#-Rewriting Systems in Relation to Simple Matrix Grammars

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  - Linear Grammars
  - Context-Free Grammars

Definition

A simple matrix grammar of degree n, $n \geq 1$, is $(n+3)$-tuple:

$$G = (N_1, N_2, \ldots, N_n, T, M, S)$$

- $N_1, N_2, \ldots N_n$ are the alphabets of nonterminals,
- T is the alphabet of terminals, $T \cap N_i = \emptyset$, all $1 \leq i \leq n$,
- M is the set of rewriting matrices in the form:
  1. $(S \rightarrow x)$, $x \in T^*$,
  2. $(S \rightarrow A_1 A_2 \ldots A_n)$, $A_i \in N_i$, $1 \leq i \leq n$,
  3. $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \ldots, A_n \rightarrow x_n)$, $x_i \in (N_i \cup T)$,
- S is the start symbol, $S \notin T \cup \{N_1, N_2, \ldots, N_n\}$.

Derivation Step

Let $u = \alpha_1 A_1 \beta_i \alpha_2 A_2 \beta_2 \ldots \alpha_n A_n \beta_n$ and $v = \alpha_1 x_1 \beta_2 \alpha_2 x_2 \beta_2 \ldots \alpha_n x_n \beta_n$, where $\alpha_i \in T^*$, $A_i \in N_i$, $\beta_i \in (N_i \cup T)^*$ for all $1 \leq i \leq n$.
If there exists $(A_1 \rightarrow x_1, \ldots, A_n \rightarrow x_n) \in M$, u derives v,

$$u \Rightarrow v.$$

* Reflexive and transitive closure, $\Rightarrow^*$, is defined in the usual manner.

Generated Language

$$L(G) = \{x \in T^* \mid S \Rightarrow^* x\}$$

Example

$$G = (\{A\}, \{B\}, \{a, b\}, M, S)$$

1. $(S \rightarrow AB)$
2. $(A \rightarrow aAb, B \rightarrow aBb)$
3. $(A \rightarrow ab, B \rightarrow ab)$

$$S \Rightarrow_1 AB \Rightarrow_2 aAb\, aBb \Rightarrow_2^n aa^nAb^nb\, aa^nBb^nb$$
$$\Rightarrow_3 aa^{n+1}abb^{n+1}b\, aa^{n+1}abb^{n+1}b$$

Generated Language

$$L(G) = \{a^nb^na^nb^n \mid n \geq 1\}$$
$$L(G) \notin \mathscr{L}(CF)$$

Generative Power of SMG

Determined by two properties – $\mathscr{L}(SM, X, n)$:

1. Type of grammar rules used – $X \in \{LIN, REG, CF\}$
   - $\mathscr{L}(X) = \mathscr{L}(SM, X - \varepsilon, 1)$,
   - $\mathscr{L}(SM, REG) \subseteq \mathscr{L}(SM, LIN) \subseteq \mathscr{L}(SM, CF) \subseteq \mathscr{L}(RE)$
2. Degree of the grammar – n
   - Grammar of degree n cannot simulate one of degree $(n + 1)$
   - $(A_1 \to x_1, A_2 \to x_2, \ldots, A_n \to x_n) \in M_n$,
   - $(A_1 \to x_1, A_2 \to x_2, \ldots, A_n \to x_n, T \to x_{n+1}) \in M_{n+1}$

$$\mathscr{L}(SM, X, n) \subset \mathscr{L}(SM, X, n + 1) \qquad \text{for } n \geq 1$$

* Infinite hierarchy for all grammar types and degrees

Definition

A context-free #-rewriting system is the quadruple:

$$H = (Q, \Sigma, s, R)$$

- Q is the finite set of states,
- Σ is the alphabet, $\# \in \Sigma$, $Q \cap \Sigma = \emptyset$,
- s is the starting state, $s \in Q$
- R is the set of rewriting rules, $R \subseteq Q \times \mathbb{N} \times \{\#\} \times Q \times \Sigma^*$

Notation

A rule, $(p, n, \#, q, x) \in R$, where $p, q \in Q$, $n \in \mathbb{N}$, $x \in \Sigma^*$, is written as: $p\,{}_n\# \rightarrow q\,x$.

Derivation Step

Let $x = p\, u\#v$ and $y = q\, uwv$, where $p, q \in Q$, $u, v, w \in \Sigma^*$, such that $occur(\#, u) = n - 1$.
If there exists $p\,_n\# \rightarrow q\, w \in R$, then x derives y,

$$x \Rightarrow y.$$

* Reflexive and transitive closure, $\Rightarrow^*$, is defined in the usual manner.

