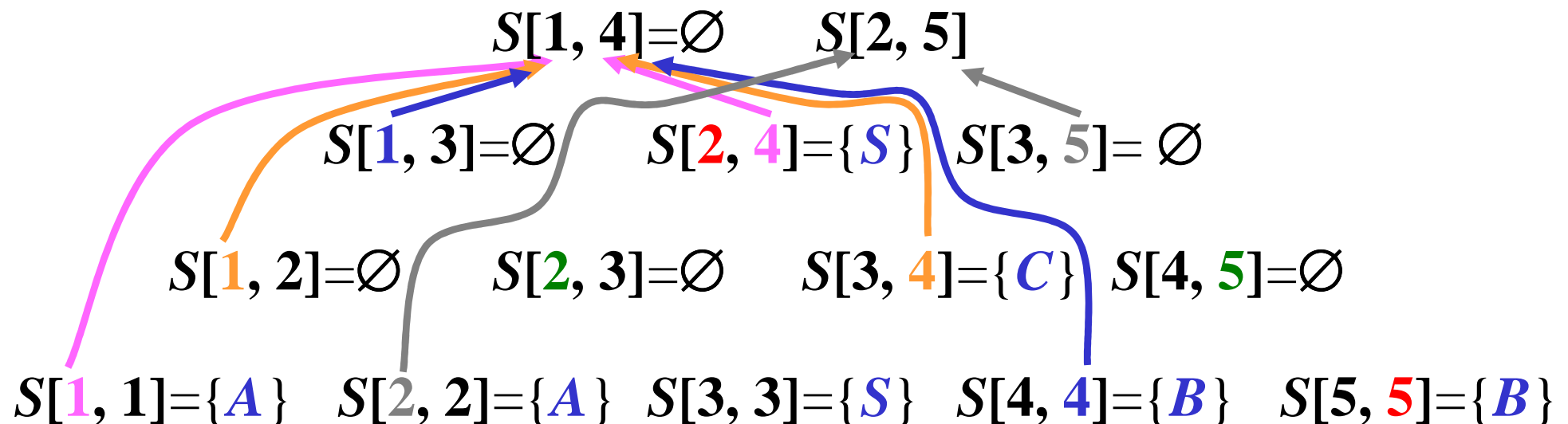
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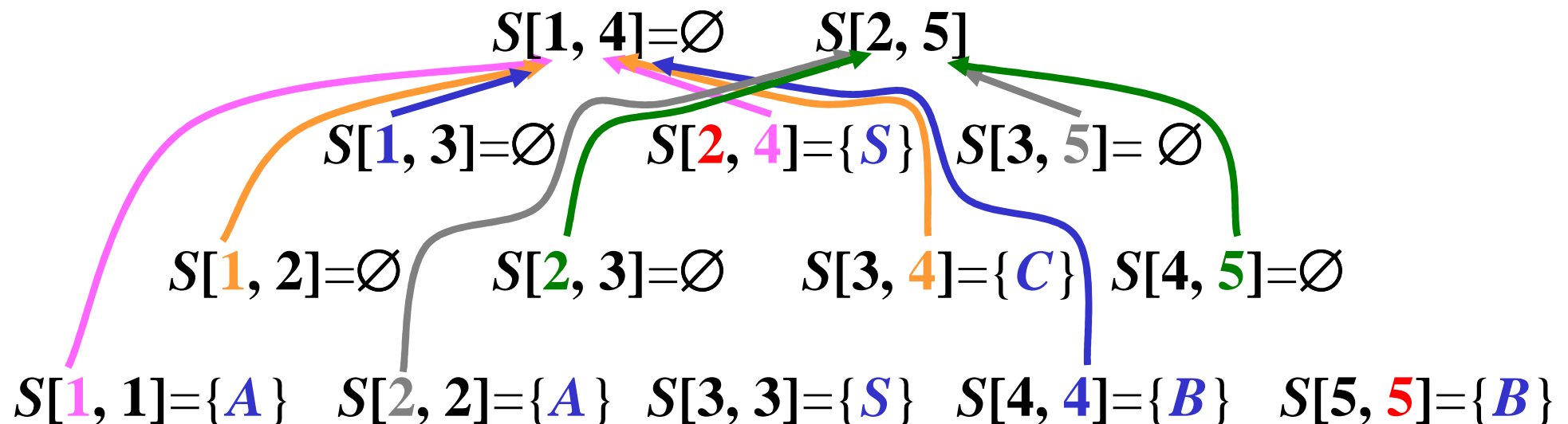
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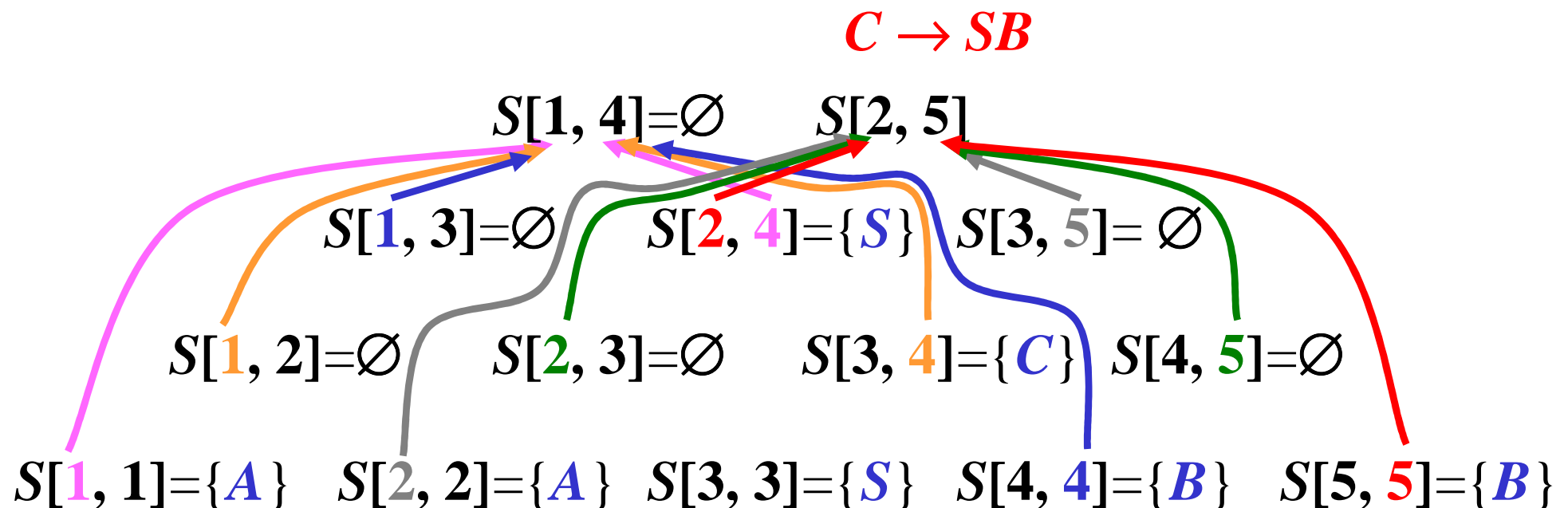
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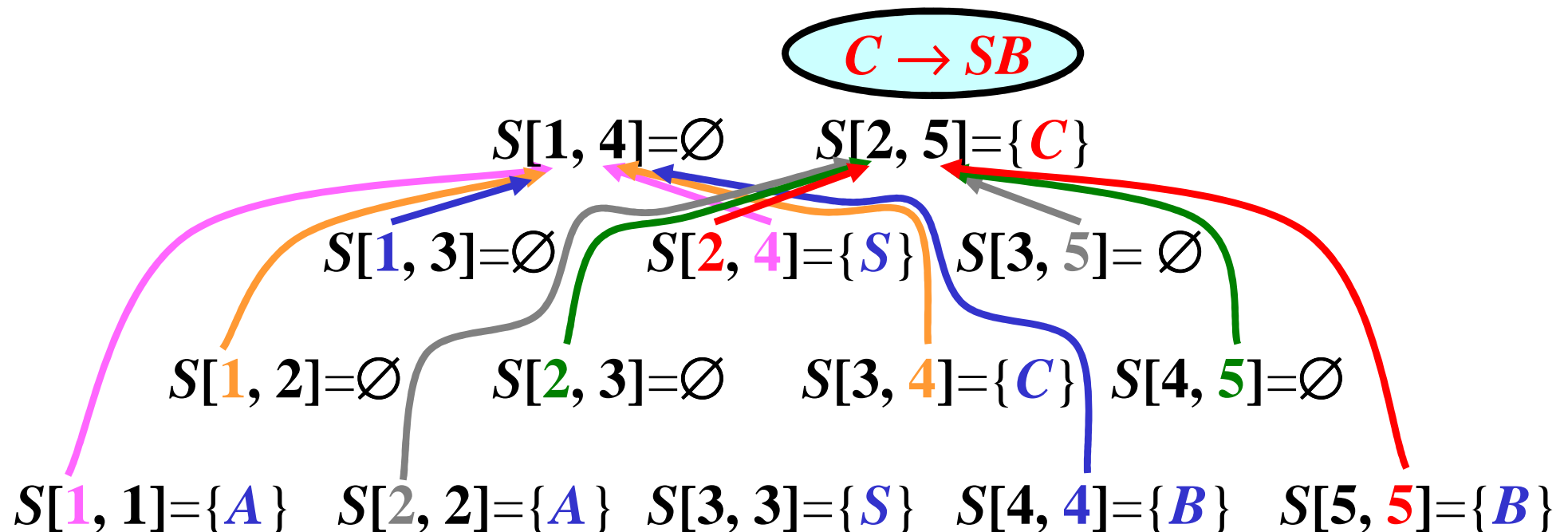
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Question: $aacbb \in L(G)$?

