Self-Regulating Finite Automata

PRLG vs. first-SFA

n-Parallel Right-Linear Grammar (n-PRLG)

Definition

Definition

An n-PRLG, $n > 0$, is an $(n+3)$-tuple

$$G = (N_1, \ldots, N_n, T, S, P),$$

where

- N_i are mutually disjoint nonterminal alphabets, $1 \leq i \leq n$,
- T is a terminal alphabet,
- $S \notin N_1 \cup \cdots \cup N_n$,
- P contains three kinds of rules:
  1. $S \to X_1 \ldots X_n$ $\quad X_i \in N_i$, $1 \leq i \leq n$,
  2. $X \to wY$ $\quad X, Y \in N_i$ for some i, $1 \leq i \leq n$, $w \in T^*$, and
  3. $X \to w$ $\quad X \in N$, $w \in T^*$.

n-Parallel Right-Linear Grammar (n-PRLG)

Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $S \to y \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\downarrow \quad \downarrow \dots \quad \downarrow$
   $y = y_1 x_1 y_2 x_2 \dots y_n x_n$
   $y_i \in T^*$, $x_i \in T^* N \cup T^*$, $X_i \in N_i$, $X_i \to x_i \in P$, $1 \leq i \leq n$.

$\mathcal{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

$R_n = \{\mathcal{L}(G) : G \text{ is an } n\text{-PRLG}\}$.

n-Parallel Right-Linear Grammar (n-PRLG)

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if and only if

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   $\downarrow \quad \downarrow \ldots \quad \downarrow$

   $y = y_1\, x_1\, y_2\, x_2 \ldots y_n\, x_n$

   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $X_i \rightarrow x_i \in P$, $1 \leq i \leq n$.

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   $y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n$
   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $X_i \to x_i \in P$, $1 \leq i \leq n$.

$\mathcal{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

$R_n = \{\mathcal{L}(G) : G \text{ is an } n\text{-PRLG}\}$.

n-PRLG

Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S$$

n-PRLG

Example

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Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB$$

n-PRLG

Example

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- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB$$

n-PRLG

Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aaAbbB$$

n-PRLG

Example

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- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aaAbbB \Rightarrow a^3b^3$$

n-PRLG

Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$\mathcal{L}(G) = \{a^n b^n : n \geq 1\}$$

Finite Automaton

Definition

Definition

A finite automaton is a 5-tuple, $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set of states,
- Σ is a finite input alphabet,
- δ is a finite set of rules,
- $q_0 \in Q$ is an initial state,
- $F \subseteq Q$ is a set of final states.

Let $\psi : \delta \to \Psi$ be a bijection (Ψ is a set of rule labels).

$$r.qw \to p \text{ means } \psi(qw \to p) = r$$

$$qwy \Rightarrow py\,[r] \text{ if } qwy \in Q\Sigma^*,\ r.qw \to p \in \delta$$

$$\mathcal{L}(M) = \{w \in \Sigma^* : q_0 w \Rightarrow^* f,\ f \in F\}$$

n-turn First-Move Self-Regulating Finite Automaton

Definition

Definition

An n-first-SFA, $n \geq 0$, M, is a 7-tuple

$$M = (Q, \Sigma, \delta, q_0, q_t, F, R),$$

where

- $(Q, \Sigma, \delta, q_0, F)$ is a finite automaton,
- $q_t \in Q$ is a turn state,
- $R \subseteq \Psi \times \Psi$ is a finite relation on the alphabet of rule labels.

n-turn First-Move Self-Regulating Finite Automaton

Definition

Definition

M accepts w if there is $q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, such that

$$\mu = \underbrace{r_1^0 \ldots r_k^0}_{k \text{ rules}} \underbrace{r_1^1 \ldots r_k^1}_{k \text{ rules}} \ldots \underbrace{r_1^n \ldots r_k^n}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, r_k^0 is the first rule of the form $qx \rightarrow q_t$, for some $q \in Q$, $x \in \Sigma^*$, and

$$(r_1^j, r_1^{j+1}) \in R$$

for all $0 \leq j < n$.

$$\mathcal{L}(M) = \{w \in \Sigma^* : q_0 w \Rightarrow^* f,\ f \in F\}.$$

The family of languages accepted by n-first-SFAs is denoted W_n.

