Closure Properties of Linear Languages under Operations of Linear Deletion

Random Parallel Deletion

Definition

$L, K \subseteq T^*$ two languages

$$[\perp, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in T^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L,$$
$$x_i \in K,\ 1 \leq i \leq n,\ n \geq 1\}$$

Example

- $[\perp, \{ababaaba\}, \{aba\}] = \{baba, \dots\}$
- $[\perp, \{abababa\}, \{aba\}] = \{baba, abba, \dots\}$
- $[\perp, \{ababaaba\}, \{aba\}] = \{baba, abba, abab, \dots\}$
- $[\perp, \{abababa\}, \{aba\}] = \{baba, abba, abab, b\}$

Parallel Deletion

Definition

$L, K \subseteq T^*$ two languages

$$[\perp_a, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in T^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L,$$
$$x_j \in K, \{u_i\} \cap T^*(K - \{\varepsilon\})T^* = \emptyset,$$
$$1 \leq i \leq n+1, 1 \leq j \leq n, n \geq 1\}$$

Example

- $[\perp_a, \{aababaaa\}, \{aba\}] = \{aba, \dots\}$
- $[\perp_a, \{aababaaa\}, \{aba\}] = \{aba, aabbaa\}$

Sequential Deletion

Definition

$L, K \subseteq T^*$ two languages

$$[\perp_1, L, K] = \{u_1 u_2 \in T^* : u_1 x u_2 \in L,\ x \in K\}$$

Example

- $[\perp_1, \{ababaaba\}, \{aba\}] = \{baba, \dots\}$
- $[\perp_1, \{abababa\}, \{aba\}] = \{baba, abba, \dots\}$
- $[\perp_1, \{ababaaba\}, \{aba\}] = \{baba, abba, abab\}$
- $[\perp_1, \{aba\}, \{aba\}] = \{\varepsilon\}$
- $[\perp_1, \{ab\}, \{aba\}] = \emptyset$

Scattered Sequential Deletion

Definition

$L, K \subseteq T^*$ two languages

$$[\perp_{1s}, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in T^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L,$$
$$x_1 x_2 \dots x_n \in K,\ n \geq 1\}.$$

Example

- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, \dots\}$
- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, adbe\}$

Multiple Scattered Sequential Deletion

Definition

$L, K \subseteq T^*$ two languages

$$[\perp_s, L, K] = \{u_1 u_2 \dots u_n u_{n+1} \in T^* : u_1 x_1 u_2 \dots u_n x_n u_{n+1} \in L,$$
$$x_1 x_2 \dots x_n \in K^+, n \geq 1\}.$$

Example

- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, \dots\}$
- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, adbe, \dots\}$
- $[\perp_{1s}, \{adbcea\}, \{ab, ca\}] = \{dcea, adbe, de\}$

Family of Languages

LIN = family of linear languages

$\mathcal{X}$ is a family of languages

Definition

$$\langle z, LIN, \mathcal{X} \rangle = \{[z, L, K] : L \in LIN,\ K \in \mathcal{X}\}$$

$$z \in \{\perp, \perp_a, \perp_1, \perp_{1s}, \perp_s\}$$

Regular deletion signifies $\mathcal{X} = REG$.

Linear deletion signifies $\mathcal{X} = LIN$.

Regular Deletion

First Main Result

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Linear Languages are closed under Operations of Regular Deletion.

Theorem

$\langle z, LIN, REG \rangle = LIN \qquad z \in \{\perp, \perp_1, \perp_{1s}, \perp_s, \perp_a\}$

First Main Result

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Linear Languages are closed under Operations of Regular Deletion.

Theorem

$\langle z, LIN, REG \rangle = LIN \qquad z \in \{\perp, \perp_1, \perp_{1s}, \perp_s, \perp_a\}$

$\supseteq$:

$L \in LIN$, then $L = [z, L, \{\varepsilon\}] \in \langle z, LIN, REG \rangle$.

Proof Idea – Random Parallel Deletion

$\subseteq$:

$L \in LIN$, $K \in REG$.

