On the Relation Between Right-Linear #-Rewriting Systems and Simple Matrix Grammars

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Outline

- **Simple Matrix Grammars**
  - Properties and Generative Power
- **#-Rewriting Systems**
  - n-Right-Linear #-Rewriting Systems
- **Relation Between Right-Linear SMG and #RS**
  - Proof of $\mathscr{L}(n\text{-}RLIN\#RS) \subseteq \mathscr{L}(SM, RLIN, n)$
  - Proof of $\mathscr{L}(SM, RLIN, n) \subseteq \mathscr{L}(n\text{-}RLIN\#RS)$

Definition

A simple matrix grammar of degree n, $n \geq 1$, is $(n+3)$-tuple:

$$G = (N_1, N_2, \dots, N_n, T, M, S)$$

- $N_1, N_2, \dots N_n$ are the alphabets of nonterminals,
- T is the alphabet of terminals, $T \cap N_i = \emptyset$, all $1 \leq i \leq n$,
- M is the set of rewriting matrices in the form:
  1. $(S \to x)$, $x \in T^*$,
  2. $(S \to A_1 A_2 \dots A_n)$, $A_i \in N_i$, $1 \leq i \leq n$,
  3. $(A_1 \to x_1, A_2 \to x_2, \dots, A_n \to x_n)$, $x_i \in (N_i \cup T)$,
- S is the start symbol, $S \notin T \cup \{N_1, N_2, \dots, N_n\}$.

- Based on the type of the type of grammar rules used, G is:
  - Regular (without ε-rules)
  - Linear
  - Context-Free (without ε-rules)

Derivation Step

Let $u = \alpha_1 A_1 \beta_i \alpha_2 A_2 \beta_2 \dots \alpha_n A_n \beta_n$ and $v = \alpha_1 x_1 \beta_2 \alpha_2 x_2 \beta_2 \dots \alpha_n x_n \beta_n$, where $\alpha_i \in T^*$, $A_i \in N_i$, $\beta_i \in (N_i \cup T)^*$ for all $1 \leq i \leq n$.
If there exists $(A_1 \rightarrow x_1, \dots, A_n \rightarrow x_n) \in M$, u derives v,

$$u \Rightarrow v.$$

Reflexive and transitive closure, $\Rightarrow^*$, is defined in the usual manner.

Generated Language

$$L(G) = \{x \in T^* \mid S \Rightarrow^* x\}$$

Example

$$G = (\{A\}, \{B\}, \{a, b\}, M, S)$$

1. $(S \rightarrow AB)$
2. $(A \rightarrow aAb, B \rightarrow aBb)$
3. $(A \rightarrow ab, B \rightarrow ab)$

$$S \Rightarrow_1 AB \Rightarrow_2 aAb\, aBb \Rightarrow_2^n aa^nAb^nb\, aa^nBb^nb$$
$$\Rightarrow_3 aa^{n+1}abb^{n+1}b\, aa^{n+1}abb^{n+1}b$$

Generated Language

$$L(G) = \{a^nb^na^nb^n \mid n \geq 1\}$$
$$L(G) \notin \mathscr{L}(CF)$$

Determined by two properties – $\mathscr{L}(SM, X, n)$:

1. Type of grammar rules used – $X \in \{LIN, REG, CF\}$
   - $\mathscr{L}(X) = \mathscr{L}(SM, X - \varepsilon, 1)$,
   - $\mathscr{L}(SM, REG) \subseteq \mathscr{L}(SM, LIN) \subseteq \mathscr{L}(SM, CF) \subseteq \mathscr{L}(RE)$
2. Degree of the grammar – n
   - Grammar of degree n cannot simulate one of degree $(n + 1)$
   - $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \ldots, A_n \rightarrow x_n) \in M_n$,
   - $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \ldots, A_n \rightarrow x_n, T \rightarrow x_{n+1}) \in M_{n+1}$

$$\mathscr{L}(SM, X, n) \subset \mathscr{L}(SM, X, n + 1) \qquad \text{for } n \geq 1$$

* Infinite hierarchy for all grammar types and degrees

Definition

A context-free #-rewriting system is the quadruple:

$$H = (Q, \Sigma, s, R)$$

- Q is the finite set of states,
- Σ is the alphabet, $\# \in \Sigma, Q \cap \Sigma = \emptyset$,
- s is the starting state, $s \in Q$
- R is the set of rewriting rules, $R \subseteq Q \times \mathbb{N} \times \{\#\} \times Q \times \Sigma^*$

Notation

A rule, $(p, n, \#, q, x) \in R$, where $p, q \in Q$, $n \in \mathbb{N}$, $x \in \Sigma^*$, is written as: $p\,{}_n\# \rightarrow q\,x$.