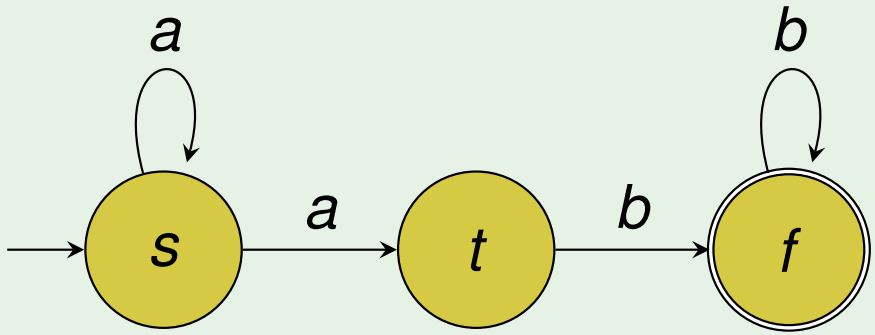
Example

Consider a 1-first-SFA

$$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 3)\})$$

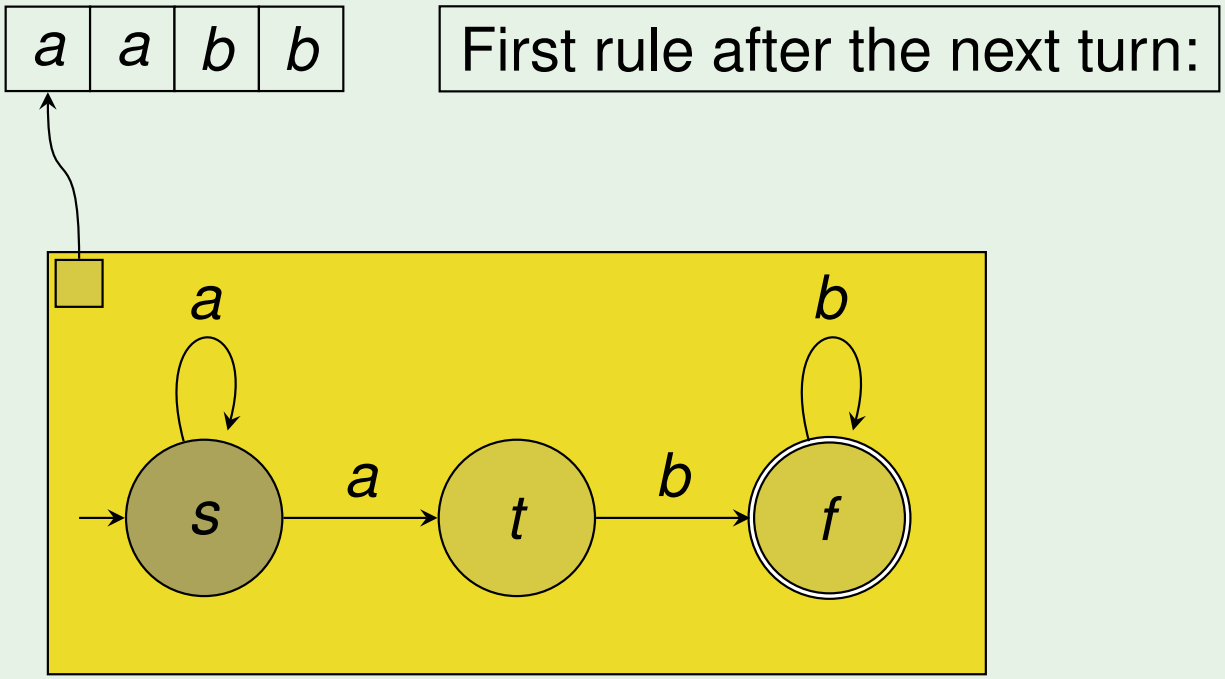
with δ containing rules

- $1.sa \to s$
- $2.sa \to t$
- $3.tb \to f$
- $4.fb \to f$



Example

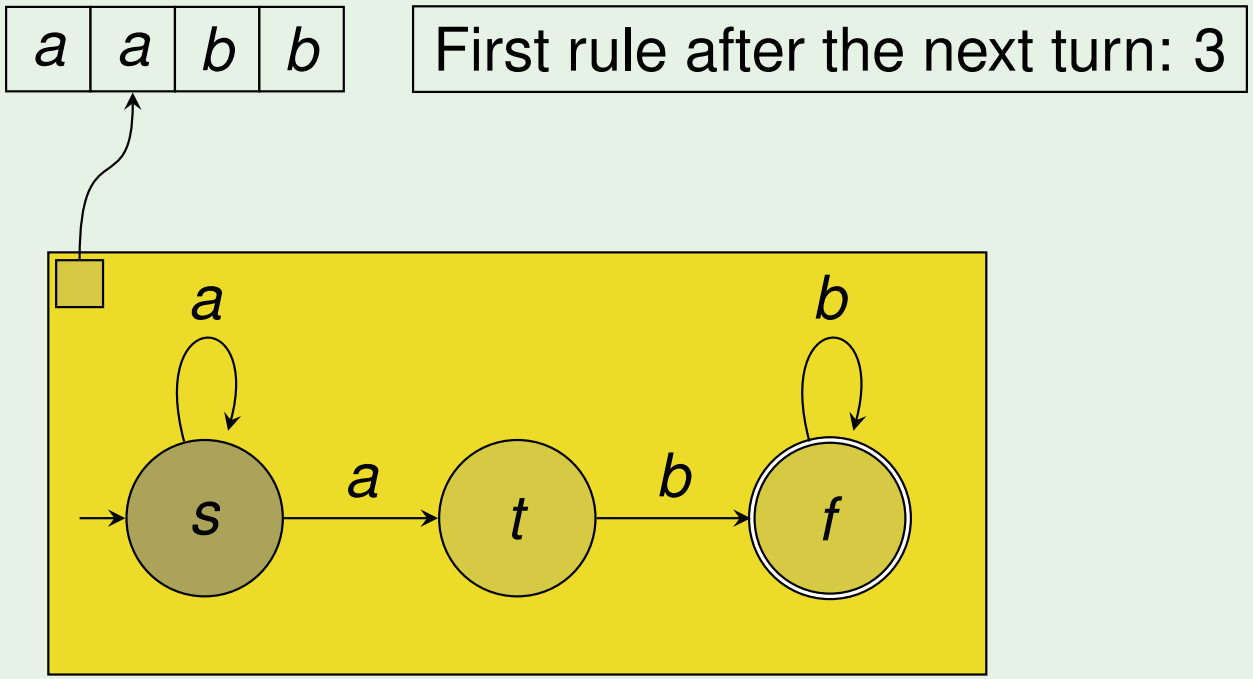
$M = (\{s, t, f\}, \{a, b\}, \{1, 2, 3, 4\}, s, t, \{f\}, \{(1, 3)\})$



saabb

Example

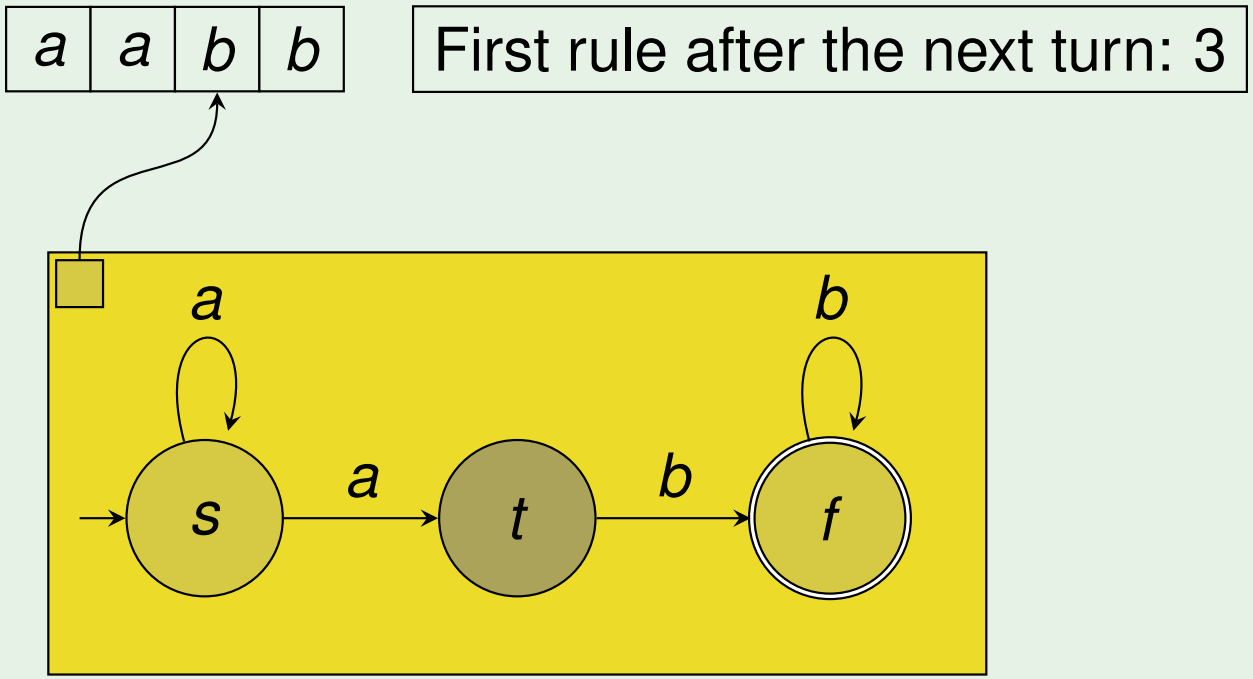
$M = (\{s, t, f\}, \{a, b\}, \{1, 2, 3, 4\}, s, t, \{f\}, \{(1, 3)\})$



$saabb \Rightarrow sabb$ [1]

