Vehicle Routing Problem and Its Relation to the Graph Theory

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Vehicle routing problem (VRP) is a generic name given to combinatorial optimization problems which focus on determining optimal set of routes for a fleet of vehicles originating in one or several depots and traversing through number of geographically dispersed customers. The classic definition states that each node can be visited only once and exactly by one vehicle. For various applications it can be beneficial to extend this definition by introducing new constraints such as limited vehicle capacity and maximum travel time. First definition of VRP dates back to late 1950s and it has been used mainly in transportation and logistic industry since.

To describe VRP rigorously we can define it on a graph $G=(V, A)$ where $V=D\cup C$ is a set of vertices representing depots (D) and customers (C) and A is a set of arcs. For every arc $a=(i, j)$, where i, j represent nodes connected by arc a and $i\neq j$, is defined a non-negative cost c_{ij} usually interpreted as travel time or travel cost. At every depot $d\in D$ is $m_{min}<m<m_{max}$ vehicles. VRP then consists of finding vehicle routes composed of a sequence of arcs with minimal total cost in such way that:

- every customer $c\in C$ is visited exactly once
- all vehicle routes end in its originating depot
- all side constraints are satisfied

Real applications extend this definition by using directed asymmetrical graphs to model one way routes and add other important nodes such as intersections to V. Some side constraints that are usually used are vehicle capacity, time and precedence restrictions.

Probably the most challenging task of VRP is route planning which is responsible for finding shortest path between a start and destination point on a road map. The road map can be represented by the very same graph as used in VRP description, where start and destination points are two vertices from the set V. The challenge of route planning lies mainly in processing very large road maps with many additional inputs including traffic and road conditions, traffic predictions and vehicle characteristics.

For intelligent car navigation it is important that the algorithm reflects real-time changes in the environment which is usually achieved by using dynamic networks. Dynamic network is such graph where road evaluation c_{ij} can vary over time.

Exact algorithms for route planning are based on graph theory and are always optimal as they compute every possible solution until the shortest path is found. The most well-known representative of those algorithms is Dijkstra's algorithm whose main advantage is that it terminates once it labels the destination node and does not construct the full shortest path tree with every possible route. This helps the algorithm achieve complexity of $O(n^2)$ where n is the number of nodes. Other representatives include Bellman-Ford algorithm or incremental graph. As VRP is known to be NP-hard problem, no exact algorithm can ensure computation in reasonable computing time and that is the reason why heuristic algorithms are preferred in real-world use.

The heuristic algorithm cannot guarantee finding an optimal solution, but despite that they typically produce good quality solutions in noticeably shorter time than exact algorithms thanks to limiting the search space. A* algorithm is a variant of Dijkstra that uses heuristic function to limit possible paths in each step to those leading to the destination with shortest remaining path. In the car navigation is the heuristic function computed based on location and destination distance. Some other heuristic algorithms, such as ant colony and genetic algorithms, use natural metaphors instead of mathematical constructions for its inspiration.

Some papers suggest hybrid approach by combining heuristic and exact algorithms to improve efficiency or features and such algorithms can then for example solve dynamic multiple-objective problems such as finding the solution for both shortest path and time.