Improved Results on the Descriptional Complexity of Scattered Context Grammars

Scattered Context Grammar (**SCG**)

Scattered context grammar $G = (V, T, P, S)$

V is a finite alphabet

T is a set of terminals, $T \subset V$

S is a starting symbol, $S \in (V - T)$

P is a finite set of productions of the form $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)$; $A_1, \ldots, A_n \in (V - T)$; $x_1, \ldots, x_n \in V^*$

Note

- for a production, p, $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)$, $\pi(p) = n$

If $\pi(p)$

$= 1$ then the production is context-free

≥ 2 then the production is context-sensitive

SCG – Generated Language

Derivation step

If

- $(A_1, \ldots, A_n) \to (x_1, \ldots, x_n) \in P$
- $u = u_1 A_1 \ldots u_n A_n u_{n+1}$
- $v = u_1 x_1 \ldots u_n x_n u_{n+1}$

then $u \Rightarrow v\ [(A_1, \ldots, A_n) \to (x_1, \ldots, x_n)]$

Generated language

$L(G) = \{x \mid x \in T^*, S \Rightarrow {}^*x\}$

Generative power

$\mathcal{L}_{\mathbf{SCG}} = \mathcal{L}_{\mathbf{RE}}$

SCG – Example

Example

$G_1 = (V_1, T_1, P_1, S)$, where
$V_1 = \{a, b, c, A, B, C, S\}$
$T_1 = \{a, b, c\}$
$P_1 = \{(S) \to (ABC),$
$(A, B, C) \to (aA, bB, cC),$
$(A, B, C) \to (\varepsilon, \varepsilon, \varepsilon)\}$

$S \Rightarrow ABC \Rightarrow aAbBcC \Rightarrow aaAbbBccC \Rightarrow aaaAbbbBcccC \Rightarrow aaabbbccc$
$L(G_1) = \{a^n b^n c^n \mid n \geq 0\}$

$\pi((S) \to (ABC)) = 1$ – context-free production
$\pi((A, B, C) \to (aA, bB, cC)) = 3$ – context-sensitive production

Descriptional Complexity – Notation

Notation

Let $G = (V, T, P, S)$ be a **SCG**, $p_1, \ldots, p_n \in P$

Nonterminal complexity: number of nonterminals in G

Degree of context-sensitivity (δ-CS(G)): number of context-sensitive productions

Maximum context-sensitivity (max-CS(G)): $\max(\{\pi(p_i) - 1 \mid 1 \leq i \leq |P|\})$

Overall context-sensitivity (sum-CS(G)): $\Sigma(\{\pi(p_i) - 1 \mid 1 \leq i \leq |P|\})$

Example

$G_1 = (\{a, b, c, A, B, C, S\}, \{a, b, c\}, \{(S) \to (ABC),$
$(A, B, C) \to (aA, bB, cC), (A, B, C) \to (\varepsilon, \varepsilon, \varepsilon)\}, S)$

G_1's nonterminal complexity $= 4$ $\quad \delta$-CS(G_1) $= 2$

max-CS(G_1) $= 2$ $\quad$ sum-CS(G_1) $= 0 + 2 + 2 = 4$

Descriptional Complexity – Notation, Cont.

$\mathcal{L}(\mathbf{SCG}, k, l, m, n)$

$L \in \mathcal{L}(\mathbf{SCG}, k, l, m, n)$ iff there exists a **SCG**, G, $L(G) = L$ and

- G's nonterminal complexity $\leq k$
- δ-CS$(G) \leq l$
- max-CS$(G) \leq m$
- sum-CS$(G) \leq n$