Generated Language

$$L(H) = \{w \mid s\,\# \Rightarrow^* q\,w, q \in Q, w \in (\Sigma - \#)^*\}$$

Index of #-Rewriting System, $\mathscr{L}_n(\#RS)$

A #-rewriting system, H, is of index n if for every configuration, $s\,\# \Rightarrow^* qy$, $occur(\#, y) \leq n$.

Example

$$H = (\{s, p, q, f\}, \{a, b, c, \#\}, s, R)$$

1. $s\ {}_1\# \rightarrow p\ \#\#$
2. $p\ {}_1\# \rightarrow q\ a\#b$
3. $q\ {}_2\# \rightarrow p\ \#c$
4. $p\ {}_1\# \rightarrow f\ ab$
5. $f\ {}_1\# \rightarrow f\ c$

$$s\ \# \Rightarrow_1 p\ \#\# \Rightarrow_2 q\ a\#b\# \Rightarrow_3 p\ a\#b\#c \Rightarrow_4 f\ ab\#c \Rightarrow_5 f\ abc$$

Generated Language

$$L(H) = \{a^n b^n c^n \mid n \geq 1\}$$
$$L(H) \notin \mathscr{L}(CF)$$

n-Linear #-Rewriting System

Let $H = (Q, \Sigma, s, R)$ be a context-free #-rewriting system and, in addition, R satisfies

$$R \subseteq Q \times \mathbb{N} \times \{\#\} \times Q \times ((\Sigma - \{\#\})^* \{\#\} (\Sigma - \{\#\})^* \cup (\Sigma - \{\#\})^*),$$

then H is an n-linear #-rewriting system, n-LIN#RS.

n-Right-Linear #-Rewriting System

Let $H = (Q, \Sigma, s, R)$ be a context-free #-rewriting system and, in addition, R satisfies

$$R \subseteq Q \times \mathbb{N} \times \{\#\} \times Q \times ((\Sigma - \{\#\})^* \{\#\} \cup (\Sigma - \{\#\})^*),$$

then H is an n-right-linear #-rewriting system, n-RLIN#RS.

Generated Language

$$L(H) = \{w \mid s\,\#^n \Rightarrow^* q\,w, q \in Q, w \in (\Sigma - \#)^*\}$$

Relation Between SMG and #RS

FIT

Theorem

$$\mathscr{L}(n\text{-}LIN\#RS) = \mathscr{L}(SM, LIN, n) \quad (1)$$

$$\mathscr{L}_n(CF\#RS) = \mathscr{L}(SM, CF, n) \quad (2)$$

Idea of Proof

$X \in \{n\text{-}LIN, CF\}$

1. $\mathscr{L}(X\#RS) \subseteq \mathscr{L}(SM, X, n)$
   - Construct an equivalent SMG for every #RS
2. $\mathscr{L}(SM, X, n) \subseteq \mathscr{L}(X\#RS)$
   - Construct an equivalent #RS for every SMG
3. $\mathscr{L}(X\#RS) = \mathscr{L}(SM, X, n)$

Lemma

$$\mathscr{L}(n\text{-}RLIN\#RS) = \mathscr{L}(SM, RLIN, n)$$

Proof

Let $H = (Q, \Sigma, s, R)$ be an n-linear #-rewriting system.
Construct a linear simple matrix grammar,

$$G = (N_1, \dots, N_n, \Sigma, M, \langle s \rangle),$$

$N_i \subseteq (Q \times \mathbb{N}_0)$ for all $1 \leq i \leq n$. M is constructed as follows:

1. $(\langle s \rangle \rightarrow \langle s, 1 \rangle \langle s, 2 \rangle \dots \langle s, n \rangle)$,
2. $(u, \langle p, i \rangle \rightarrow w_1 \langle q, i \rangle w_2, v)$ for every rule $p\,{}_i\# \rightarrow q\, w_1 \# w_2 \in R$,
   - u, v defined analogously with RLIN grammars

   such that $|u| + |v| = n - 1$,

Proof Contd.

3. $(u, \langle p, i\rangle \rightarrow \langle q, 0\rangle, v)$ for every rule $p\ {}_i\# \rightarrow q\ w \in R$
   - u, v defined analogously with RLIN grammars
   - ordinal numbers of nonterminals in v are decreased

   such that $|u| + |v| = n - 1$,
4. $(\langle q, 0\rangle \rightarrow \varepsilon)^n$ for every $q \in Q$.