 a a c b b

GP Based on CNF: Example 5/5

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Question: $aacbb \in L(G)$?

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a

a

c

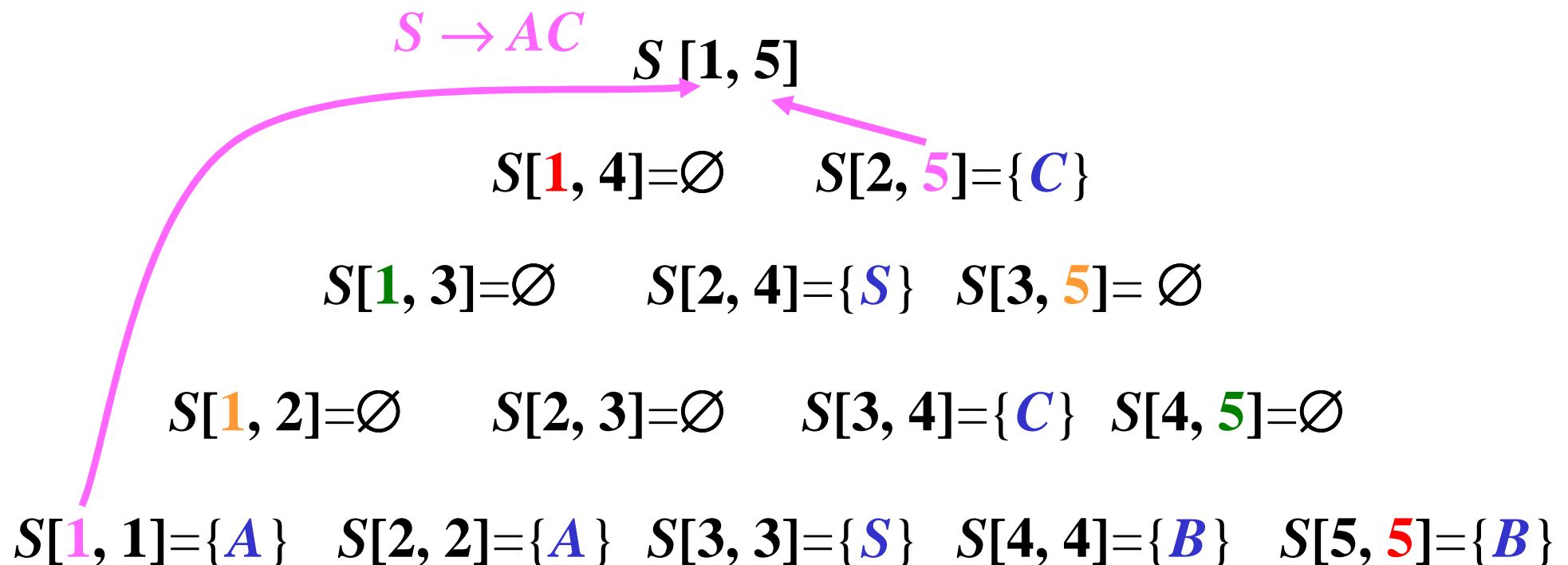
b

b

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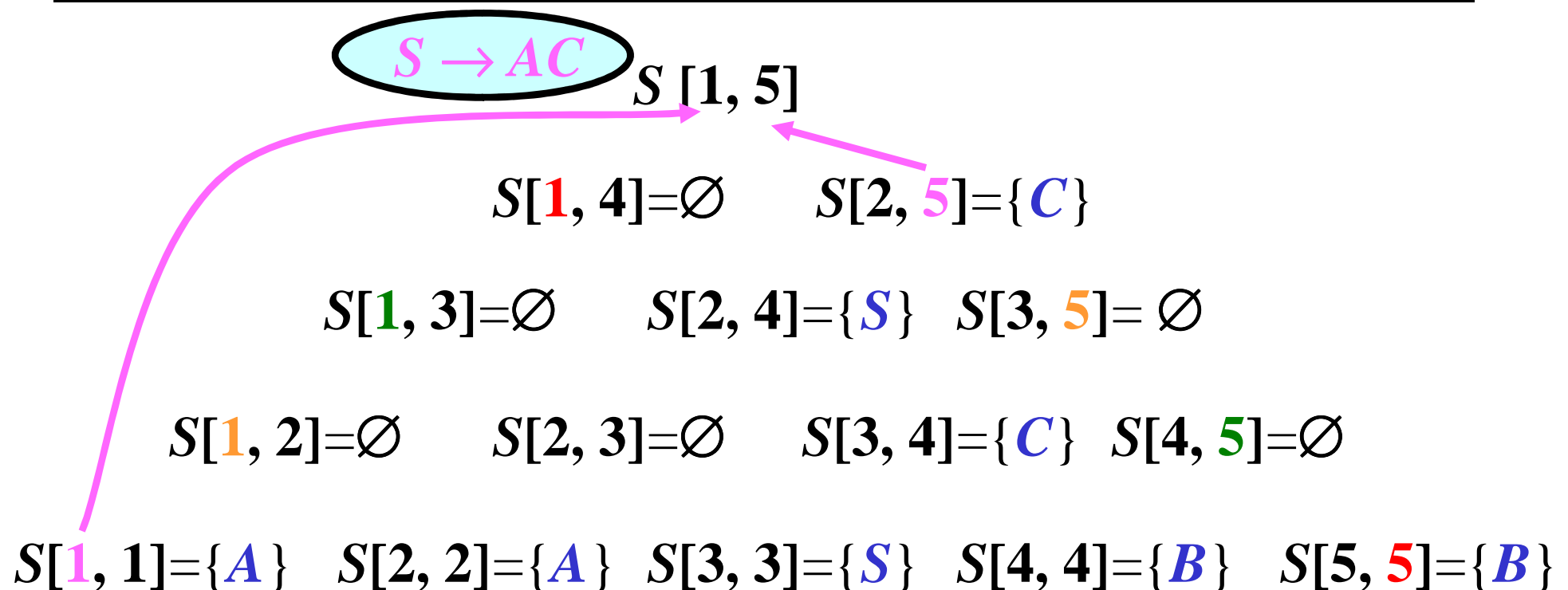