Derivation Step

Let $x = p\,u\#v$ and $y = q\,uwv$, where $p, q \in Q$, $u, v, w \in \Sigma^*$, such that $occur(\#, u) = n - 1$.
If there exists $p\,_n\# \rightarrow q\,w \in R$, then x derives y,

$$x \Rightarrow y.$$

Reflexive and transitive closure, $\Rightarrow^*$, is defined in the usual manner.

Generated Language

$$L(H) = \{w \mid s\# \Rightarrow^* q\,w, q \in Q, w \in (\Sigma - \#)^*\}$$

Example

$$H = (\{s, p, q, f\}, \{a, b, c, \#\}, s, R)$$

1. $s\ {}_1\# \rightarrow p\ \#\#$
2. $p\ {}_1\# \rightarrow q\ a\#b$
3. $q\ {}_2\# \rightarrow p\ \#c$
4. $p\ {}_1\# \rightarrow f\ ab$
5. $f\ {}_1\# \rightarrow f\ c$

$$s\,\# \Rightarrow_1 p\,\#\# \Rightarrow_2 q\,a\#b\# \Rightarrow_3 p\,a\#b\#c \Rightarrow_4 f\,ab\#c \Rightarrow_5 f\,abc$$

Generated Language

$$L(H) = \{a^n b^n c^n \mid n \geq 1\}$$
$$L(H) \notin \mathscr{L}(CF)$$

n-Right-Linear #-Rewriting System

Let $H = (Q, \Sigma, s, R)$ be a context-free #-rewriting system and, in addition, R satisfies

$$R \subseteq Q \times \mathbb{N} \times \{\#\} \times Q \times ((\Sigma - \{\#\})^*\{\#\} \cup (\Sigma - \{\#\})^*),$$

then H is an n-right-linear #-rewriting system, n-RLIN#RS.

Generated Language

$$L(H) = \{w \mid s\,\#^n \Rightarrow^* q\,w, q \in Q, w \in (\Sigma - \#)^*\}$$

Theorem

$$\mathscr{L}(n\text{-}RLIN\#RS) = \mathscr{L}(SM, RLIN, n)$$

Idea of Proof

1. $\mathscr{L}(n\text{-}RLIN\#RS) \subseteq \mathscr{L}(SM, RLIN, n)$
   - Construct an equivalent RLIN-SMG for every RL#RS
2. $\mathscr{L}(SM, RLIN, n) \subseteq \mathscr{L}(n\text{-}RLIN\#RS)$
   - Construct an equivalent RL#RS for every RLIN-SMG
3. $\mathscr{L}(n\text{-}RLIN\#RS) = \mathscr{L}(SM, RLIN, n)$

Claim 1

$$\mathscr{L}(n\text{-}RLIN\#RS) \subseteq \mathscr{L}(SM, RLIN, n)$$

Proof

Let $H = (Q, \Sigma, s, R)$ be an n-right-linear #-rewriting system. Construct an SMG, $G = (N_1, \ldots, N_n, \Sigma, M, \langle s \rangle)$, $N_i \subseteq (Q \times \mathbb{N}_0)$ for all $1 \leq i \leq n$. M is constructed in the following way:

1. $(\langle s \rangle \to \langle s, 1 \rangle \langle s, 2 \rangle \ldots \langle s, n \rangle)$,
2. $(u, \langle p, i \rangle \to w \langle q, i \rangle, v)$ for every rule $p\,{}_i\# \to q\,w\# \in R$ where

$$u = (\langle p, j_1 \rangle \to \langle q, j_1 \rangle, \ldots, \langle p, j_{i-1} \rangle \to \langle q, j_{i-1} \rangle), \qquad 0 \leq j_k < i$$
$$v = (\langle p, j'_1 \rangle \to \langle q, j'_1 \rangle, \ldots, \langle p, j'_n \rangle \to \langle q, j'_n \rangle), \quad i < j'_{k'} \leq n \vee j'_{k'} = 0$$

such that $|u| + |v| = n - 1$,

Proof Contd.

