Closure Properties of Trios under Operations of Regular Deletion

Definitions

Basic Definitions

- Σ^* – free monoid generated by finite set Σ
- $\varepsilon \in \Sigma^*$ – unit
- $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$
- homomorphism $h : \Sigma^* \to \Delta^*$, $h(uv) = h(u)h(v)$, i.e., if $w = a_1 a_2 \dots a_n$, then $h(w) = h(a_1)h(a_2)\dots h(a_n)$
- $\mathcal{L}$ is a family of languages if
  - $\mathcal{L}$ is a set of languages
  - $\mathcal{L}$ contains nonempty language
- FIN, REG, LIN, CF, CS, REC, RE

Definitions

Trios and Full Trios

- A trio is a family of languages closed under
  - ε-free homomorphism $(h(a) \neq \varepsilon \; \forall a \in \Sigma)$
  - inverse homomorphism $(h^{-1}(L) = \{w : h(w) \in L\})$
  - intersection with regular language
- REG, LIN, CF, CS, REC, RE

- A full trio is a trio closed under homomorphism
- REG, LIN, CF, RE

Operation

Random Parallel Deletion

$L, K \subseteq \Sigma^*$ two languages

$$[\perp, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in \Sigma^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L, \\ x_i \in K,\ 1 \leq i \leq n,\ n \geq 1\}$$

Example

- $[\perp, \{abababa\}, \{aba\}] = \{baba, \dots\}$
- $[\perp, \{abababa\}, \{aba\}] = \{baba, abba, \dots\}$
- $[\perp, \{abababa\}, \{aba\}] = \{baba, abba, abab, \dots\}$
- $[\perp, \{abababa\}, \{aba\}] = \{baba, abba, abab, b\}$

Operation

Parallel Deletion

$L, K \subseteq \Sigma^*$ two languages

$$[\perp_a, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in \Sigma^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L,$$
$$x_j \in K, \{u_i\} \cap \Sigma^*(K \setminus \{\varepsilon\})\Sigma^* = \emptyset,$$
$$1 \leq i \leq n+1, 1 \leq j \leq n, n \geq 1\}$$

Example

- $[\perp_a, \{aababaaa\}, \{aba\}] = \{aba, \dots\}$.
- $[\perp_a, \{aababaaa\}, \{aba\}] = \{aba, aabbaa\}$.

Operation

Sequential Deletion

$L, K \subseteq \Sigma^*$ two languages

$$[\perp_1, L, K] = \{u_1 u_2 \in \Sigma^* : u_1 x u_2 \in L,\ x \in K\}.$$

- $[\perp_1, \{ababaaba\}, \{aba\}] = \{baba, \dots\}$.
- $[\perp_1, \{abababa\}, \{aba\}] = \{baba, abba, \dots\}$.
- $[\perp_1, \{ababaaba\}, \{aba\}] = \{baba, abba, abab\}$.
- $[\perp_1, \{aba\}, \{aba\}] = \{\varepsilon\}$.
- $[\perp_1, \{ab\}, \{aba\}] = \emptyset$.

Operation

Scattered Sequential Deletion

$L, K \subseteq \Sigma^*$ two languages

$$[\perp_{1s}, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in \Sigma^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L, \\ x_1 x_2 \dots x_n \in K,\ n \geq 1\}.$$

- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, \dots\}$.
- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, adbe\}$.

Operation

Multiple Scattered Sequential Deletion

$L, K \subseteq \Sigma^*$ two languages

$$[\perp_s, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in \Sigma^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L, \\ x_1 x_2 \dots x_n \in K^+, n \geq 1\}.$$

- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, \dots\}$.
- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, adbe, \dots\}$.
- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, adbe, de\}$.

Notation

$\mathcal{X}$, $\mathcal{Y}$ families of languages

$$\langle x, \mathcal{X}, \mathcal{Y} \rangle = \{[x, L, K] : L \in \mathcal{X}, K \in \mathcal{Y}\}$$

$$x \in \{\perp, \perp_a, \perp_1, \perp_{1s}, \perp_s\}$$

Regular deletion signifies $\mathcal{Y} = REG$.

Auxiliary Lemma

Lemma

Let $\mathcal{C}$ be a family of languages. Then,

$$\mathcal{C} \subseteq \langle x, \mathcal{C}, REG \rangle$$

$x \in \{\perp_1, \perp, \perp_a, \perp_{1s}, \perp_s\}$.

Proof.