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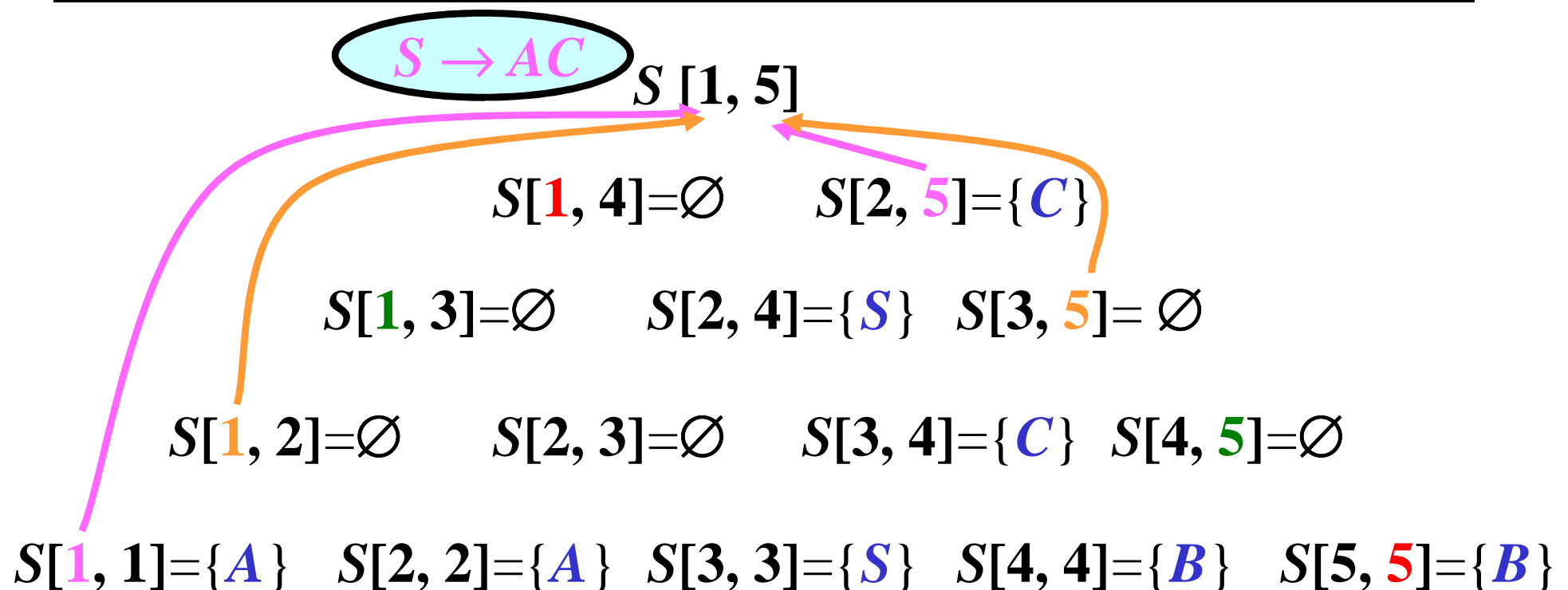
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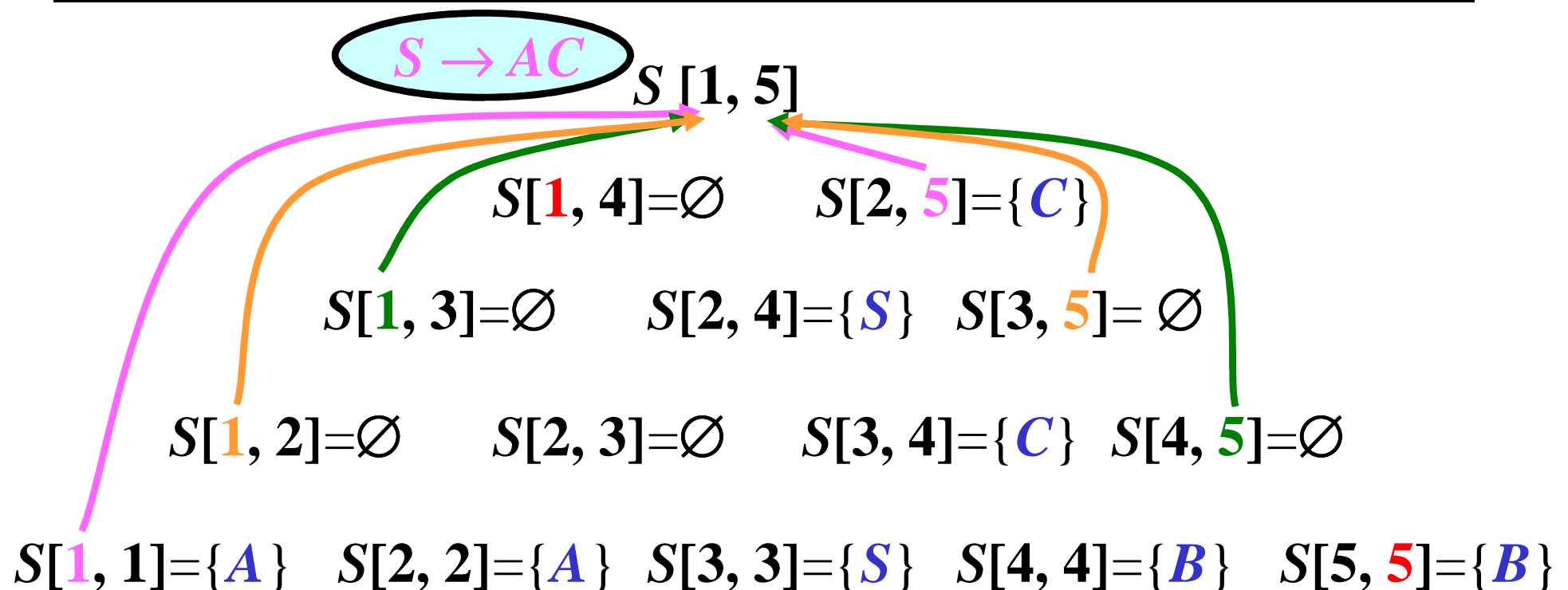
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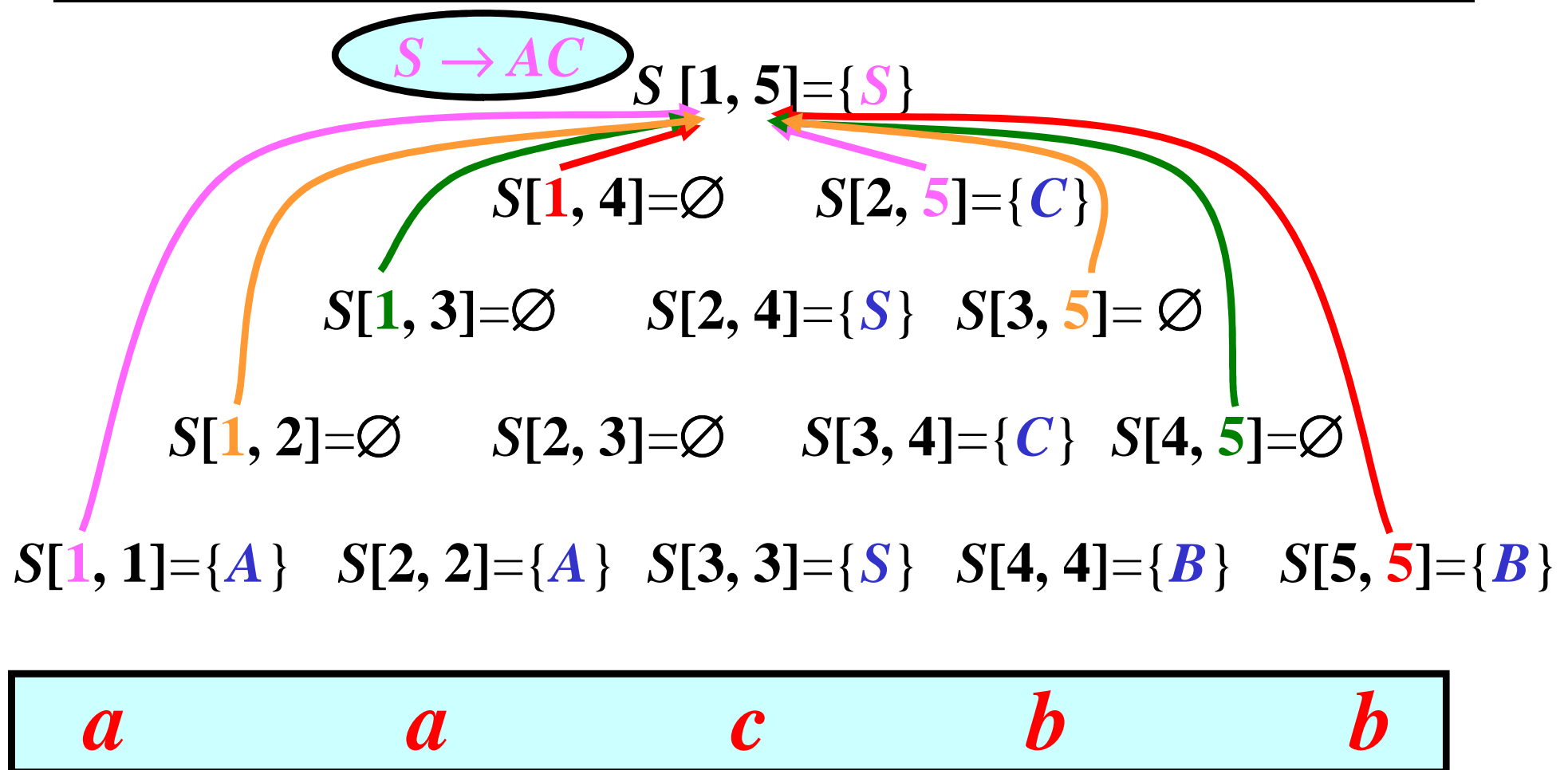
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 a a c b b