3. $(u, \langle p, i\rangle \rightarrow \langle q, 0\rangle, v)$ for every rule $p\,{}_i\# \rightarrow q\,w \in R$ where

$$u = (\langle p, j_1\rangle \rightarrow \langle q, j_1\rangle, \dots, \langle p, j_{i-1}\rangle \rightarrow \langle q, j_{i-1}\rangle), \qquad 0 \leq j_k < i$$
$$v = (\langle p, j'_1\rangle \rightarrow \langle q, j'_1\rangle, \dots, \langle p, j'_n\rangle \rightarrow \langle q, j'_{n-1}\rangle),$$
$$i \leq j'_{k'} \leq n \vee j'_{k'} = 0$$

   such that $|u| + |v| = n - 1$,
4. $(\langle q, 0\rangle \rightarrow \varepsilon)^n$ for every $q \in Q$.

Observation

$$L(H) = L(G)$$

Claim 2

$$\mathscr{L}(SM, RLIN, n) \subseteq \mathscr{L}(n\text{-}RLIN\#RS)$$

Proof

Let $G = (N_1, N_2, \dots, N_n, T, M, S)$ be a right-linear SMG.
Construct an n-right-linear #-RS, $H = (Q, T, \langle \Delta, S \rangle, R)$,

$$Q \subseteq (label(P) \cup \{\Delta\}) \times (((N_1 \cup \overline{N_1}) \times (N_2 \cup \overline{N_2}) \times \cdots \times (N_n \cup \overline{N_n})) \cup \{\Delta\})$$

for $1 \leq i \leq n$ and R is constructed in the following way:

1. $\langle \Delta, S \rangle\ {}_1\# \rightarrow \langle \Delta, A_1, A_2, \dots, A_n \rangle\ \#$ for every $p \in M$, $lhs(p) = S$,
2. $\langle \Delta, A_1, A_2, \dots, A_n \rangle\ {}_1\# \rightarrow \langle p, \overline{A_1}, A_2, \dots, A_n \rangle x_1\#$
   $\langle p, \overline{A_1}, \overline{A_2}, \dots, A_i, \dots, A_n \rangle\ {}_i\# \rightarrow \langle p, \overline{A_1}, \overline{A_2}, \dots, \overline{A_i}, \dots, A_n \rangle x_i\#$
   for all $1 < i < n$
   $\langle p, \overline{A_1}, \dots, \overline{A_{n-1}}, A_n \rangle\ {}_n\# \rightarrow \langle \Delta, B_1, B_2, \dots, B_n \rangle x_n\#$
   for every rule p: $(A_1 \rightarrow x_1 B_1, A_2 \rightarrow x_2 B_2, \dots, A_n \rightarrow x_n B_n) \in M$.

Proof Contd.

3. $\langle \Delta, A_1, A_2, \ldots, A_n\rangle\ {}_1\# \rightarrow \langle p, \overline{A_1}, A_2, \ldots, A_n\rangle x_1$
$\langle p, \overline{A_1}, \overline{A_2}, \ldots, A_i, \ldots, A_n\rangle\ {}_1\# \rightarrow \langle p, \overline{A_1}, \overline{A_2}, \ldots, \overline{A_i}, \ldots, A_n\rangle x_i$
for all $1 < i < n$
$\langle p, \overline{A_1}, \ldots, \overline{A_{n-1}}, A_n\rangle\ {}_1\# \rightarrow \langle \Delta, \Delta\rangle x_n$
for every rule p: $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \ldots, A_n \rightarrow x_n) \in M$.

Observation

$$L(G) = L(H)$$

Conclusion

$$\mathscr{L}(n\text{-}RLIN\#RS) = \mathscr{L}(SM, RLIN, n)$$

Conclusion

Overview

- Simple Matrix Grammars
- Infinite hierarchy of SMG based on degree
- #-Rewriting Systems, n-Right-Linear #-Rewriting Systems
- Equivalence of presented families
- Existence of infinite hierarchy for RL#RS

Future Research

- Equivalence of additional SMG and #RS families

Bibliography

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