- $L = \mathcal{L}(G_L)$, $G_L = (N_L, T_L, P_L, S_L)$ is a proper linear grammar
- $K = \mathcal{L}(G_K)$, $G_K = (N_K, T_K, P_K, S_K)$ is a regular grammar such that S_K does not occur on the right-hand side of any rule

Construct linear grammar $G = (N, T_L, P, S)$, where

- $N = \{S, \langle x, B, y, X, Y\rangle : x, y \in T_L^*, B \in N_L \cup \{\varepsilon\}, X, Y \in N_K \cup \{\varepsilon\}, |x|, |y| \leq \max\{|u|, |v| : A \to uBv \in P_L\}\}$,
- P contains rules of the following forms

Proof Idea – Random Parallel Deletion

Rules

1. $S \to \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle$
2. $\langle \varepsilon, A, \varepsilon, X, Y \rangle \to \langle x, B, y, X, Y \rangle$ if $A \to xBy \in P_L$
3. $X \in \{S_K, \varepsilon\}$
   a) $\langle ax, A, yb, S_K, Y \rangle \to a \langle x, A, yb, S_K, Y \rangle$
   b) $\langle ax, A, yb, \varepsilon, Y \rangle \to a \langle x, A, yb, \varepsilon, Y \rangle$
   c) $\langle ax, A, yb, \varepsilon, Y \rangle \to \langle ax, A, yb, S_K, Y \rangle$
4. $Y \in \{S_K, \varepsilon\}$
   a) $\langle ax, A, yb, X, \varepsilon \rangle \to \langle ax, A, y, X, \varepsilon \rangle b$
   b) $\langle ax, A, yb, X, S_K \rangle \to \langle ax, A, y, X, S_K \rangle b$
   c) $\langle ax, A, yb, X, S_K \rangle \to \langle ax, A, yb, X, \varepsilon \rangle$
5. $\langle ax, A, yb, X, Y \rangle \to \langle x, A, yb, V, Y \rangle$ if $X \to aV \in P_K$, $V \in N_K \cup \{\varepsilon\}$
6. $\langle ax, A, yb, X, Y \rangle \to \langle ax, A, y, X, V \rangle$ if $V \to bY \in P_K$, $V \in N_K$
7. $\langle \varepsilon, \varepsilon, \varepsilon, X, X \rangle \to \varepsilon$

Example

Example

1. $S_L \rightarrow abcdef$

1. $S_K \rightarrow a$
2. $S_K \rightarrow cD$
3. $D \rightarrow dE$
4. $E \rightarrow e$

$S \Rightarrow$

Used rule:

Example

Example

1. $S_L \rightarrow abcdef$

1. $S_K \rightarrow a$
2. $S_K \rightarrow cD$
3. $D \rightarrow dE$
4. $E \rightarrow e$

$$S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle$$

Used rule:
1) $S \rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle \Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle$$

Used rule:
2) $\langle \varepsilon, A, \varepsilon, X, Y \rangle \to \langle x, B, y, X, Y \rangle$ if $A \to xBy \in P_L$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle \Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow$$
$$\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle$$

Used rule:

5) $\langle ax, A, yb, X, Y \rangle \to \langle x, A, yb, V, Y \rangle$ if $X \to aV \in P_K$, $V \in N^*$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle \Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow$$
$$\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle$$

Used rule:
3b) $\langle ax, A, yb, \varepsilon, Y \rangle \to a \langle x, A, yb, \varepsilon, Y \rangle$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$
\begin{aligned}
S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle &\Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow \\
\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle &\Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow \\
b \langle c, \varepsilon, def, S_K, \varepsilon \rangle &
\end{aligned}
$$

Used rule:
3c) $\langle ax, A, yb, \varepsilon, Y \rangle \to \langle ax, A, yb, S_K, Y \rangle$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$
\begin{aligned}
S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle &\Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow \\
\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle &\Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow \\
b \langle c, \varepsilon, def, S_K, \varepsilon \rangle &\Rightarrow b \langle \varepsilon, \varepsilon, def, D, \varepsilon \rangle
\end{aligned}
$$