GP Based on CNF: Example 5/5

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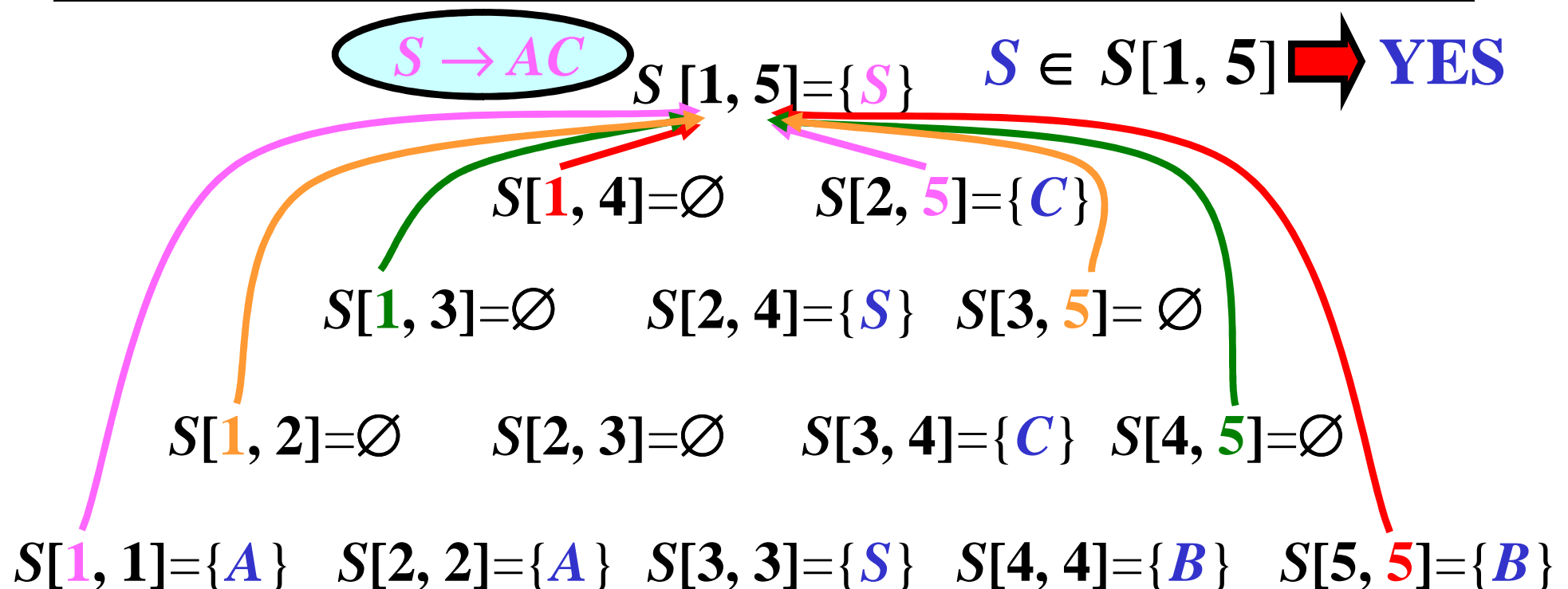
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Question: $aacbb \in L(G)$?



Pumping Lemma for CFL

- Let L be CFL. Then, there exists $k \geq 1$ such that:
if $z \in L$ and $|z| \geq k$ **then** there exist u, v, w, x, y so
 $z = uvwxy$ and
1) $vx \neq \varepsilon$ **2)** $|vwx| \leq k$ **3)** for each $m \geq 0$, $uv^mwx^my \in L$

Example:

$G = (\{S, A\}, \{a, b, c\}, \{S \rightarrow aAa, A \rightarrow bAb, A \rightarrow c\}, S)$
 generate $L(G) = \{ab^n cb^n a : n \geq 0\}$, so $L(G)$ is CFL.

There is $k = 5$ such that **1)**, **2)** and **3)** holds:

- for $z = \textcolor{brown}{a}\textcolor{blue}{b}\textcolor{green}{c}\textcolor{blue}{b}\textcolor{brown}{a}$: $z \in L(G)$ and $|z| \geq 5$:

$$\begin{array}{c} \textcolor{brown}{a}\textcolor{blue}{b}\textcolor{green}{c}\textcolor{blue}{b}\textcolor{brown}{a} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ uvwx y \end{array}$$

$$uv^0wx^0y = \textcolor{brown}{a}\textcolor{blue}{b}^0\textcolor{green}{c}\textcolor{blue}{b}^0\textcolor{brown}{a} = \textcolor{brown}{a}\textcolor{green}{c}\textcolor{brown}{a} \in L(G)$$

$$uv^1wx^1y = \textcolor{brown}{a}\textcolor{blue}{b}^1\textcolor{green}{c}\textcolor{blue}{b}^1\textcolor{brown}{a} = \textcolor{brown}{a}\textcolor{blue}{b}\textcolor{green}{c}\textcolor{blue}{b}\textcolor{brown}{a} \in L(G)$$

$$uv^2wx^2y = \textcolor{brown}{a}\textcolor{blue}{b}^2\textcolor{green}{c}\textcolor{blue}{b}^2\textcolor{brown}{a} = \textcolor{brown}{a}\textcolor{blue}{b}\textcolor{blue}{b}\textcolor{green}{c}\textcolor{blue}{b}\textcolor{blue}{b}\textcolor{brown}{a} \in L(G)$$

$$\vdots$$

$$vx = \textcolor{blue}{b}\textcolor{blue}{b} \neq \varepsilon$$

$$|vwx| = \textcolor{violet}{3}: 1 \leq \textcolor{violet}{3} \leq \textcolor{red}{5}$$
- for $z = \textcolor{brown}{a}\textcolor{blue}{b}\textcolor{blue}{b}\textcolor{green}{c}\textcolor{blue}{b}\textcolor{blue}{b}\textcolor{brown}{a}$: $z \in L(G)$ and $|z| \geq 5$:

