Ramsey theory - Ramsey theorem

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Content

- Intro
- Ramsey-theorems
- Applications
- Open questions

Let's play a game

- Imagine we have two cans of two colors, red and blue
- Take every 2-combination of $\mathbb{N}$ and color them red or blue. Then print every such combination on the wall (Fig. 1).
- Take all members of red 2-combinations from the wall and make a set R
- Question: If we create all 2-combinations from R, are they all red?

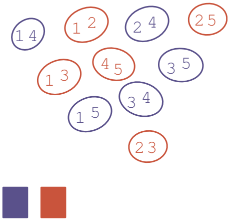

Figure: 2-colored combinations of the first 5 positive integeres.

Second question

- Let's try diffrent coloring, what about now? In Fig. 2, the set $\{1, 2, 3, 4\}$ do the thing
- Now let's try something harder and ask ourselves second question.
- Question: No matter how we color 2-combinations, is there always a subset of $\mathbb{N}$ of size k with all its 2-combinations having the same color on the wall?

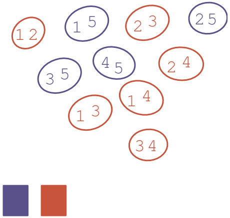

Figure: 2-colored combinations of the first 5 positive integeres in diffrent way.

Frank Plumley Ramsey

Figure: Frank Plumpton Ramsey [?]

- British mathematician
- In 1928 *On a problem of formal logic*

Theorem

Let Γ be an infinite class, and μ and τ positive integers; let all τ-combinations of the members of Γ be divided in any manner into μ mutually exclusive classes $C_i (i = 1, 2, ..., \mu)$, so that every τ-combination is a member of one and only one C_i; then Γ contains an infinite sub-class Δ such that all the τ-combinations of the members of Δ belong to the same C_i.

- We have 5 points on the plane, no free points on the same line
- Questions: is it possible no matter how we set the points on the plane to find a convex 4-gon?

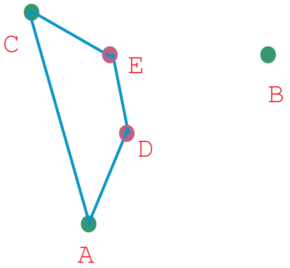

Figure: convex 4-gon among 5 points

Szekeres and Erdos

- Second question: generally what is the number of points n on the plane (no matter how we set them, no free collinear) needed to find convex k-gon? Denoted $n(k)$.
- Proved by Szekeres and Erdos in 1935, Ramsey-theorems boom
- $n(k)$ is called Ramsey Number, grows exponentialy

Third example

- We have party of 6 people who like each other and who do not, Fig. 5 (like blue, don't like red)
- Questions: Can we find always find at least three people who like each other or do not (one of them)? What about at least k people, how big the party must be?

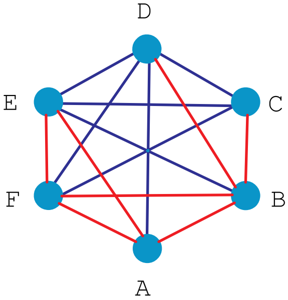

Figure: party of 6 people as a graph

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Van der Waerden

Theorem

If I r-color the positive integers then there are arbitrarily large monochromatic arithmetical progressions

Theorem

Given k and r, there is an n so large that if we r-color the integers $1, \ldots, n$ then there is a monochromatic arithmetical progression of size at least k

More theorems

- Hales-Jewett (nice example, analogy to vector spaces)
- Graham, Leeb, Rothschild (finite field)
- Rados' Theorem (homogenous system of linear equations)
- Hindman's Theorem (finite sums)

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Applications

- Logic:
  - Godel's incompleteness theorem. Does it matter to ordinary mathematics? Yes (special form of Ramsey Theorem)
  - Theorem proving
  - Computability
- Complexity:
  - estimating of lower bounds

Application - Number theory

Theorem

Schur's theorem: if $\mathbb{N}$ *is partitioned into a nite number of classes, at least one partition class contains a solution to the equation* $x + y = z$.

Application - Others

- Graph theory
- Planar geometry
- Convex and computational geometry
- VLSI design wire routing

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Open questions

- The growth of R(n): R(n) is the smallest integer such that in graph consisting of R(n) vertices there is clique or independent set of size n.
- Van der Waerden's Theorem: prove that for all $n, W(n) < 2^{n^2}$, where W(n) is Van der Waerden's number
- chromatic number χ is minimum number of colors needed for coloring points in Euclidean plane $\mathbb{N}^2$ such a way that any two points in distance 1 have dirent colors. The task is to prove that: $\chi(\mathbb{N}^2) \geq 5$ and $\chi(\mathbb{N}^2) \leq 6$.