Used rule:

5) $\langle ax, A, yb, X, Y \rangle \to \langle x, A, yb, V, Y \rangle$ if $X \to aV \in P_K$, $V \in N^*$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$
\begin{aligned}
S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle &\Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow \\
\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle &\Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow \\
b \langle c, \varepsilon, def, S_K, \varepsilon \rangle &\Rightarrow b \langle \varepsilon, \varepsilon, def, D, \varepsilon \rangle \Rightarrow \\
b \langle \varepsilon, \varepsilon, de, D, \varepsilon \rangle\, f &
\end{aligned}
$$

Used rule:

4a) $\langle ax, A, yb, X, \varepsilon \rangle \to \langle ax, A, y, X, \varepsilon \rangle\, b$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$
\begin{aligned}
S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle &\Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow \\
\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle &\Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow \\
b \langle c, \varepsilon, def, S_K, \varepsilon \rangle &\Rightarrow b \langle \varepsilon, \varepsilon, def, D, \varepsilon \rangle \Rightarrow \\
b \langle \varepsilon, \varepsilon, de, D, \varepsilon \rangle f &\Rightarrow b \langle \varepsilon, \varepsilon, d, D, E \rangle f
\end{aligned}
$$

Used rule:

6) $\langle ax, A, yb, X, Y \rangle \to \langle ax, A, y, X, V \rangle$ if $V \to bY \in P_K$, $V \in N_K$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$
\begin{aligned}
S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle &\Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow \\
\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle &\Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow \\
b \langle c, \varepsilon, def, S_K, \varepsilon \rangle &\Rightarrow b \langle \varepsilon, \varepsilon, def, D, \varepsilon \rangle \Rightarrow \\
b \langle \varepsilon, \varepsilon, de, D, \varepsilon \rangle f &\Rightarrow b \langle \varepsilon, \varepsilon, d, D, E \rangle f \Rightarrow \\
b \langle \varepsilon, \varepsilon, \varepsilon, D, D \rangle f &
\end{aligned}
$$

Used rule:

6) $\langle ax, A, yb, X, Y \rangle \to \langle ax, A, y, X, V \rangle$ if $V \to bY \in P_K$, $V \in N_K$

Example

Example

1. $S_L \to abcdef$

1. $S_K \to a$
2. $S_K \to cD$
3. $D \to dE$
4. $E \to e$

$$
\begin{aligned}
S \Rightarrow \langle \varepsilon, S_L, \varepsilon, S_K, \varepsilon \rangle &\Rightarrow \langle abc, \varepsilon, def, S_K, \varepsilon \rangle \Rightarrow \\
\langle bc, \varepsilon, def, \varepsilon, \varepsilon \rangle &\Rightarrow b \langle c, \varepsilon, def, \varepsilon, \varepsilon \rangle \Rightarrow \\
b \langle c, \varepsilon, def, S_K, \varepsilon \rangle &\Rightarrow b \langle \varepsilon, \varepsilon, def, D, \varepsilon \rangle \Rightarrow \\
b \langle \varepsilon, \varepsilon, de, D, \varepsilon \rangle f &\Rightarrow b \langle \varepsilon, \varepsilon, d, D, E \rangle f \Rightarrow \\
b \langle \varepsilon, \varepsilon, \varepsilon, D, D \rangle f &\Rightarrow bf
\end{aligned}
$$

Used rule:

7) $\langle \varepsilon, \varepsilon, \varepsilon, X, X \rangle \to \varepsilon$

Linear Deletion

(Random) Parallel Deletion, Sequential Deletion

Second Main Result

Linear Languages are not closed under Operations of Linear Deletion.

Theorem

$\langle z, LIN, LIN \rangle = RE \qquad z \in \{\perp, \perp_a, \perp_1\}$

(Random) Parallel Deletion, Sequential Deletion

Second Main Result

Linear Languages are not closed under Operations of Linear Deletion.

Theorem

$\langle z, LIN, LIN \rangle = RE \qquad z \in \{\perp, \perp_a, \perp_1\}$

$\subseteq$:

It is not hard to construct a Turing machine accepting $\langle z, LIN, LIN \rangle$, $z \in \{\perp, \perp_a, \perp_1\}$.

Extenden Post Correspondence

$\supseteq$:

- Suppose $L \in RE$, $L \subseteq T^*$, $T = \{a_1, \ldots, a_n\}$
- Extended Post Correspondence (EPC)
  $P = (\{(u_1, v_1), \ldots, (u_r, v_r)\}, (z_{a_1}, \ldots, z_{a_n})), \quad u_i, v_i, z_a \in \{0,1\}^*$
- $\mathcal{L}(P) = \{x_1 x_2 \ldots x_n \in T^* : \exists s_1, \ldots, s_l \in \{1, \ldots, r\}, l \geq 1,$
  $v_{s_1} \ldots v_{s_l} = u_{s_1} \ldots u_{s_l} z_{x_1} \ldots z_{x_n}\}$
- For every $L \in RE$, there is an EPC, P, such that $\mathcal{L}(P) = L$

String Generation

$\supseteq$:

Generate $x_1 x_2 \ldots x_n$ as follows:

S'

String Generation

$\supseteq$:

Generate $x_1 x_2 \ldots x_n$ as follows:

$$S' \Rightarrow \$S$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \ldots x_n$ as follows:

$$S' \Rightarrow \$S \Rightarrow \$z_{x_n}^R S x_n$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \ldots x_n$ as follows:

$$S' \Rightarrow \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \ldots x_n$ as follows:

$$S' \Rightarrow \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n$$
$$\Rightarrow^* \$z_{x_n}^R z_{x_{n-1}}^R \ldots z_{x_1}^R S x_1 \ldots x_{n-1} x_n$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \dots x_n$ as follows:

$$
\begin{aligned}
S' \Rightarrow\ & \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R S x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R A\$ x_1 \dots x_{n-1} x_n
\end{aligned}
$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \dots x_n$ as follows:

$$
\begin{aligned}
S' \Rightarrow\ & \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R S x_1 \dots x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R A \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R A v_{s_l} \$ x_1 \dots x_{n-1} x_n
\end{aligned}
$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \dots x_n$ as follows:

$$
\begin{aligned}
S' \Rightarrow\ & \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R S x_1 \dots x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R A \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R A v_{s_l} \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^* & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R \dots u_{s_1}^R \# v_{s_1} \dots v_{s_l} \$ x_1 \dots x_{n-1} x_n
\end{aligned}
$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \dots x_n$ as follows:

$$
\begin{aligned}
S' \Rightarrow\ & \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R S x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R A \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R A v_{s_l} \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R \dots u_{s_1}^R \# v_{s_1} \dots v_{s_l} \$ x_1 \dots x_{n-1} x_n \\
=\ & \$(u_{s_1} \dots u_{s_l} z_{x_1} \dots z_{x_n})^R \#(v_{s_1} \dots v_{s_l}) \$ x_1 \dots x_n
\end{aligned}
$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \ldots x_n$ as follows:

$$
\begin{aligned}
S' \Rightarrow\ & \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \ldots z_{x_1}^R S x_1 \ldots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \ldots z_{x_1}^R A \$ x_1 \ldots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \ldots z_{x_1}^R u_{s_l}^R A v_{s_l} \$ x_1 \ldots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \ldots z_{x_1}^R u_{s_l}^R \ldots u_{s_1}^R \# v_{s_1} \ldots v_{s_l} \$ x_1 \ldots x_{n-1} x_n \\
=\ & \$(u_{s_1} \ldots u_{s_l} z_{x_1} \ldots z_{x_n})^R \#(v_{s_1} \ldots v_{s_l}) \$ x_1 \ldots x_n \\
=\ & \$ w_1^R \# w_2 \$ x_1 x_2 \ldots x_n
\end{aligned}
$$