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Pumping Lemma: Illustration

- L = any context-free language:
-

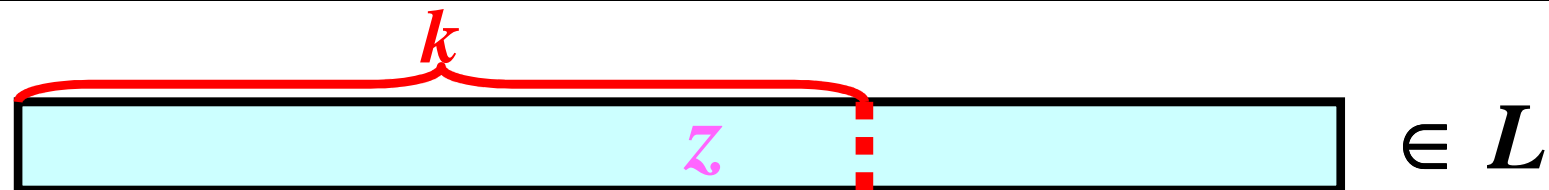
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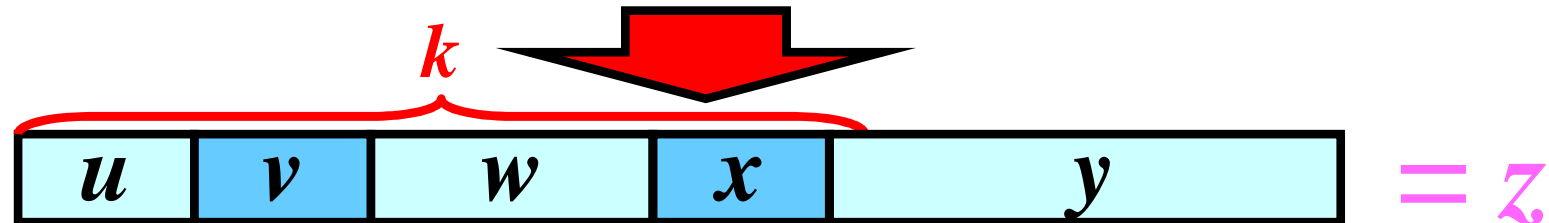
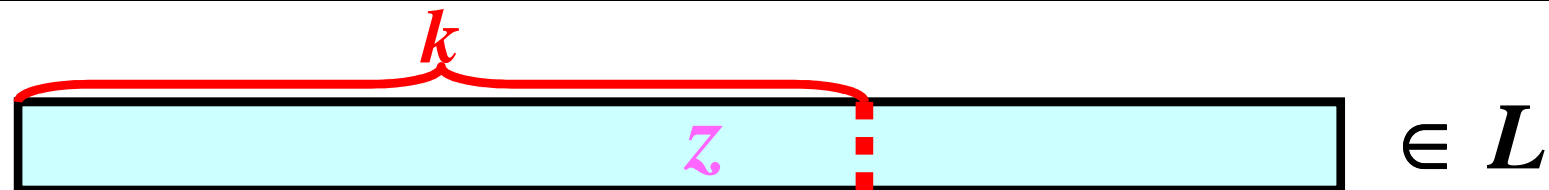
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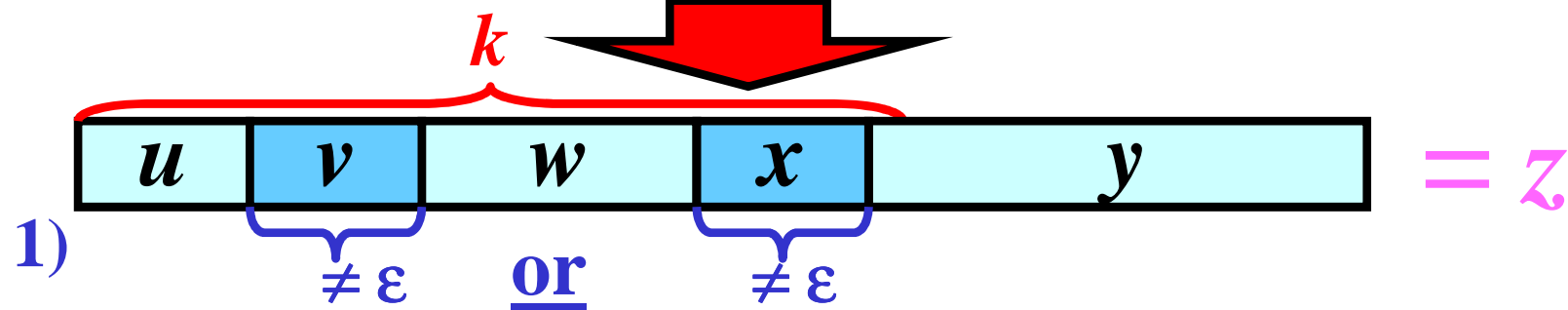
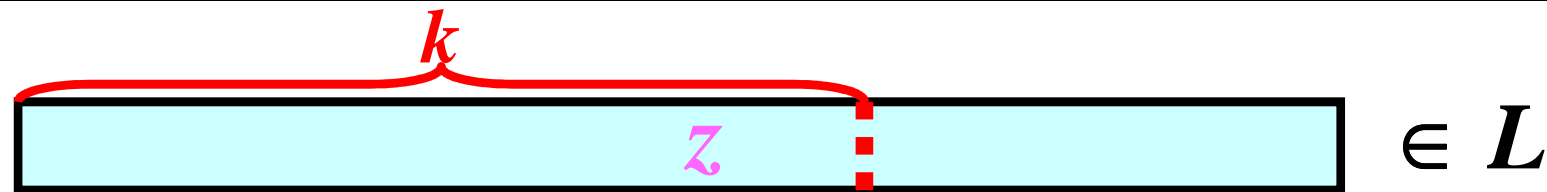
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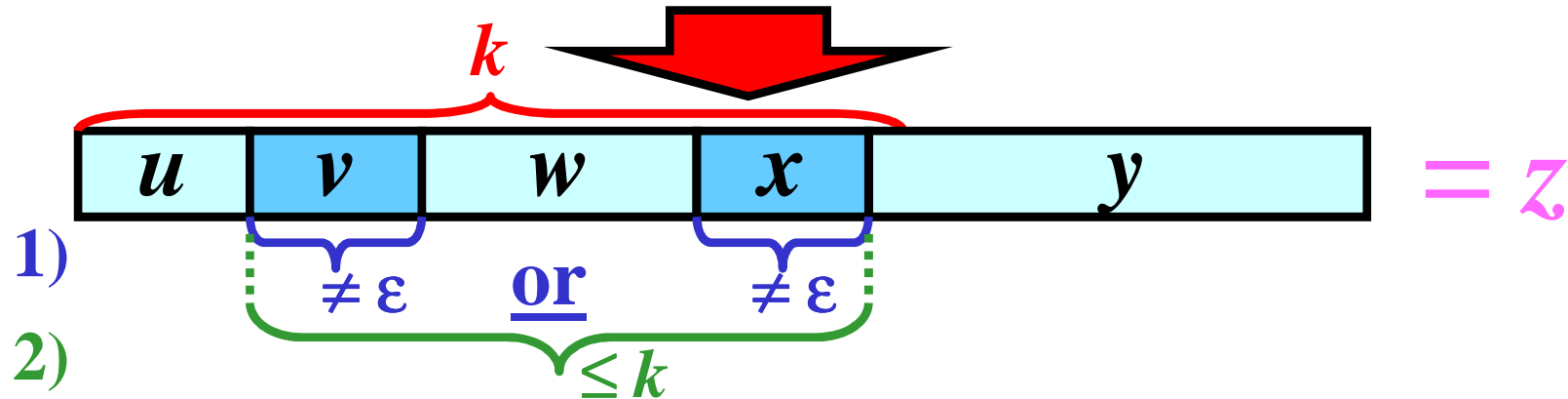
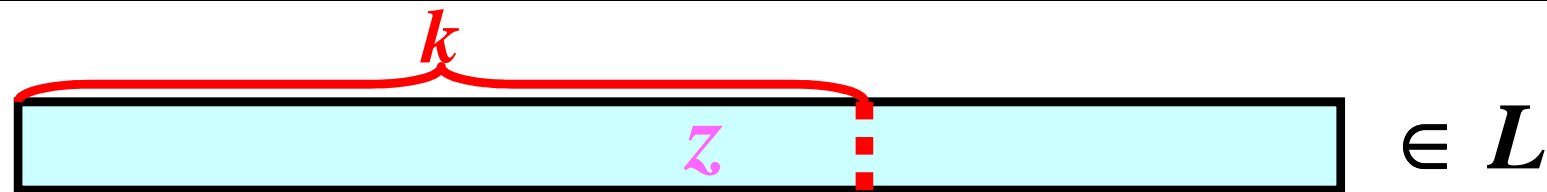
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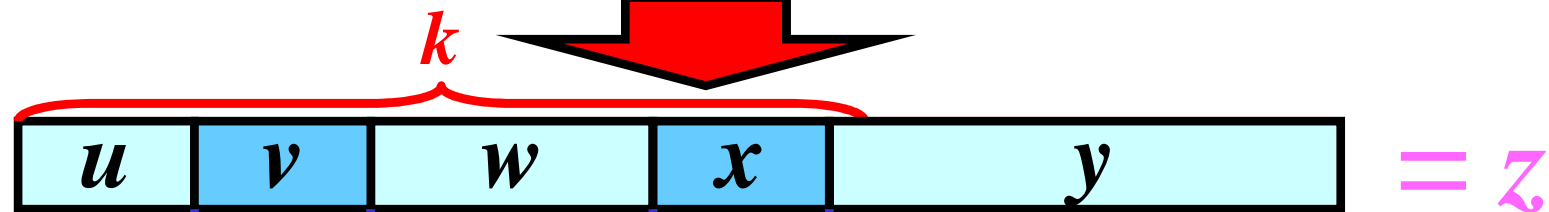
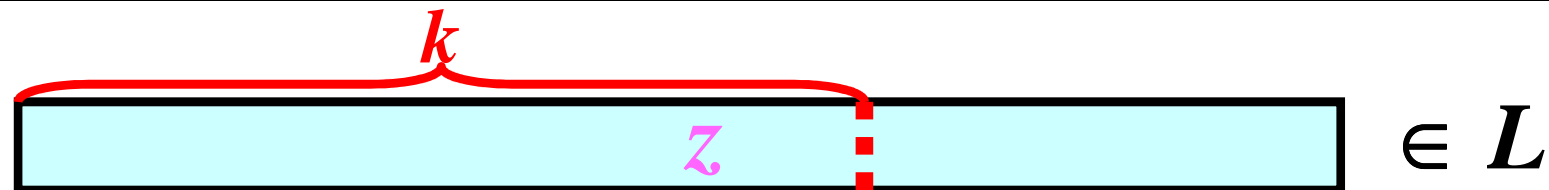
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Pumping Lemma: Illustration