Recent results

1. $\mathcal{L}(\mathbf{SCG}, 3, \infty, \infty, \infty) = \mathcal{L}_{\mathbf{RE}}$ (Meduna, 2000, [2])
2. $\mathcal{L}(\mathbf{SCG}, \infty, 2, 3, 6) = \mathcal{L}_{\mathbf{RE}}$ (Meduna, Fernau, 2003, [3])
3. $\mathcal{L}(\mathbf{SCG}, 7, 5, 8, 29) = \mathcal{L}_{\mathbf{RE}}$ (Meduna, Fernau, 2003, [4])
4. $\mathcal{L}(\mathbf{SCG}, 6, 6, 12, 44) = \mathcal{L}_{\mathbf{RE}}$ (Meduna, Fernau, 2003, [4])

Main Result

Theorem

$\mathcal{L}(\mathbf{SCG}, 5, 2, 3, 6)$ *coincides with the family of recursively enumerable languages.*

- context-sensitive productions of the form
  - $(0, \$, \$, 0) \rightarrow (\$, \varepsilon, \varepsilon, \$)$
  - $(1, \$, \$, 1) \rightarrow (\$, \varepsilon, \varepsilon, \$)$
- improves results 2, 3, 4 from previous slide

Corollary

$\mathcal{L}(\mathbf{SCG}, 8, 6, 1, 6)$ *coincides with the family of recursively enumerable languages.*

Post Correspondence Problem

Post correspondence problem (**PCP**)

- alphabet $\Sigma = \{0, 1\}$
- **PCP**, $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$, $u_i, v_i \in \Sigma^*$
- solution is a sequence of indexes $s_1, s_2, \ldots, s_l \in \{1, \ldots, r\}$ such that $u_{s_1} u_{s_2} \ldots u_{s_l} = v_{s_1} v_{s_2} \ldots v_{s_l}$
- undecidable problem

Example

$P_1 = \{(1, 111), (10111, 10), (10, 0)\}$
Solution: $2, 1, 1, 3$
$101111110 = 101111110$

$P_2 = \{(01, 011), (1, 10), (1, 11)\}$
No solution ($|u_i| < |v_i|$)

Extended Post Correspondence Problem

Extended Post correspondence problem (**EPC**)

- $L \in \mathcal{L}_{\mathbf{RE}}$ over $T = \{a_1, \ldots, a_n\}$
- alphabet $\Sigma = \{0, 1\}$
- **EPC**, $E = \{\{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}, (z_{a_1}, \ldots, z_{a_n})\}$, $u_i, v_i, z_{a_j} \in \Sigma^*$
- $L(E)$ – language defined by **EPC**, E
- $L(E) = \{b_1 \ldots b_k \in T^* \mid \text{ exists } s_1, \ldots, s_l \in \{1, \ldots, r\}, v_{s_1} v_{s_2} \ldots v_{s_l} = u_{s_1} u_{s_2} \ldots u_{s_l} z_{b_1} \ldots z_{b_k}\}$

Example

$E_1 = \{\{(01, 011), (1, 10), (1, 11)\}, (1(a), 01(b))\}$, $T = \{a, b\}$

$0111011 = 0111011$ $(123ba)$

$ba \in L(E_1)$

Basic Idea

Theorem

Every recursively enumerable language can be represented by **EPC** *(see [1]).*

Basic idea

1. represent $L \in \mathcal{L}_{\mathbf{RE}}$ by **EPC**
2. use context-free productions to generate a possible solution of **EPC**
3. verify this solution by context-sensitive productions

Note

Similar idea used by Geffert in [1]

Theorem Proof

- let $L \in \mathcal{L}_{\mathbf{RE}}$ over $T = \{a_1, \ldots, a_n\}$
- **EPC**, $E = (\{(u_1, v_1), \ldots, (u_r, v_r)\}, (z_{a_1}, \ldots, z_{a_n}))$, $L(E) = L$

Define the **SCG**, $G = (\{S, A, 0, 1, \$\} \cup T, T, P, S)$

Context-free productions

1. For every $a \in T$, add
   1. $(S) \rightarrow (\text{rev}(z_a)Sa)$, and
   2. $(S) \rightarrow (\text{rev}(z_a)Aa)$ to P
2. For every $(u_i, v_i) \in E$, add $(A) \rightarrow (\text{rev}(u_i)Av_i)$ to P
3. Add $(A) \rightarrow (\$\$)$ to P

Example

$E_1 = \{\{(01, 011), (1, 10), (1, 11)\}, (1, 01)\}$, $T = \{a, b\}$
S

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$1Sa$

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Theorem Proof, Cont.

Context-sensitive productions

4. Add
   1. $(0, \$, \$, 0) \to (\$, \varepsilon, \varepsilon, \$)$,
   2. $(1, \$, \$, 1) \to (\$, \varepsilon, \varepsilon, \$)$ to P

One more context-free production

5. Add $(\$) \to (\varepsilon)$ to P

Example

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Theorem Proof, Cont.

Context-sensitive productions

4. Add
   1. $(0, \$, \$, 0) \rightarrow (\$, \varepsilon, \varepsilon, \$)$,
   2. $(1, \$, \$, 1) \rightarrow (\$, \varepsilon, \varepsilon, \$)$ to P

One more context-free production

5. Add $(\$) \rightarrow (\varepsilon)$ to P

Example

$E_1 = \{\{(01, 011), (1, 10), (1, 11)\}, (1, 01)\}$, $T = \{a, b\}$
$ba \in L(E_1)$
$00ab \notin L(E_1)$

Corollary Proof

Idea

- simulate every context-sensitive production, p, with $\pi(p) = 4$ by 3 productions p_1, p_2, p_3 with $\pi(p_1) = \pi(p_2) = \pi(p_3) = 2$ in 3 derivation steps

Define the **SCG**, $\bar{G} = (\{S, A, 0, 1, \$_L, \$_R, \$_0, \$_1\} \cup T, T, \bar{P}, S)$ with $\bar{P}$ initialized with productions from steps (1) and (2).

3. Add $(A) \longrightarrow (\$_L \$_R)$ to $\bar{P}$
4. Add
   1. $(0, \$_L) \longrightarrow (\$_0, \varepsilon)$, $(\$_R, 0) \longrightarrow (\varepsilon, \$_0)$, $(\$_0, \$_0) \longrightarrow (\$_L, \$_R)$,
   2. $(1, \$_L) \longrightarrow (\$_1, \varepsilon)$, $(\$_R, 1) \longrightarrow (\varepsilon, \$_1)$, $(\$_1, \$_1) \longrightarrow (\$_L, \$_R)$ to $\bar{P}$
5. Add $(\$_L) \longrightarrow (\varepsilon)$, and $(\$_R) \longrightarrow (\varepsilon)$ to $\bar{P}$

Open Problems

- We know that
  - $\mathcal{L}(\mathbf{SCG}, 1, \infty, \infty, \infty) \subset \mathcal{L}_{\mathbf{RE}}$
  - $\mathcal{L}(\mathbf{SCG}, 3, \infty, \infty, \infty) = \mathcal{L}_{\mathbf{RE}}$

  What is the power of $\mathcal{L}(\mathbf{SCG}, 2, \infty, \infty, \infty)$?
- Can we generate any recursively enumerable language with only 1 context-sensitive production?

Bibliography

V. Geffert.
Context-free-like forms for the phrase-structure grammars.
In *Proceedings of the Mathematical Foundations of Computer Science 1988*, pages 309–317, New York, 1988. Springer-Verlag.

A. Meduna.
Generative power of three-nonterminal scattered context grammars.
Theoretical Computer Science, 246:276–284, 2000.

A. Meduna and H. Fernau.
On the degree of scattered context-sensitivity.
Theoretical Computer Science, 290:2121–2124, 2003.

A. Meduna and H. Fernau.
A simultaneous reduction of several measures of descriptional complexity in scattered context grammars.
Information Processing Letters, 86:235–240, 2003.