String Generation

$\supseteq$:

Generate $x_1 x_2 \dots x_n$ as follows:

$$
\begin{aligned}
S' \Rightarrow\ & \$S \Rightarrow \$z_{x_n}^R S x_n \Rightarrow \$z_{x_n}^R z_{x_{n-1}}^R S x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R S x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R A \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R A v_{s_l} \$ x_1 \dots x_{n-1} x_n \\
\Rightarrow^*\ & \$z_{x_n}^R z_{x_{n-1}}^R \dots z_{x_1}^R u_{s_l}^R \dots u_{s_1}^R \# v_{s_1} \dots v_{s_l} \$ x_1 \dots x_{n-1} x_n \\
=\ & \$(u_{s_1} \dots u_{s_l} z_{x_1} \dots z_{x_n})^R \#(v_{s_1} \dots v_{s_l}) \$ x_1 \dots x_n \\
=\ & \$ w_1^R \# w_2 \$ x_1 x_2 \dots x_n
\end{aligned}
$$

and $x_1 x_2 \dots x_n \in L = \mathcal{L}(P)$ iff $w_1 = w_2$, $\quad \$, \# \notin T \cup \{0, 1\}$

(Random) Parallel Deletion, Sequential Deletion

$\supseteq$:

Denote the linear grammar generating previous language by G, i.e.

$$\mathcal{L}(G) = \{\$w_1^R \# w_2 \$ x_1 \dots x_n : w_1, w_2 \in \{0,1\}^*, x_i \in T\}$$

(Random) Parallel Deletion, Sequential Deletion

$\supseteq$:

Denote the linear grammar generating previous language by G, i.e.

$$\mathcal{L}(G) = \{\$w_1^R \# w_2 \$ x_1 \dots x_n : w_1, w_2 \in \{0,1\}^*, x_i \in T\}$$

In addition, there is a linear grammar G' such that

$$\mathcal{L}(G') = \{\$w^R \# w\$: w \in \{0,1\}^*\}$$

(Random) Parallel Deletion, Sequential Deletion

$\supseteq$:

Denote the linear grammar generating previous language by G, i.e.

$$\mathcal{L}(G) = \{\$w_1^R \# w_2 \$ x_1 \dots x_n : w_1, w_2 \in \{0,1\}^*,\ x_i \in T\}$$

In addition, there is a linear grammar G' such that

$$\mathcal{L}(G') = \{\$w^R \# w\$: w \in \{0,1\}^*\}$$

Thus,

$$L = [z, \mathcal{L}(G), \mathcal{L}(G')],$$

$z \in \{\perp, \perp_a, \perp_1\}$. □

(Multiple) Scattered Sequential Deletion

Theorem

$\langle z, LIN, LIN\rangle \not\subseteq REC \qquad z \in \{\perp_{1s}, \perp_s\}$

Proof.

- $L \in RE$, $L \subseteq T^*$, $T \cap \{0,1\} = \emptyset$.
- The proof follows from the previous theorem since $L = [z, \mathcal{L}(G), \mathcal{L}(G')] \cap T^*$.
  - $\mathcal{L}(G) = \{\$w_1^R \# w_2 \$ x_1 \dots x_n : w_1, w_2 \in \{0,1\}^*, x_i \in T\}$
  - $\mathcal{L}(G') = \{\$w^R \# w\$: w \in \{0,1\}^*\}$
- If $[z, \mathcal{L}(G), \mathcal{L}(G')]$ is recursive, then so is L.
- For $L \in RE - REC$ language $[z, \mathcal{L}(G), \mathcal{L}(G')]$ is not recursive.

Summary

- Linear Languages are closed under Operations of Regular Deletion.
- Linear Languages are not closed under Operations of Linear Deletion.

- Here we summarize two open problems:
  1. Is it true that $\langle \perp_{1s}, LIN, LIN \rangle = RE$?
  2. Is it true that $\langle \perp_{s}, LIN, LIN \rangle = RE$?