Let $L \in \mathcal{C}$. Then, $L = [x, L, \{\varepsilon\}] \in \langle x, \mathcal{C}, REG \rangle$. □

Full Trios

First Main Result

Theorem

Let $\mathcal{T}$ be a full trio. Then,

$$\langle x, \mathcal{T}, REG \rangle = \mathcal{T}$$

$x \in \{\perp_1, \perp, \perp_a, \perp_{1s}, \perp_s\}$.

Full Trios

Proof (Sequential Deletion)

Theorem

Let $\mathcal{T}$ be a full trio. Then, $\langle \perp_1, \mathcal{T}, REG \rangle = \mathcal{T}$.

Proof.

- $[\perp_1, L, R] \in \langle \perp_1, \mathcal{T}, REG \rangle$, $L \subseteq \Sigma^*$, $R \subseteq \Delta^*$, $\Delta \subseteq \Sigma$
- $[\perp_1, L, R] = \{uv \in \Sigma^* : uxv \in L,\ x \in R\}$
- We want $L' \in \mathcal{T}$ such that $L' = [\perp_1, L, R]$

Full Trios

Proof (Sequential Deletion)

Theorem

Let $\mathcal{T}$ be a full trio. Then, $\langle \perp_1, \mathcal{T}, REG \rangle = \mathcal{T}$.

Proof.

- $[\perp_1, L, R] = \{uv \in \Sigma^* : uxv \in L, x \in R\}$
- $h : (\Delta' \cup \Sigma)^* \to \Sigma^* \qquad \Delta' = \{a' : a \in \Delta\} \qquad \Delta' \cap \Sigma = \emptyset$
  $h(a') = h(a) = a \qquad a' \in \Delta', a \in \Sigma$
- $h(ux'v) = uxv \qquad (x' = a'_1 \dots a'_n \text{ if } x = a_1 \dots a_n)$
- $ux'v \in h^{-1}(L) \cap \Sigma^* R' \Sigma^* \qquad x' \in R' \subseteq \Delta'^*$

Full Trios

Proof (Sequential Deletion)

Theorem

Let $\mathcal{T}$ be a full trio. Then, $\langle \perp_1, \mathcal{T}, REG \rangle = \mathcal{T}$.

Proof.

- $ux'v \in h^{-1}(L) \cap \Sigma^* R' \Sigma^* \qquad x' \in R' \subseteq \Delta'^*, \Delta' \cap \Sigma = \emptyset$
- $g : (\Delta' \cup \Sigma)^* \to \Sigma^*$
  $g(a') = \varepsilon \qquad a' \in \Delta'$
  $g(a) = a \qquad a \in \Sigma$
- $g(ux'v) = uv \qquad uv \in g(h^{-1}(L) \cap \Sigma^* R' \Sigma^*)$

Full Trios

Corollary

Theorem

Let $\mathcal{T}$ be a full trio. Then, $\langle x, \mathcal{T}, REG\rangle = \mathcal{T}$, $x \in \{\perp_1, \perp, \perp_a, \perp_{1s}, \perp_s\}$.

Proof.

$$[\perp_1, L, R] = g(h^{-1}(L) \cap \Sigma^* R' \Sigma^*)$$

The proofs for the other cases are analogous. □

Corollary

REG, LIN, CF and RE are closed under all these operations.

Full Trios

Corollary

Theorem

Let $\mathcal{T}$ be a full trio. Then, $\langle x, \mathcal{T}, REG\rangle = \mathcal{T}$, $x \in \{\perp_1, \perp, \perp_a, \perp_{1s}, \perp_s\}$.

Proof.

$$[\perp_1, L, R] = g(h^{-1}(L) \cap \Sigma^* R' \Sigma^*)$$

The proofs for the other cases are analogous. □

Corollary

REG, LIN, CF and RE are closed under all these operations.

Full Trios

Only Sufficient Condition for Deletion Closure

Example

The reverse implication does not hold:

$$\langle x, FIN, FIN \rangle = FIN$$

$x \in \{\perp_1, \perp, \perp_a, \perp_{1s}, \perp_s\}$, and FIN is not trio.

Non-full Trios

Second Main Result

Theorem

Let $\mathcal{C}$ be a non-full trio. Then,

$$\mathcal{C} \subset \langle x, \mathcal{C}, REG \rangle$$

$x \in \{\perp, \perp_a, \perp_{1s}, \perp_s\}$.

Non-full Trios

Proof (Parallel Deletion)

Theorem

Let $\mathcal{C}$ be a non-full trio. Then, $\mathcal{C} \subset \langle \perp, \mathcal{C}, REG \rangle$.