Observation

$$L(H) = L(G)$$

Claim 2

$$\mathscr{L}(SM, LIN, n) \subseteq \mathscr{L}(n\text{-}LIN\#RS)$$

Proof

Let $G = (N_1, N_2, \ldots, N_n, T, M, S)$ be a linear SMG.
Construct an n-linear #-rewriting system,

$$H = (Q, T, \langle \Delta, S \rangle, R),$$

where Q is defined analogously with RLIN grammars, and R is constructed in the following way:

1. $\langle \Delta, S \rangle \ {}_1\# \rightarrow \langle \Delta, A_1, A_2, \ldots, A_n \rangle \ \#$
2. $\langle \Delta, A_1, A_2, \ldots, A_n \rangle \ {}_1\# \rightarrow \langle p, \overline{A_1}, A_2, \ldots, A_n \rangle x_{i1} \# x_{i2}$
   $\langle p, \overline{A_1}, \overline{A_2}, \ldots, A_i, \ldots, A_n \rangle \ {}_i\# \rightarrow \langle p, \overline{A_1}, \overline{A_2}, \ldots, \overline{A_i}, \ldots, A_n \rangle x_{i1} \# x_{i2}$
   for all $1 < i < n$
   $\langle p, \overline{A_1}, \ldots, \overline{A_{n-1}}, A_n \rangle \ {}_n\# \rightarrow \langle \Delta, B_1, B_2, \ldots, B_n \rangle x_{n1} \# x_{n2}$
   for every p: $(A_1 \rightarrow x_{11} B_1 x_{12}, \ldots, A_n \rightarrow x_{n1} B_n x_{n2}) \in M$.

Proof Contd.

3. $\langle \Delta, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_n} \rangle\ _1\# \rightarrow \langle p, \overline{A_1}, A_2, \underline{\ldots}, \underline{A_n} \rangle x_1$
   $\langle p, \overline{A_1}, \overline{A_2}, \ldots, A_i, \ldots, A_n \rangle\ _1\# \rightarrow \langle p, \overline{A_1}, \overline{A_2}, \ldots, \overline{A_i}, \ldots, A_n \rangle x_i$
   for all $1 < i < n$
   $\langle p, \overline{A_1}, \ldots, \overline{A_{n-1}}, A_n \rangle\ _1\# \rightarrow \langle \Delta, \Delta \rangle x_n$
   for every rule p: $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \ldots, A_n \rightarrow x_n) \in M$.

Observation

$$L(G) = L(H)$$

Conclusion

$$\mathscr{L}(n\text{-}LIN\#RS) = \mathscr{L}(SM, LIN, n)$$

Claim

$$\mathscr{L}_n(CF\#RS) \subseteq \mathscr{L}(SM, CF, \textit{left-}n)$$

Proof

Let $H = (Q, \Sigma, s, R)$ be a #-rewriting system of index n.
Construct a context-free simple matrix grammar,

$$G = (N_1, \ldots, N_n, \Sigma, M, \langle s \rangle),$$

$N_i \subseteq (Q \times \mathbb{N}_0)$ for all $1 \leq i \leq n$. M is constructed as follows:

1. $(\langle s \rangle \rightarrow \langle s, 1 \rangle \langle s, 0 \rangle^{n-1})$
2. – 4. Defined analogously with LIN grammars
5. $(u, \langle p, i \rangle \rightarrow \langle q, i \rangle \ldots \langle q, i + i' \rangle, v)$ for every $p\,{}_i\# \rightarrow q\, w\#^{i'+1} \in R$
   - u defined analogously with LIN grammars
   - $v = (\langle p, j'_1 \rangle \rightarrow w_{j'_1}, \ldots, \langle p, j'_n \rangle \rightarrow w_{j'_n})$ $\quad i + i' < j'_{k'} \leq n \vee j'_{k'} = 0$
   - i' leftmost $\langle p, 0 \rangle$ nonterminals in v are erased, $\langle p, 0 \rangle \rightarrow \varepsilon$

Proof Contd.

6. $(u, \langle p, i\rangle \to \langle q, 0\rangle, v)$ for every $p\ {}_i\# \to q \in R$
   * u defined analogously with LIN grammars
   * $v = (\langle p, j_1'\rangle \to \langle q, i\rangle, \ldots, \langle p, j_n'\rangle \to \langle p, j_{n-1}'\rangle) \qquad i \leq j_{k'}' \leq n \vee j_{k'}' = 0$
7. $(u, \langle p, 0\rangle \to \varepsilon, v)$
   * u, v defined analogously with LIN grammars
8. $(u, \langle p, i\rangle \to \langle p, i\rangle\langle p, 0\rangle, v)$
   * u, v defined analogously with LIN grammars

Observation

$$L(H) = L(G)$$

* Proof of $\mathscr{L}(SM, CF, left\text{-}n) \subseteq \mathscr{L}_n(CF\#RS)$ will be omitted.

Conclusion

$$\mathscr{L}(SM, CF, left\text{-}n) = \mathscr{L}_n(CF\#RS)$$

Conclusion

Overview

- Simple Matrix Grammars – infinite hierarchy
- #-Rewriting Systems
- Equivalence of presented families
- Existence of infinite hierarchy for LIN#RS, CF#RS

Future Research

- Relation between CF#RS and non-leftmost SMG

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