Multi-Island Finite Automata and Their Even Computation

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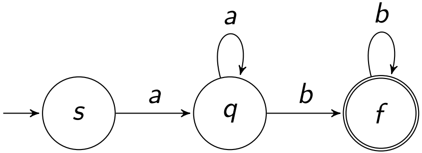
The GFA

$$M = (\{s, q, f\}, \{a, b\}, \{sa \to q, qa \to q, qb \to f, fb \to b\}, s, f)$$

can be represented as:



The language accepted by this automaton is

$$L(M) = \{a^n b^m \mid n, m \geq 1\}.$$

Finite Automata: Definition

A *generalized finite automaton* (GFA) is a 5-tuple $M = (Q, \Sigma, R, s, f)$, where

- Q – a finite *set of states*,
- Σ – a finite, nonempty *input alphabet*,
- $R \subseteq Q \times \Sigma^* \times Q$ – a finite set of *rules*:
  - $(p, w, q) \in R$ written as $pw \to q$,
- $s \in Q$ – the *initial state*,
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Note these peculiarities:

- The model is non-deterministic;
- The production rules allow reading entire strings;
- There is only a single final state.

Transition Graph of a GFA

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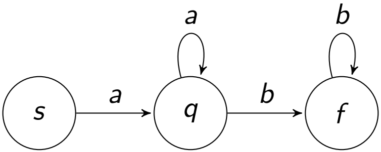
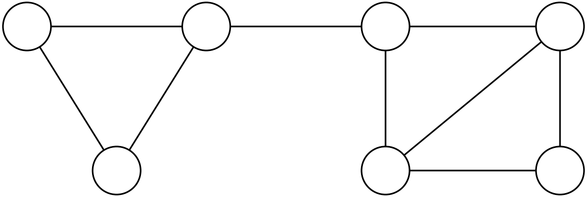


- $V = \{s, q, f\}$,
- $E = \{(s, q), (q, q), (q, f), (f, f)\}$,
  - $W(s, q) = \{a\}$,
  - $W(q, q) = \{a\}$,
  - $W(q, f) = \{b\}$,
  - $W(f, f) = \{b\}$

Bridges and Islands

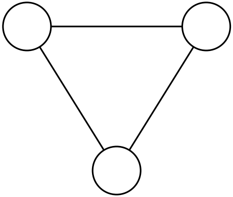
Connected graph

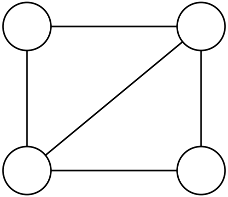
Connected graph: Any two nodes are connected by an undirected path.



Disconnected graph

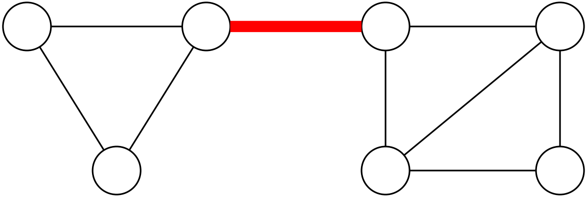
Connected graph: Any two nodes are connected by an undirected path.





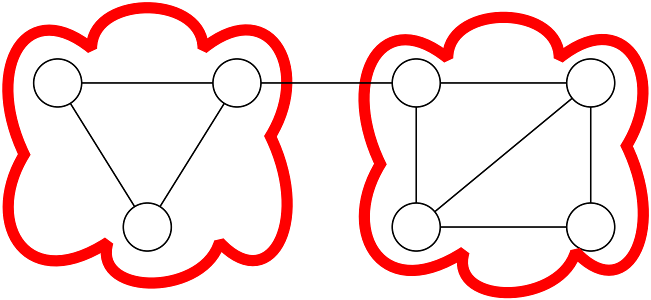
Bridge

Bridge: an edge such that when it is removed, the graph is no longer connected.



Island

A *bridgeless island* = a maximal bridgeless connected component



Every node and edge is either a bridge or contained in exactly one bridgeless island.