Proof.

- Let $L \in \mathcal{C}$, $L \subseteq \Sigma^*$
- $h : \Sigma^* \to \Delta^*$, $\Sigma \cap \Delta = \emptyset$, such that $h(L) \notin \mathcal{C}$
- $A = \{a \in \Sigma : h(a) = \varepsilon\}$
- $g : \Sigma^* \to (\Delta \cup A)^*$
  $g(a) = h(a) \quad a \in \Sigma \setminus A$
  $g(a) = a \quad a \in A$
- $g(L) \in \mathcal{C}$

Non-full Trios

Proof (Parallel Deletion)

Theorem

Let $\mathcal{C}$ be a non-full trio. Then, $\mathcal{C} \subset \langle \perp, \mathcal{C}, REG \rangle$.

Proof.

- $h(v) \in h(L)$ if and only if
- $g(v) = g(u_1)x_1 \dots g(u_n)x_n g(u_{n+1}) \in g(L)$, $g(u_i) \in \Delta^*$,
  $h(v) = g(u_1) \dots g(u_n)g(u_{n+1})$ if and only if
- $h(v) \in [\perp, g(L), A^*] \cap \Delta^*$
- $v = u_1 x_1 \dots u_n x_n u_{n+1} \in L$, $n \geq 1$, $u_i \in (\Sigma \setminus A)^*$, $x_j \in A^*$

Non-full Trios

Proof (Parallel Deletion)

Theorem

Let $\mathcal{C}$ be a non-full trio. Then, $\mathcal{C} \subset \langle \perp, \mathcal{C}, REG \rangle$.

Proof.

- $h(L) = [\perp, g(L), A^*] \cap \Delta^*$
- $[\perp, g(L), A^*] \in \mathcal{C}$ implies $h(L) = [\perp, g(L), A^*] \cap \Delta^* \in \mathcal{C}$ – contradiction
- $[\perp, g(L), A^*] \notin \mathcal{C}$

Non-full Trios

Corollary

Theorem

Let $\mathcal{C}$ be a non-full trio. Then,

$$\mathcal{C} \subset \langle x, \mathcal{C}, REG \rangle$$

$x \in \{\perp, \perp_a, \perp_{1s}, \perp_s\}$.

Corollary

CS and REC are not closed under these operations.

Non-full Trios

Sequential Deletion

Problem

1. CS ? $\langle \perp_1, CS, REG \rangle$
2. REC ? $\langle \perp_1, REC, REG \rangle$

Solution

1. $CS \subset \langle \perp_1, CS, REG \rangle$
2. $REC \subset \langle \perp_1, REC, REG \rangle$

Theorem

There is a non-full trio $\mathcal{C}$ such that $\mathcal{C} \subset \langle \perp_1, \mathcal{C}, REG \rangle$.

Non-full Trios

Sequential Deletion

Problem

1. $CS \; ? \; \langle \perp_1, CS, REG \rangle$
2. $REC \; ? \; \langle \perp_1, REC, REG \rangle$

Solution

1. $CS \subset \langle \perp_1, CS, REG \rangle$
2. $REC \subset \langle \perp_1, REC, REG \rangle$

Theorem

There is a non-full trio $\mathcal{C}$ such that $\mathcal{C} \subset \langle \perp_1, \mathcal{C}, REG \rangle$.

Non-full Trios

Sequential Deletion

Theorem

$\langle \perp_1, CS, REG \rangle = RE$.

Proof.

- $L \in RE$, $L \subseteq \Sigma^*$, $a, b, c \notin \Sigma$
- There is $L' \in CS$ such that $L' \subseteq Lba^*c$
  - $\alpha \to \beta$ if $|\beta| \geq |\alpha|$
  - $\alpha \to \beta X^{|\alpha|-|\beta|}$ if $|\beta| < |\alpha|$
  - $S' \to Sbc$
  - $X\alpha \to \alpha X$ if $\alpha \in V \cup \{b\}$
  - $bX \to ba$
- $L = [\perp_1, L', ba^*c] \in \langle \perp_1, CS, REG \rangle$

Summary

- Every full trio is closed under all these operations.
- Except for sequential deletion, any non-full trio is not closed under these operations.

- Open Problem
  - Is there a non-full trio $\mathcal{C}$ such that $\mathcal{C} = \langle \perp_1, \mathcal{C}, \mathit{REG} \rangle$?