Example

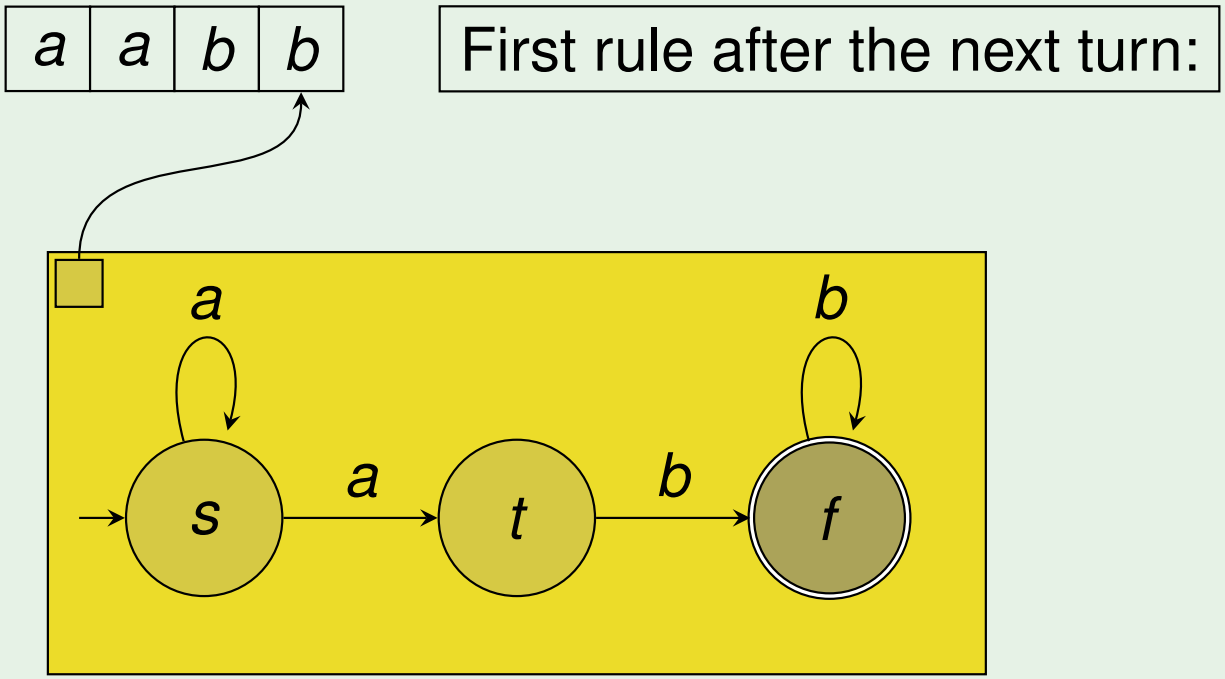
$M = (\{s, t, f\}, \{a, b\}, \{1, 2, 3, 4\}, s, t, \{f\}, \{(1, 3)\})$



$saabb \Rightarrow sabb\ [1] \Rightarrow tbb\ [2]$

Example

$M = (\{s, t, f\}, \{a, b\}, \{1, 2, 3, 4\}, s, t, \{f\}, \{(1, 3)\})$



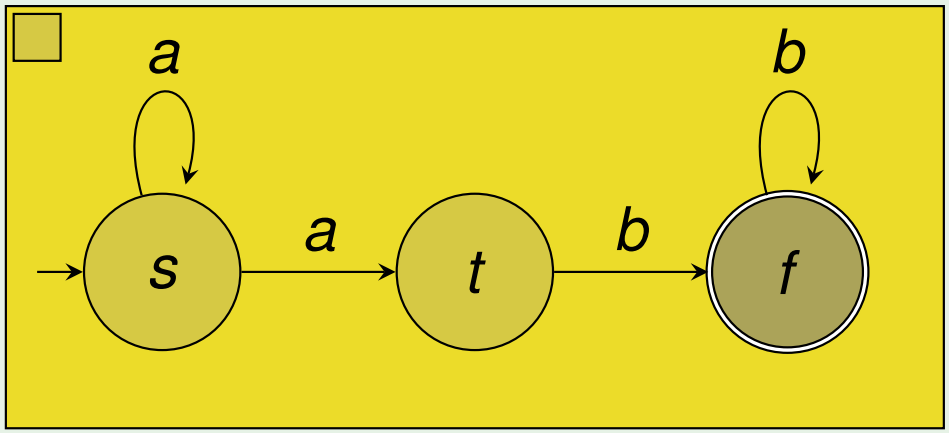
$saabb \Rightarrow sabb\ [1] \Rightarrow tbb\ [2] \Rightarrow fb\ [3]$

Example

$M = (\{s, t, f\}, \{a, b\}, \{1, 2, 3, 4\}, s, t, \{f\}, \{(1, 3)\})$

a	a	b	b

First rule after the next turn:



$saabb \Rightarrow sabb$ [1] $\Rightarrow tbb$ [2] $\Rightarrow fb$ [3] $\Rightarrow f$ [4]

Example

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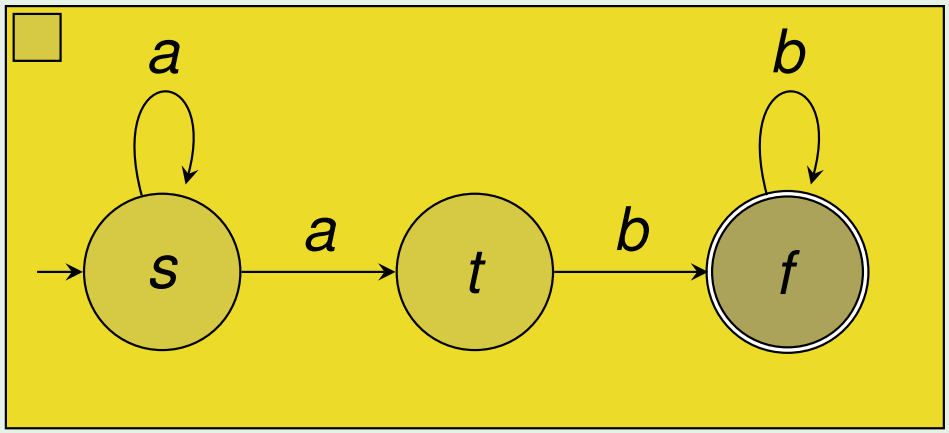
First rule after the next turn:



$$saabb \Rightarrow sabb\ [1] \Rightarrow tbb\ [2] \Rightarrow fb\ [3] \Rightarrow f\ [4]$$

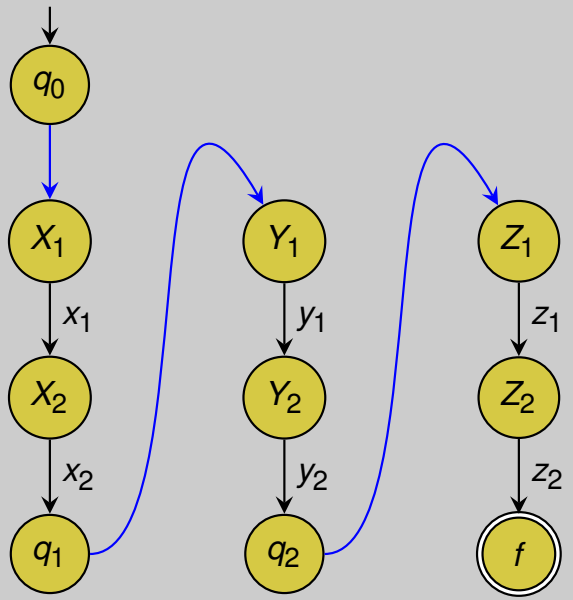
$\mathcal{L}(M) = \{a^n b^n : n \geq 1\} \in CF - REG.$

Lemma

Let G be a 3-PRLG. There is a 2-first-SFA M such that $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof idea.

$$S \Downarrow X_1 Y_1 Z_1 \Downarrow x_1 X_2 y_1 Y_2 z_1 Z_2 \Downarrow x_1 x_2 y_1 y_2 z_1 z_2$$



Problem

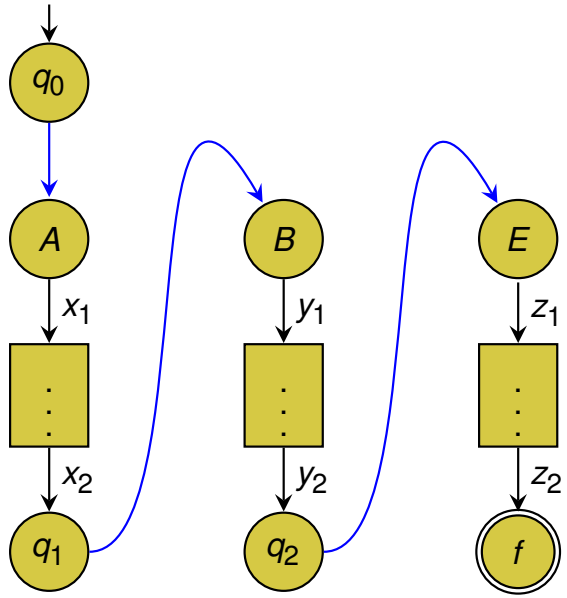
Grammar

$S \rightarrow ABC \mid DBE \in P$ and $S \rightarrow ABE \notin P$. Then M has transitions

- $1.q_0 \rightarrow A$
- $2.q_1 \rightarrow B$
- $3.q_2 \rightarrow C$
- $4.q_0 \rightarrow D$
- $5.q_2 \rightarrow E$

and the relation

$R = \{(1,2),(2,3),(4,2),(2,5),\dots\}$



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- $4.q_0 \rightarrow D$
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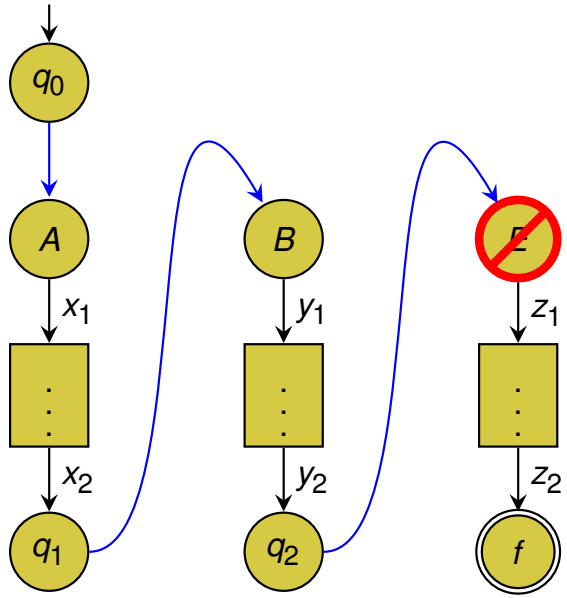


Problem – solution

Lemma

Let G be an n-PRLG. There is an equivalent n-PRLG G' such that

1. *if $S \to X_1 \dots X_n$, then X_i does not occur on the right-hand side of any rule, $1 \leq i \leq n$;*
2. *if $S \to \alpha$, $S \to \beta$ and $\alpha \neq \beta$, then $alph(\alpha) \cap alph(\beta) = \emptyset$.*

Proof.

$S \to ABC, S \to DBE \in P$ *and* $alph(ABC) \cap alph(DBE) = \{B\}$.
Replace them by

$$S \to A'B'C', S \to D''B''E'',$$

$$A' \to A, B' \to B, C' \to C, D'' \to D, B'' \to B, E'' \to E.$$

□

No Problem

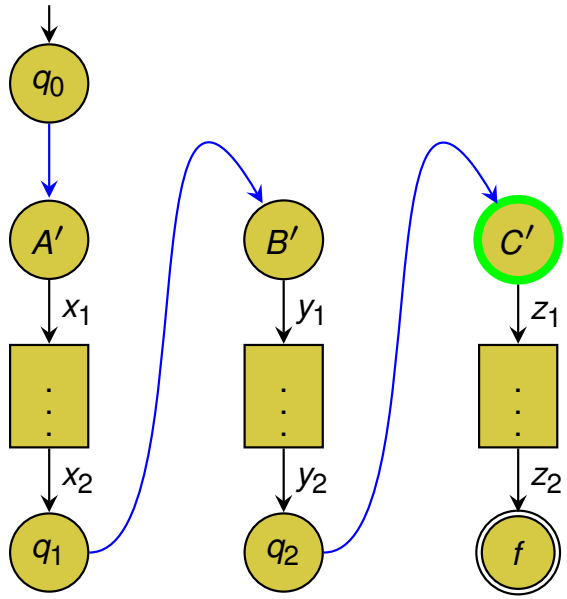
Grammar

$S \longrightarrow A'B'C' \mid D''B''E'' \in P$. Then M has transitions

- $1.q_0 \longrightarrow A'$
- $2.q_1 \longrightarrow B'$
- $3.q_2 \longrightarrow C'$
- $4.q_0 \longrightarrow D''$
- $5.q_1 \longrightarrow B''$
- $6.q_2 \longrightarrow E''$

and the relation

$R = \{(1,2),(2,3),(4,5),(5,6),\ldots\}$



Lemma

Let M be an 2-first-SFA. There is an 3-PRLG G s. t. $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof idea.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
$(q_ty_0 \to r_1), (r_1y_1 \to r_2), (r_2y_2 \to q_i),$
$(q_iz_0 \to s_1), (s_1z_1 \to s_2), (s_2z_2 \to q_f).$

be an acceptance of $x_0x_1x_2y_0y_1y_2z_0z_1z_2z_2$ in M. Then,

$$
\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
&\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\
&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
&\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\
&\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2
\end{aligned}
$$

□

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Let M be an 2-first-SFA. There is an 3-PRLG G s. t. $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof idea.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
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\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
&\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\
&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
&\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\
&\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2
\end{aligned}
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Lemma

Let M be an 2-first-SFA. There is an 3-PRLG G s. t. $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof idea.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
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be an acceptance of $x_0x_1x_2y_0y_1y_2z_0z_1z_2z_2$ in M. Then,

$$
\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
&\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\
&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
&\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\
&\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2
\end{aligned}
$$

□

Lemma

Let M be an 2-first-SFA. There is an 3-PRLG G s. t. $\mathcal{L}(G) = \mathcal{L}(M)$.

Proof idea.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
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\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
&\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\
&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
&\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\
&\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2
\end{aligned}
$$

□

PRLG vs. first-SFA

Equivalence

Lemma

Let G be an n-PRLG. There is an $(n-1)$-first-SFA M such that $\mathcal{L}(G) = \mathcal{L}(M)$.

Lemma

Let M be an n-first-SFA. There is an $(n+1)$-PRLG G such that $\mathcal{L}(G) = \mathcal{L}(M)$.

Theorem

For all $n \geq 0$, $W_n = R_{n+1}$.

Corollary

1. $REG = W_0 \subset W_1 \subset W_2 \subset \cdots \subset CS$.
2. $W_1 \subset CF$.
3. $W_2 \not\subseteq CF$.
4. $CF \not\subseteq W_n$ *for any* $n \geq 0$.
5. *For all* $n \geq 0$, W_n *is closed under union, finite substitution, homomorphism, intersection with a regular language and right quotient with a regular language.*
6. *For all* $n \geq 1$, W_n *is not closed under intersection,complement and inverse homomorphism.*

RLSMG vs. all-SFA

n-Right-Linear Simple Matrix Grammar (n-RLSMG)

Definition

Definition

An n-RLSMG, is an $(n+3)$-tuple

$$G = (N_1, \ldots, N_n, T, S, P),$$

where

- N_i are mutually disjoint nonterminal alphabets, $1 \leq i \leq n$,
- T is a terminal alphabet,
- $S \notin N_1 \cup \cdots \cup N_n$,
- P contains three kinds of matrix rules:
  1. $[S \to X_1 \ldots X_n]$ $\quad X_i \in N_i,\ 1 \leq i \leq n,$
  2. $[X_1 \to w_1 Y_1, \ldots, X_n \to w_n Y_n]$ $\quad w_i \in T^*,\ X_i, Y_i \in N_i,\ 1 \leq i \leq n,$
  3. $[X_1 \to w_1, \ldots, X_n \to w_n]$ $\quad X_i \in N_i,\ w_i \in T^*,\ 1 \leq i \leq n.$

n-Right-Linear Simple Matrix Grammar (n-RLSMG)

Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $[S \to y] \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\downarrow \quad \downarrow \dots \quad \downarrow$
   $y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n$
   $y_i \in T^*$, $x_i \in T^* N \cup T^*$, $X_i \in N_i$, $1 \leq i \leq n$,
   $[X_1 \to x_1, \dots, X_n \to x_n] \in P$.

$\mathcal{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ is defined as usual.

$R_{[n]} = \{\mathcal{L}(G) : G \text{ is an } n\text{-RLSMG}\}$.

n-Right-Linear Simple Matrix Grammar (n-RLSMG)

Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $[S \to y] \in P$,
2. $$\begin{array}{l} x = y_1 X_1 y_2 X_2 \dots y_n X_n \\ \qquad\ \ \downarrow \qquad \downarrow \ \dots \quad \downarrow \\ y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n \end{array}$$

   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $1 \leq i \leq n$,
   $[X_1 \to x_1, \dots, X_n \to x_n] \in P$.

$\mathcal{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ is defined as usual.

$R_{[n]} = \{\mathcal{L}(G) : G$ is an n-RLSMG$\}$.

n-Right-Linear Simple Matrix Grammar (n-RLSMG)

Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $[S \to y] \in P$,
2. $$\begin{array}{l} x = y_1 X_1 y_2 X_2 \ldots y_n X_n \\ \quad\;\;\; \downarrow \quad\;\; \downarrow \ldots \;\; \downarrow \\ y = y_1\, x_1\, y_2\, x_2 \ldots y_n\, x_n \end{array}$$
   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $1 \leq i \leq n$,
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n-RLSMG

Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $[S \to AB]$
- $[A \to aA, B \to aB]$
- $[A \to bA, B \to bB]$
- $[A \to \varepsilon, B \to \varepsilon]$

Consider a derivation in G:

$$S$$

n-RLSMG

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$$S \Rightarrow AB$$

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Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAaB$$

n-RLSMG

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Consider a derivation in G:

$$\mathcal{L}(G) = \{ww : w \in \{a, b\}^*\}$$

n-turn All-Move Self-Regulating Finite Automaton

Definition

Definition

An n-all-SFA, $n \geq 0$, M, is a 7-tuple

$$M = (Q, \Sigma, \delta, q_0, q_t, F, R),$$

M is an n-first-SFA, and M accepts w if there is $q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, s.t.

$$\mu = \underbrace{r_1^0 \dots r_k^0}_{k \text{ rules}} \underbrace{r_1^1 \dots r_k^1}_{k \text{ rules}} \dots \underbrace{r_1^n \dots r_k^n}_{k \text{ rules}},$$

and

$$(r_i^j, r_i^{j+1}) \in R$$

for all $1 \leq i \leq k,\ 0 \leq j < n$.
The family of languages accepted by n-all-SFAs is denoted S_n.