- L = any context-free language:



- 1) $v \neq \epsilon$ or $x \neq \epsilon$
- 2) $|vwx| \leq k$



...

Pumping Lemma: Application

- Based on the pumping lemma for CFL, we often make a proof by contradiction to demonstrate that a language is **not** a CFL.

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For all decompositions of z into $uvwxy$: $vx \neq \varepsilon$, $|vwx| \leq k$, show that there exists $m \geq 0$ such that $uv^mwx^my \notin L$; } **contradiction**
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Therefore,
 L is not a CFL

Pumping Lemma: Example 1/2

Prove that $L = \{a^n b^n c^n : n \geq 1\}$ is not CFL.

- 1) Assume that L is a CFL. Let $k \geq 1$ be the pumping lemma constant for L .
- 2) Let $z = a^k b^k c^k : a^k b^k c^k \in L$, $|z| = |a^k b^k c^k| = 3k \geq k$
- 3) All decompositions of z into $uvwxy$; $vx \neq \varepsilon$, $|vwx| \leq k$:

$\overbrace{aaaaa \dots a}^k \overbrace{bb \dots b}^k \overbrace{cc \dots c}^k$
 $aaaaa \dots a \underbrace{bb \dots b}_{\text{red}} \underbrace{cc \dots c}_{\text{red}}$

a) $vwx \in \{a\}^* \{b\}^*$,
 $vx \neq \varepsilon$

b) $vwx \in \{b\}^* \{c\}^*$,
 $vx \neq \varepsilon$

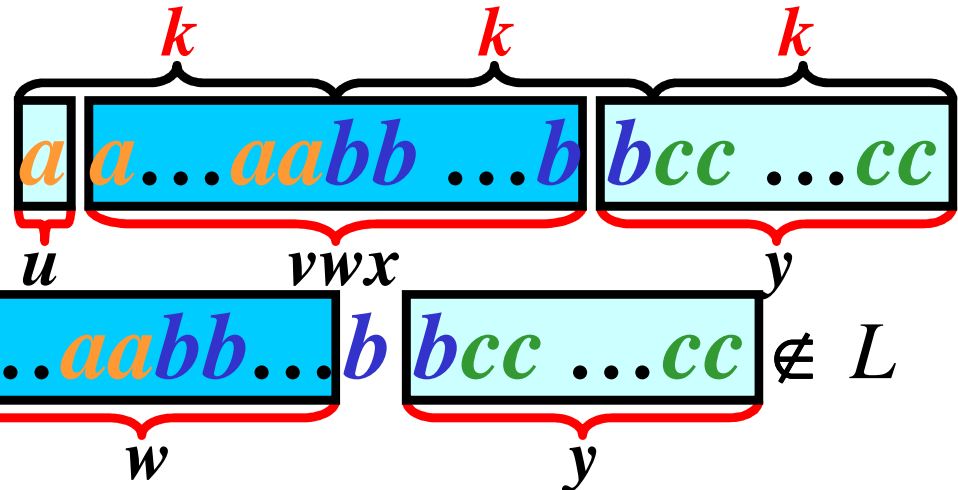
Pumping Lemma: Example 2/2

a) $vwx \in \{a\}^* \{b\}^*$:

- Pumping lemma:

$$uv^0wx^0y \in L$$

- $uv^0wx^0y = uwy = \underbrace{a}_{u} \underbrace{a \dots aabb \dots b}_{w} \underbrace{bcc \dots cc}_{y} \notin L$



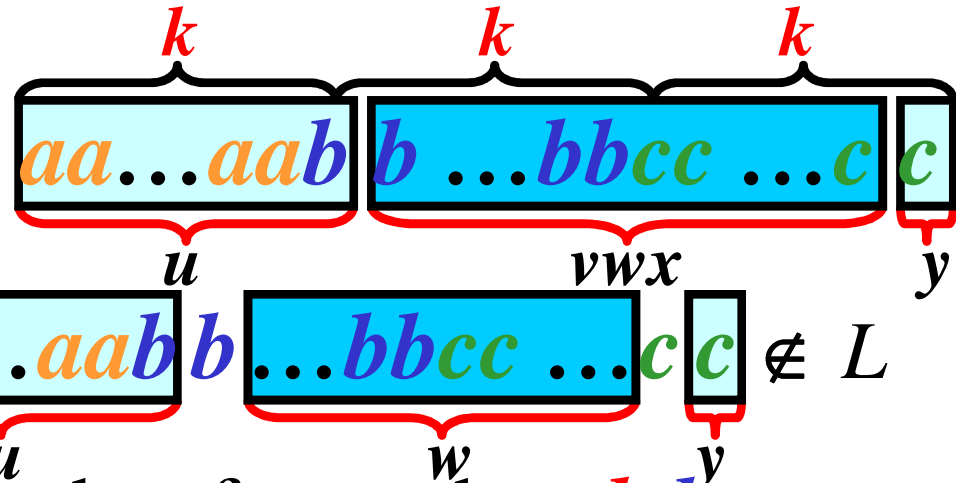
Note: uwy contains k c 's, but fewer than k a 's or b 's.

b) $vwx \in \{b\}^* \{c\}^*$:

- Pumping lemma:

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- $uv^0wx^0y = uwy = \underbrace{aa \dots aab}_{u} \underbrace{b \dots bbcc \dots c}_{w} \underbrace{c}_{y} \notin L$



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All these decompositions lead to a contradiction!

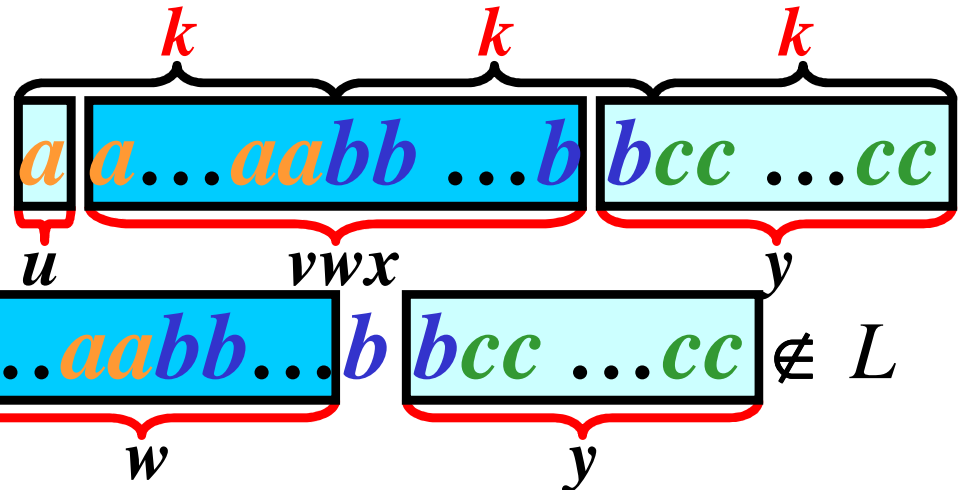
Pumping Lemma: Example 2/2

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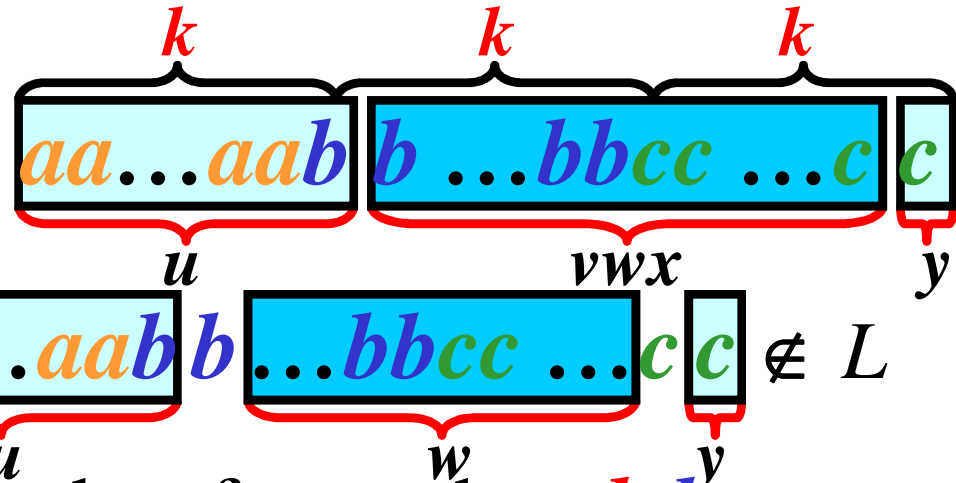
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All these decompositions lead to a contradiction!

4) Therefore, L is not a CFL.

Closure properties of CFL

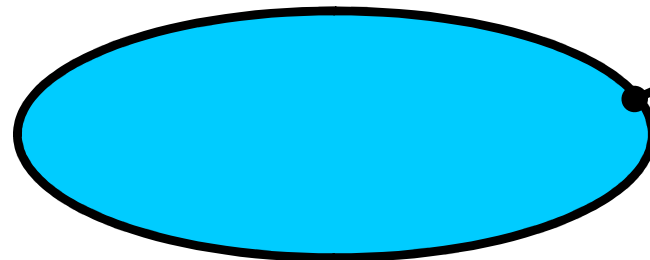
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Illustration:

- The family of CF languages is closed under *union*.
It means:



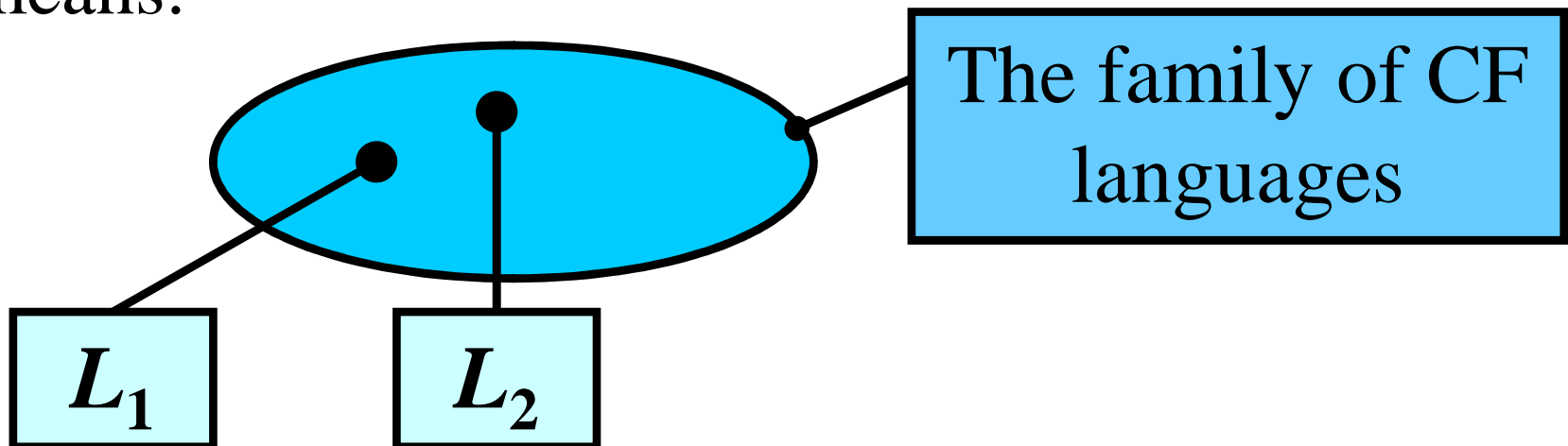
The family of CF
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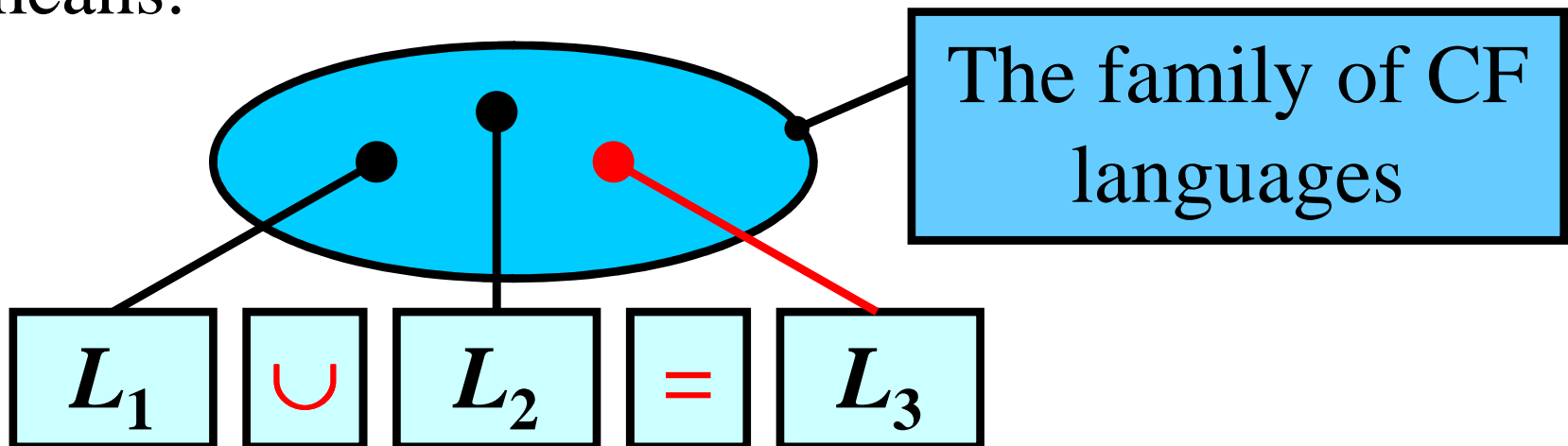


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Algorithm: CFG for Union

- **Input:** Grammars $G_1 = (N_1, T, P_1, S_1)$ and $G_2 = (N_2, T, P_2, S_2)$;
 - **Output:** Grammar $G_u = (N, T, P, S)$ such that $L(G_u) = L(G_1) \cup L(G_2)$
-

- **Method:**

- let $S \notin N_1 \cup N_2$, let $N_1 \cap N_2 = \emptyset$:
 - $N := \{S\} \cup N_1 \cup N_2$;
 - $P := \{S \rightarrow S_1, S \rightarrow S_2\} \cup P_1 \cup P_2$;

Algorithm: CFG for Concatenation

- **Input:** $G_1 = (N_1, T, P_1, S_1)$ and $G_2 = (N_2, T, P_2, S_2)$;
 - **Output:** $G_c = (N, T, P, S)$ such that $L(G_c) = L(G_1) \cdot L(G_2)$
-

- **Method:**

- let $S \notin N_1 \cup N_2$, let $N_1 \cap N_2 = \emptyset$:
 - $N := \{S\} \cup N_1 \cup N_2$;
 - $P := \{S \rightarrow S_1 S_2\} \cup P_1 \cup P_2$;

Algorithm: CFG for Iteration

- **Input:** $G = (N_1, T, P_1, S_1)$
 - **Output:** $G_i = (N, T, P, S)$ such that $L(G_i) = L(G)^*$
-
- **Method:**
 - let $S \notin N_1$:
 - $N := \{S\} \cup N_1$;
 - $P := \{S \rightarrow S_1 S, S \rightarrow \varepsilon\} \cup P_1$;

Closure properties

Theorem: The family of CFLs is closed under **union, concatenation, iteration.**

Proof:

- Let L_1, L_2 be two CFLs.
- Then, there exist two CFGs G_1, G_2 such that $L(G_1) = L_1, L(G_2) = L_2$;
- Construct grammars
 - G_u such that $L(G_u) = L(G_1) \cup L(G_2)$
 - G_c such that $L(G_c) = L(G_1) \cdot L(G_2)$
 - G_i such that $L(G_i) = L(G_1)^*$
 by using the previous three algorithms
- Every CFG denotes CFL, so
- $L_1 L_2, L_1 \cup L_2, L_1^*$ are CFLs.

Intersection: Not Closed

Theorem: The family of CFLs is **not** closed under **intersection**.

Proof:

- The intersection of some CFLs is not a CFL:
- $L_1 = \{a^m b^n c^n : m, n \geq 1\}$ is a CFL
- $L_2 = \{a^n b^n c^m : m, n \geq 1\}$ is a CFL
- $L_1 \cap L_2 = \{a^n b^n c^n : n \geq 1\}$ is not a CFL
(proof based on the pumping lemma) *QED*

Complement: Not Closed

Theorem: The family of CFLs is **not** closed under **complement**.

Proof by contradiction:

- Assume that family of CFLs is closed under complement.
- $L_1 = \{a^m b^n c^n : m, n \geq 1\}$ is a **CFL**
- $L_2 = \{a^n b^n c^m : m, n \geq 1\}$ is a **CFL**
- $\overline{L_1}, \overline{L_2}$ are **CFLs**
- $\overline{L_1} \cup \overline{L_2}$ is a **CFL** (the family of CFLs is closed under union)
- $\overline{\overline{L_1} \cup \overline{L_2}}$ is a **CFL** (assumption)
- DeMorgan's law implies $L_1 \cap L_2 = \{a^n b^n c^n : n \geq 1\}$ is a **CFL**
- $\{a^n b^n c^n : n \geq 1\}$ is not a **CFL** \Rightarrow **Contradiction**

Main Decidable Problems

1. Membership problem:

- **Instance:** CFG G , $w \in \Sigma^*$; **Question:** $w \in L(G)$?

2. Emptiness problem:

- **Instance:** CFG G ; **Question:** $L(G) = \emptyset$?

3. Finiteness problem:

- **Instance:** CFG G ; **Question:** Is $L(G)$ finite?

Algorithm: Membership

- **Input:** CFG $G = (N, T, P, S)$ in Chomsky normal form; $w \in T^+$
 - **Output:** **YES** if $w \in L(G)$
NO if $w \notin L(G)$
-

- **Method I:**

- if $S \Rightarrow^n w$, where $1 \leq n \leq 2|w| - 1$, then write ('**YES**')
else write ('**NO**')

- **Method II:**

- See: The general parsing method based on CNF
-

Summary:

The membership problem for CFLs is decidable

Accessible Symbols

Gist: Symbol X is *accessible* if $S \Rightarrow^* \dots X \dots$,
where S is the start nonterminal.

Definition: Let $G = (N, T, P, S)$ be a CFG. A symbol $X \in N \cup T$ is *accessible* if there exist $u, v \in \Sigma^*$ such that $S \Rightarrow^* uXv$; otherwise, X is *inaccessible*.

Note: Each inaccessible symbol can be removed from CFG

Example:

$G = (\{S, A, B\}, \{a, b\}, \{S \rightarrow SB, S \rightarrow a, A \rightarrow ab, B \rightarrow aB\}, S)$

S - accessible: for $u = \epsilon, v = \epsilon$: $S \Rightarrow^0 S$

A - **inaccessible**: there is no $u, v \in \Sigma^*$ such that $S \Rightarrow^* uAv$

B - accessible: for $u = S, v = \epsilon$: $S \Rightarrow^1 SB$

a - accessible: for $u = \epsilon, v = \epsilon$: $S \Rightarrow^1 a$

b - **inaccessible**: there is no $u, v \in \Sigma^*$ such that $S \Rightarrow^* ubv$

Terminating Symbols

Gist: Symbol X is *terminating* if X derives a terminal string.

Definition: Let $G = (N, T, P, S)$ be a CFG. A symbol $X \in N \cup T$ is *terminating* if there exists $w \in T^*$ such that $X \Rightarrow^* w$; otherwise, X is *nonterminating*

Note: Each nonterminating symbol can be removed from any CFG.

Example:

$G = (\{S, A, B\}, \{a, b\}, \{S \rightarrow SB, S \rightarrow a, A \rightarrow ab, B \rightarrow aB\}, S)$

Symbol S - terminating: for $w = a$: $S \Rightarrow^1 a$

Symbol A - terminating: for $w = ab$: $A \Rightarrow^1 ab$

Symbol B - **nonterminating**: there is no $w \in T^*$ such that $B \Rightarrow^* w$

Symbol a - terminating: for $w = a$: $a \Rightarrow^0 a$

Symbol b - terminating: for $w = b$: $b \Rightarrow^0 b$

Algorithm: Emptiness

- **Input:** CFG $G = (N, T, P, S)$;
 - **Output:** **YES** if $L(G) = \emptyset$
NO if $L(G) \neq \emptyset$
-
- **Method:**
 - **if** S is nonterminating **then** write (**YES**)
else write (**NO**)
-

Summary:

The emptiness problem for CFLs is decidable

Algorithm: Finiteness

- **Input:** CFG $G = (N, T, P, S)$;
 - **Output:** **YES** if $L(G)$ is finite
NO if $L(G)$ is infinite
-
- **Method:**
 - Let $k = 2^{\text{card}(N)}$
 - **if** there exist $z \in L(M)$, $k \leq |z| < 2k$ **then** write (**NO**)
else write (**YES**)
-

Summary:

The finiteness problem for CFLs is decidable

Main Undecidable Problems

1. Equivalence problem:

- **Instance:** CFGs G_1, G_2 ; **Question:** $L(G_1) = L(G_2)$?

2. Ambiguity problem:

- **Instance:** G ; **Question:** Is G ambiguous?

Note:

It is mathematically proved that there exists no algorithm, which solve these problems in finite time.