Islands in Automata

Islands in Automata: The Structure

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- Sketch of Proof:
  1. Think of an "island graph" – the nodes are islands, the edges are bridges;
  2. This graph is necessarily a tree;
  3. There must be exactly one path between I_s and I_f;
  4. All states are useful, so all islands must lie on this path.

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Islands in Automata: Number Variability

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- Idea of proof:
  - Redundant states and transitions can merge existing islands and create new ones.

Islands in Automata: An abstraction

- a *k-bridge island* in G:
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  - the merging of $k+1$ bridgeless islands and their connecting bridges

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     - $\binom{b}{n-1}$ ways to select n islands in a GFA with b bridges.

n-Island GFA

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- Sketch of proof:
  1. n-IGFA are special cases of GFA;
  2. A GFA along with $\Gamma = \emptyset$ is a 1-IGFA;
  3. An n-IGFA can be transformed into an equivalent m-IGFA.

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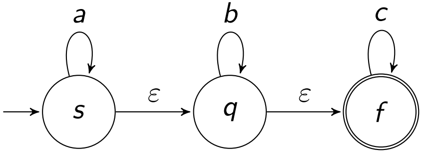
- An n-IGFA accepts the same language as the underlying GFA. . .
- . . . unless we add an additional constraint to their computation:
- A computation of an n-IGFA is *even* if **the same number of steps is taken in each island**.

Even Computations: Example (1/2)



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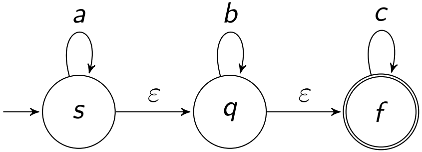

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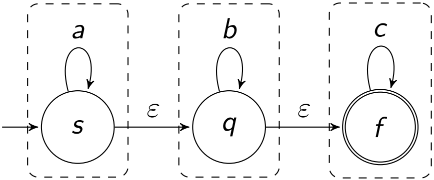

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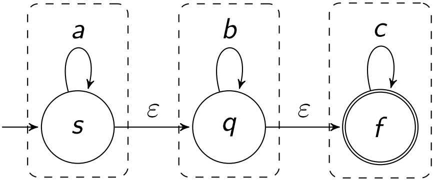

- Note: ε denotes the *empty string*;
- Let us consider islands defined by the bridges $\Gamma = \{(s, q), (q, f)\}$:
- The language accepted by this automaton *by even computations with regard to* Γ is

$$L_e(M, \Gamma) = \{a^n b^n c^n \mid n \geq 0\};$$

- $L_e(M, \Gamma) \in \text{CS} \setminus \text{CF}$.

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- Equivalent power to *n-parallel right linear grammars* (n-PRLG):
  - (N, Σ, P, S);
  - P contains rules of the forms:
    a) $S \rightarrow x$, where $x \in \Sigma^*$,
    b) $S \rightarrow A_1 \cdots A_n$, where $A_i \in N$,
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  - All nonterminals rewritten at once;
- We denote the class of languages generated by n-PRLGs by PRL_n.

n-PRLG: Example

- $G = (\{S, A, B\}, \{a, b\}, P, S)$,
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- In general, how do we deal with different initial rules of an n-PRLG?

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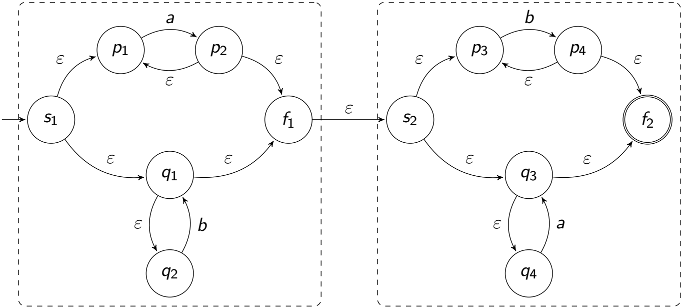