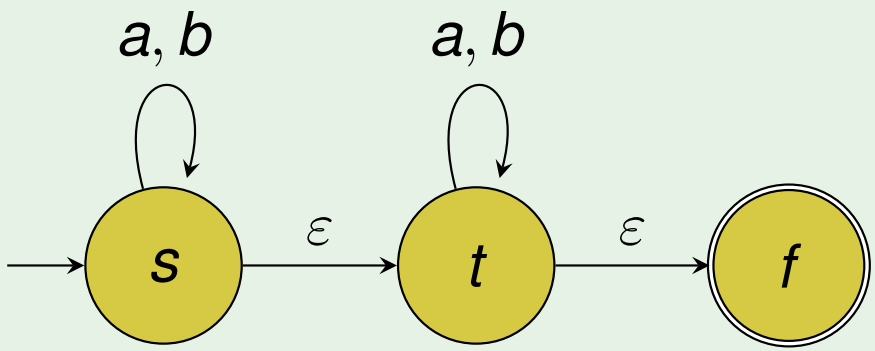
Example

Consider a 1-all-SFA

$$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$$

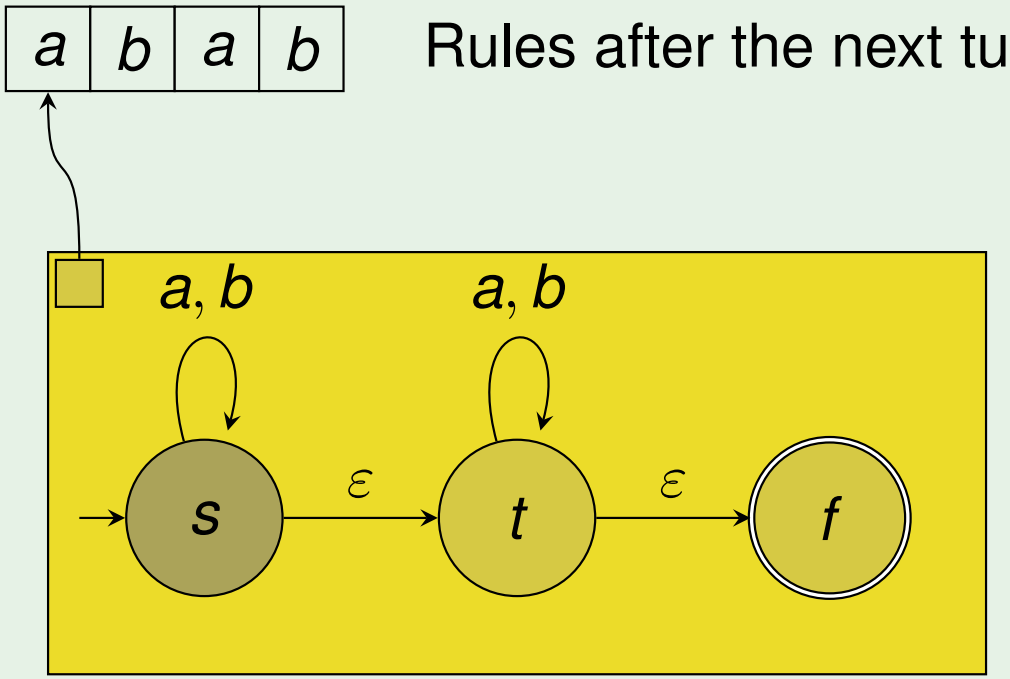
with δ containing rules

- $1.sa \rightarrow s$
- $2.sb \rightarrow s$
- $3.s \rightarrow t$
- $4.ta \rightarrow t$
- $5.tb \rightarrow t$
- $6.t \rightarrow f$



Example

$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$

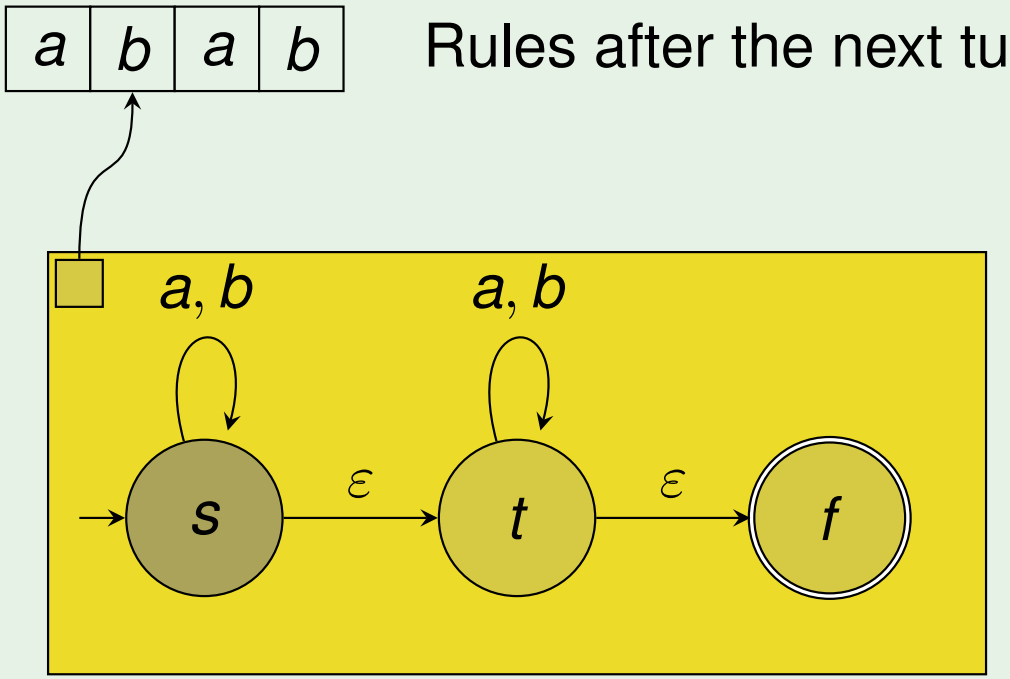


Rules after the next turn:

sabab

Example

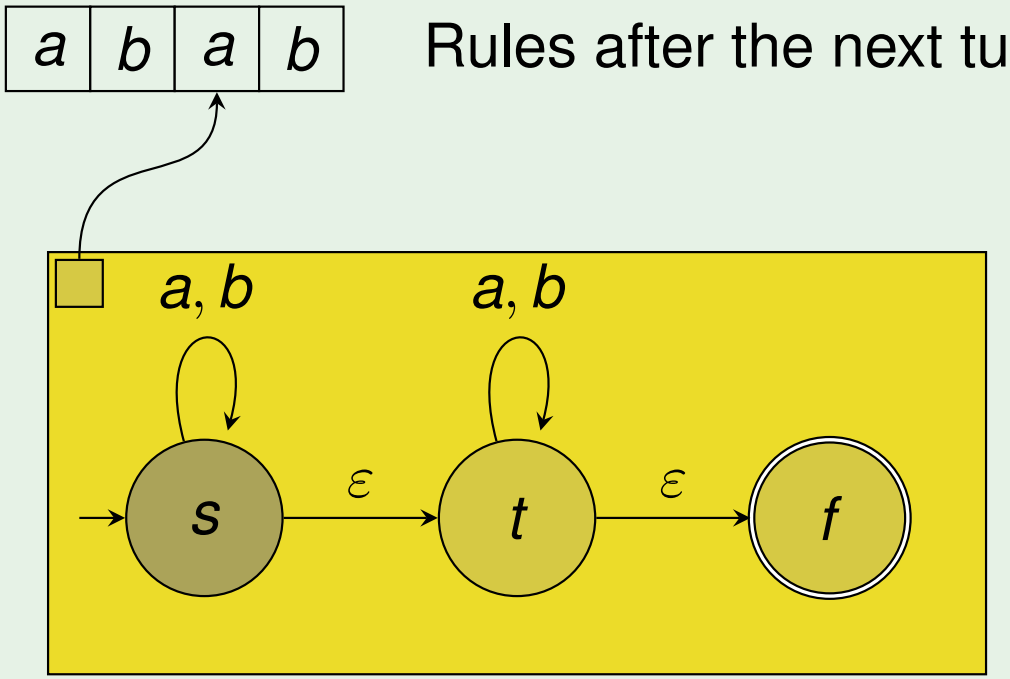
$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$



$sabab \Rightarrow sbab$ [1]

Example

$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$



$sabab \Rightarrow sbab$ [1] $\Rightarrow$ sab [2]

Example

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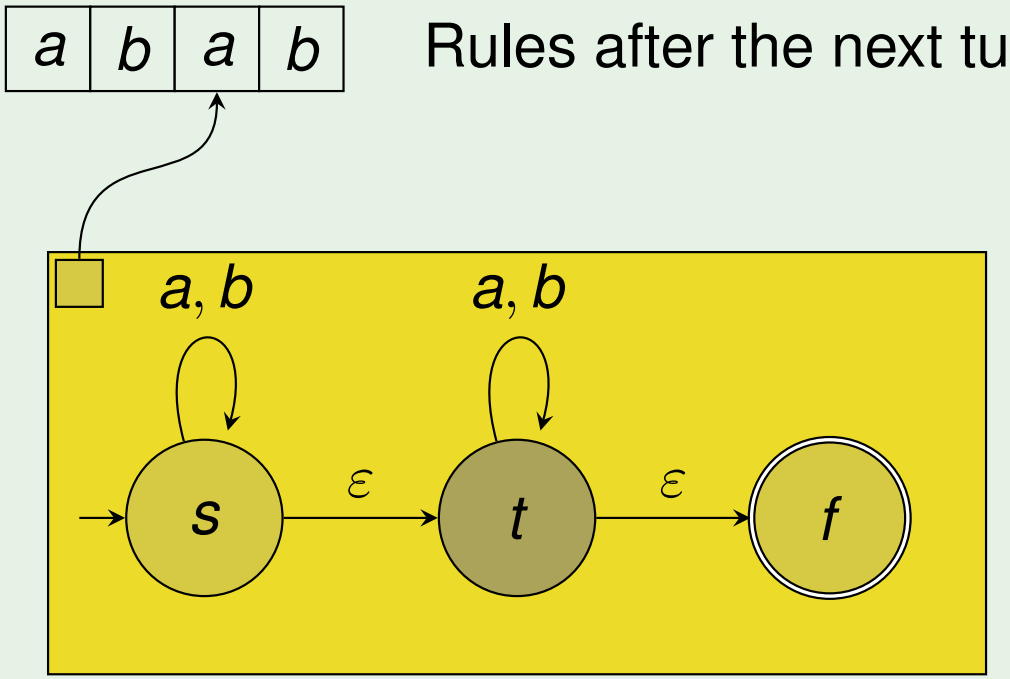
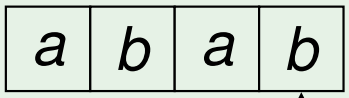


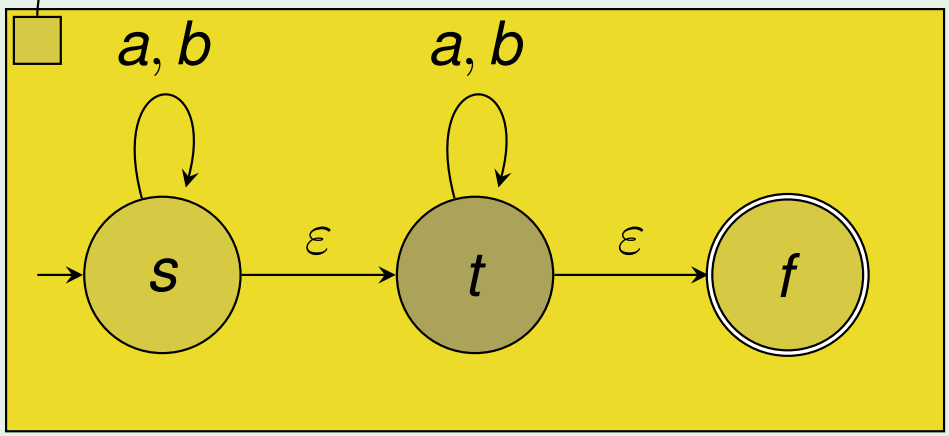
$sabab \Rightarrow sbab$ [1] $\Rightarrow sab$ [2] $\Rightarrow tab$ [3]

Example

$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$



Rules after the next turn: 4,5,6



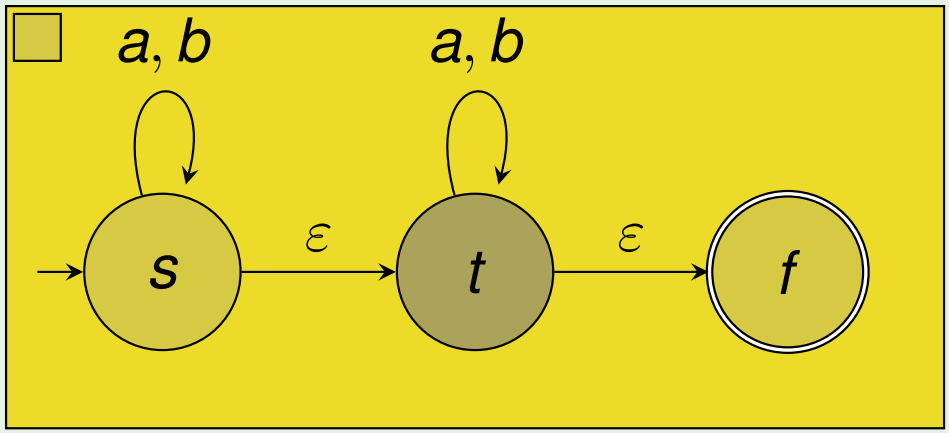
$sabab \Rightarrow sbab$ [1] $\Rightarrow sab$ [2] $\Rightarrow tab$ [3] $\Rightarrow tb$ [4]

Example

$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$

a	b	a	b

Rules after the next turn: 4,5,6



$sabab \Rightarrow sbab$ [1] $\Rightarrow sab$ [2] $\Rightarrow tab$ [3] $\Rightarrow tb$ [4] $\Rightarrow t$ [5]

Example

$M = (\{s, t, f\}, \{a, b\}, \delta, s, t, \{f\}, \{(1, 4), (2, 5), (3, 6)\})$

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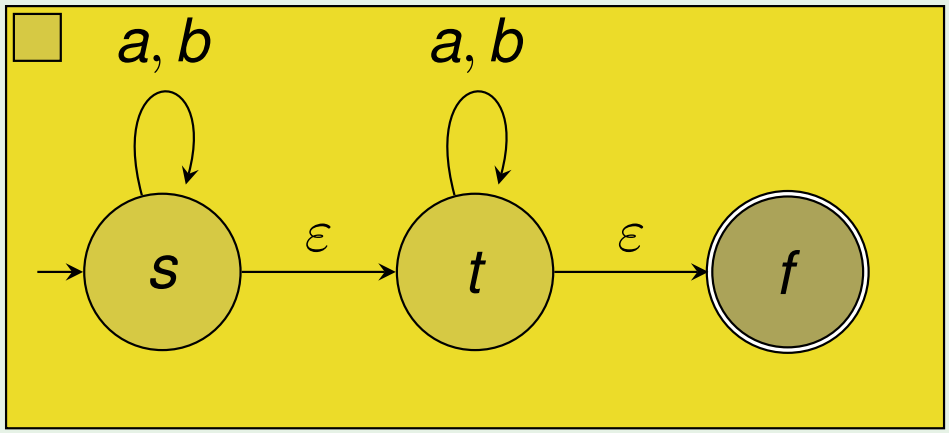

$sabab \Rightarrow sbab$ [1] $\Rightarrow sab$ [2] $\Rightarrow tab$ [3] $\Rightarrow tb$ [4] $\Rightarrow t$ [5] $\Rightarrow f$ [6]

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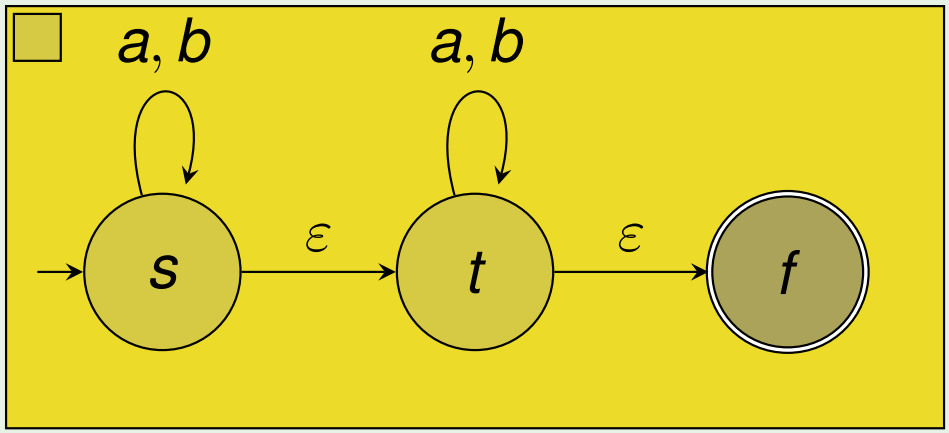


$$sabab \Rightarrow sbab\ [1] \Rightarrow sab\ [2] \Rightarrow tab\ [3] \Rightarrow tb\ [4] \Rightarrow t\ [5] \Rightarrow f\ [6]$$

$\mathcal{L}(M) = \{ww : w \in \{a, b\}^*\} \in CS - CF.$

RLSMG vs. all-SFA

Equivalence

Lemma

Let G be an n-RLSMG. There is an $(n-1)$-all-SFA M such that $\mathcal{L}(G) = \mathcal{L}(M)$.

Lemma

Let M be an n-all-SFA. There is an $(n+1)$-RLSMG G such that $\mathcal{L}(G) = \mathcal{L}(M)$.

Theorem

For all $n \geq 0$, $S_n = R_{[n+1]}$.

Corollary

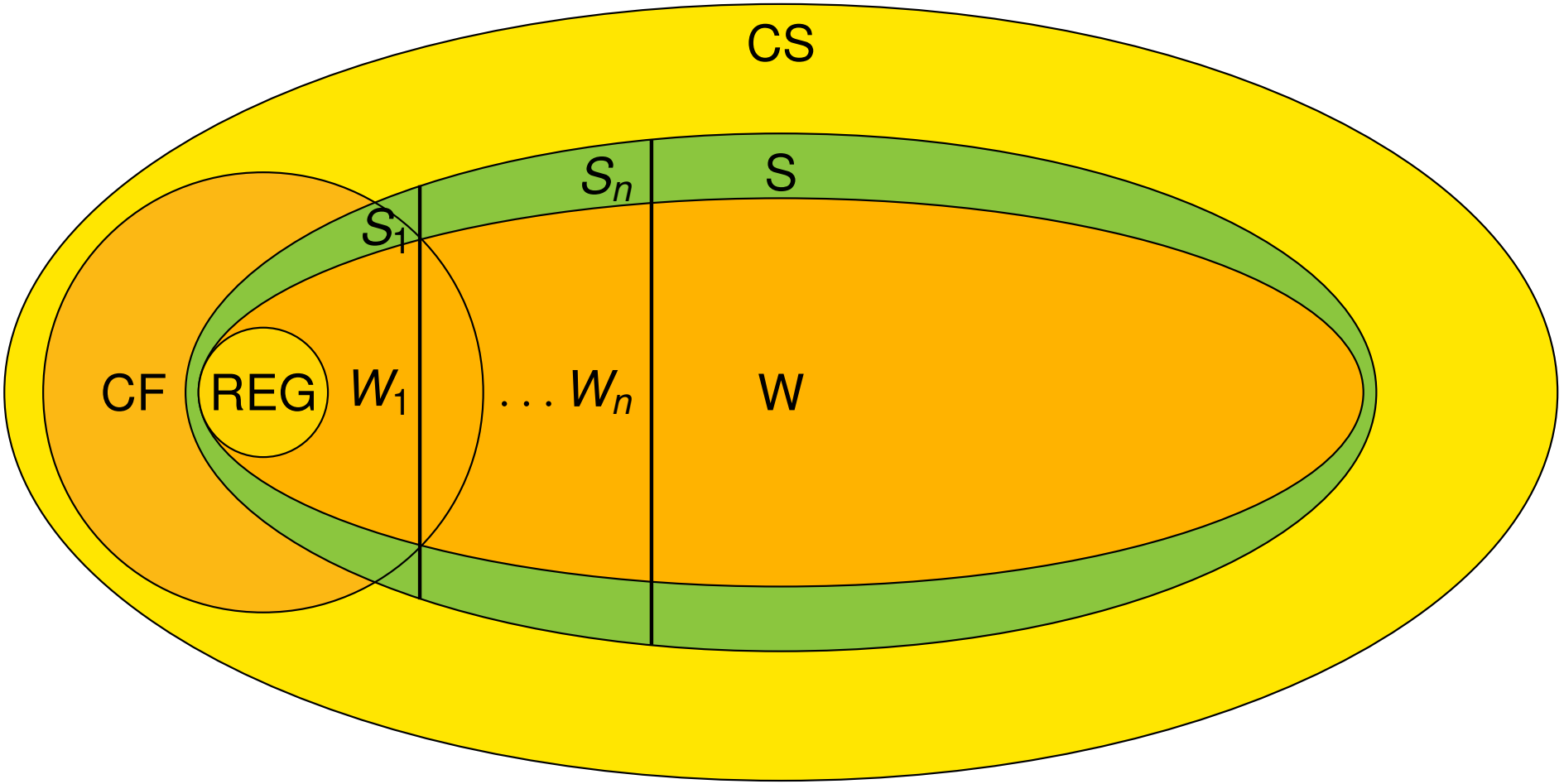
1. $REG = S_0 \subset S_1 \subset S_2 \subset \cdots \subset CS$.
2. $S_1 \not\subseteq CF$.
3. $CF \not\subseteq S_n$, *for* $n \geq 0$.
4. *For all* $n \geq 0$, S_n *is closed under union, finite substitution, homomorphism, intersection with a regular language, right quotient with a regular language and inverse homomorphism.*
5. *For all* $n \geq 0$, S_n *is full trio.*
6. *For all* $n \geq 1$, S_n *is not closed under intersection and complement.*

Comparison

Theorem

1. $W_0 = S_0 = REG$.
2. *For all* $n > 0$, $W_n \subset S_n$.
3. $W_n \not\subseteq S_{n-1}$, $n \geq 1$.
4. $S_n - W \neq \emptyset$, $n \geq 1$, *where* $W = \bigcup_{m=1}^{\infty} W_m$.

Comparison



all-SPDA

Theorem

$\mathcal{L}(0\text{-}all\text{-}SPDA) = CF.$

Theorem

$\mathcal{L}(1\text{-}all\text{-}SPDA) = RE.$