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  - Entry state s_i and exit state f_i,
  - States $\langle i, j \rangle$ where $1 \leq j \leq m$ for the initial generation of remainder j,
  - States $\langle A, i, j \rangle$ where $1 \leq j \leq m$ and $A \in N \setminus \{S\}$ to represent nonterminal A simulated with predetermined remainder j,

Proof: $\mathrm{PRL}_n \subseteq \mathcal{L}_e(\mathrm{GFA}_n)$ – General Case (Sketch, 1/2)

- Let m be the number of starting production rules of the input grammar of the form $S \to A_1 \cdots A_n$;
- Associate each of these starting rules with a remainder modulo $m+1$ with the remainder 0 reserved for starting rules of the form $S \to x$, $x \in \Sigma^*$;
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  - States $\langle A, i, j \rangle$ where $1 \leq j \leq m$ and $A \in N \setminus \{S\}$ to represent nonterminal A simulated with predetermined remainder j,
  - States $\langle i, j, k, B \rangle$ where $1 \leq j, k \leq m$ and $B \in (N \setminus \{S\}) \cup \{\varepsilon\}$, which use k as a counter to drag out the rewriting of A to B to $m+1$ moves.

Proof: $\mathrm{PRL}_n \subseteq \mathcal{L}_e(\mathrm{GFA}_n)$ – General Case (Sketch, 2/2)

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  - Rules to pair a given remainder with the corresponding computation within each island,
    - $\langle 1, m\rangle \to f_1$, $\langle i, m\rangle \to f_i$ for $S \to x \in P$,
    - $s_i \to \langle A_{i1}, i, 1\rangle$, $\langle i, j-1\rangle \to \langle A_{ij}, i, j\rangle$ for $p_j : S \to A_{1j} \cdots A_{nj} \in P$;

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  - Rules to simulate grammar rules of the form $A \to xB$ and $A \to x$, $A, B \in N$, $x \in \Sigma^*$, along with ε-rules ensuring that each rule is simulated in exactly $m+1$ steps,
    - $\langle A, i, j\rangle x \to \langle i, j, 1, B\rangle$ for $A \to xB \in P$,
    - $\langle i, j, k, B\rangle \to \langle i, j, k+1, B\rangle$, $\langle i, j, m, B\rangle \to \langle B, i, j\rangle$, $\langle i, j, m, \varepsilon\rangle \to f_i$;

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    - $\langle i, j, k, B\rangle \to \langle i, j, k+1, B\rangle$, $\langle i, j, m, B\rangle \to \langle B, i, j\rangle$, $\langle i, j, m, \varepsilon\rangle \to f_i$;
  - Bridge rules:
    - $f_i \to s_{i+1}$.

Corollary: $\mathrm{PRL}_n = \mathcal{L}_e(\mathrm{GFA}_n)$

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- Proof: See previous slides

Accepting Power

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- $\mathrm{PRL}_2 \subset \mathrm{CF}$;
- $\mathrm{PRL}_n \not\subseteq \mathrm{CF}$, $\mathrm{CF} \not\subseteq \mathrm{PRL}_n$, $n \geq 3$;

Accepting Power

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- $\mathrm{REG} = \mathrm{PRL}_1 \subset \mathrm{PRL}_k \subset \mathrm{PRL}_{k+1} \subset \mathrm{CS}$ for any $k > 1$;
- $\mathrm{PRL}_2 \subset \mathrm{CF}$;
- $\mathrm{PRL}_n \not\subseteq \mathrm{CF}$, $\mathrm{CF} \not\subseteq \mathrm{PRL}_n$, $n \geq 3$;
- Finally, $\mathrm{PRL}_n = \mathcal{L}_e(\mathrm{GFA}_n)$ for all $n \geq 1$.

Accepting Power: Summary

- $\mathcal{L}_e(\mathrm{GFA}_n)$ equivalent to languages generated by n-PRLGs:
  - An infinite hierarchy between REG and CS;
  - For $n \geq 3$ incomparable with CF.
- For compactness, EI_n will denote $\mathcal{L}_e(\mathrm{GFA}_n)$ in the following diagram: