Self-Regulating Finite Automata

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Outline

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Outline

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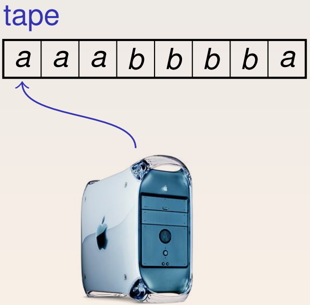
Finite Automata—Concept



Characteristics

- Finite state control
- Input cannot be modified
- Head only moves forward

Finite Automata—Definition

Definition

A finite automaton is a quintuple

$$M = (Q, \Sigma, \delta, q_0, F)$$

where

- Q is a finite set of states,
- Σ is an input alphabet,
- δ is a finite set of rules,
- $q_0 \in Q$ is an initial state,
- $F \subseteq Q$ is a set of final states.

Finite Automata—Language

Definition

A configuration is any member of $Q\Sigma^*$.

If

$$qwy \in Q\Sigma^* \text{ and } r.qw \rightarrow p \in \delta,$$

then

$$qwy \Rightarrow py\,[r].$$

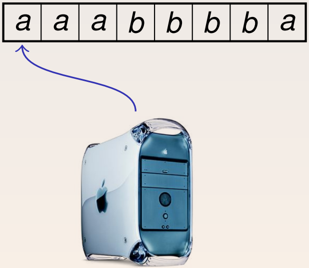
The language of M is the set

$$\mathscr{L}(M) = \{w \in \Sigma^* : q_0 w \Rightarrow^* f,\ f \in F\},$$

where $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$.

Finite Automata—Example

tape



Current state: q_a

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- δ contains
  - $1. q_a a \rightarrow q_a$
  - $2. q_a b \rightarrow q_b$
  - $3. q_b b \rightarrow q_b$
- M starts in q_a
- $F = Q$
- M cannot read the last a
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	a

Current state: q_a

Description

- $Q = \{q_a, q_b\}$
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  - $1.q_a a \to q_a$
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Finite Automata—Example

tape

a	a	a	b	b	b	b	a

Current state: q_a

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- δ contains
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- M starts in q_a
- $F = Q$
- M cannot read the last a
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	a

Current state: q_b

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- δ contains
  - $1. q_a a \rightarrow q_a$
  - $2. q_a b \rightarrow q_b$
  - $3. q_b b \rightarrow q_b$
- M starts in q_a
- $F = Q$
- M cannot read the last a
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	a

Current state: q_b

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- δ contains
  - $1. q_a a \rightarrow q_a$
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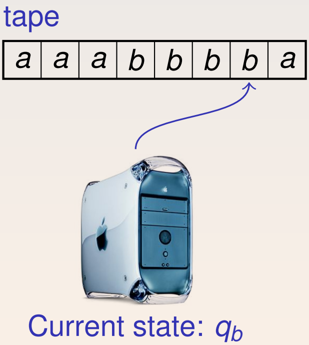

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Finite Automata—Example

tape

a	a	a	b	b	b	b	a

Current state: q_b

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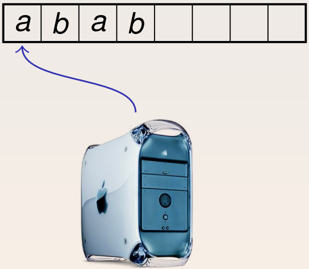
Outline

Self-Regulation—The Main Idea

tape

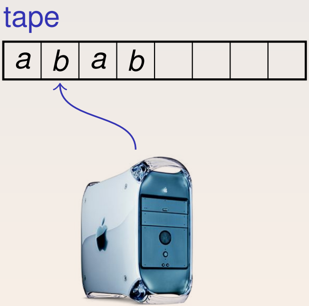


state: s, gen. rules: 4,5,6

Description

- δ contains
  - $(1.sa \rightarrow s, \{4\})$
  - $(2.sb \rightarrow s, \{5\})$
  - $(3.s \rightarrow t, \{6\})$
  - $(4.ta \rightarrow t, \emptyset)$
  - $(5.tb \rightarrow t, \emptyset)$
  - $(6.t \rightarrow f, \emptyset)$
- f is the final state
- t is the turn state
- $\mathscr{L}(M) = \{ww : w \in \{a,b\}^*\}$

Self-Regulation—The Main Idea



state: s, gen. rules: 4

Description

- δ contains
  - $(1.sa \rightarrow s, \{4\})$
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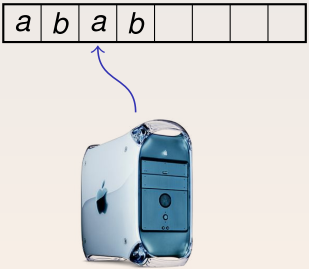
Self-Regulation—The Main Idea

tape



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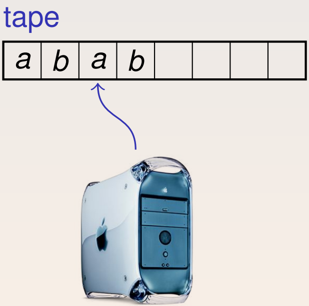

state: t, gen. rules: 4,5,6

Automaton makes a turn

Description

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- f is the final state
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Self-Regulation—The Main Idea

tape

a	*b*	*a*	*b*				

↑

state: t, gen. rules: 4,5,6

Description

- δ contains
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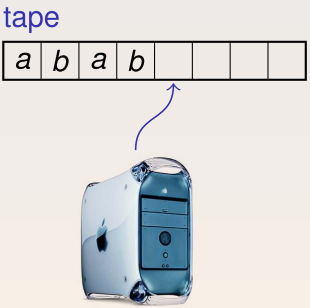
Self-Regulation—The Main Idea



state: t, gen. rules: 4,5,6

Description

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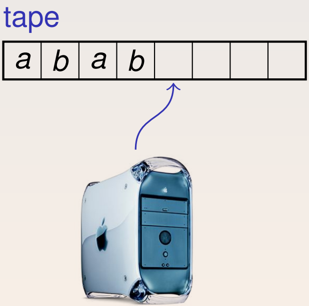
Self-Regulation—The Main Idea



state: f, gen. rules: 4,5,6

Description

- δ contains
  - $(1.sa \rightarrow s, \{4\})$
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  - $(3.s \rightarrow t, \{6\})$
  - $(4.ta \rightarrow t, \emptyset)$
  - $(5.tb \rightarrow t, \emptyset)$
  - $(6.t \rightarrow f, \emptyset)$
- f is the final state
- t is the turn state
- $\mathscr{L}(M) = \{ww : w \in \{a, b\}^*\}$

Outline

Self-Regulating Finite Automata

Definition

Let

$$N = (Q, \Sigma, \delta, q_0, F)$$

be a finite automaton.
A self-regulating finite automaton, SFA, is a triple

$$M = (N, q_t, R),$$

where

1. $q_t \in Q$ is a turn state, and
2. $R \subseteq \Psi \times \Psi$ is a finite relation on the alphabet of rule labels.

Outline

All-Move Self-Regulating Finite Automata

Definition

For $n \geq 0$, an SFA M is an *n-turn all-move SFA*, n-all-SFA, if M accepts w as follows. There is $q_0 w \Rightarrow^* f[\mu]$, $f \in F$, such that

$$\mu = \underbrace{r_1^0 r_2^0 \dots r_k^0}_{k \text{ rules}} \underbrace{r_1^1 r_2^1 \dots r_k^1}_{k \text{ rules}} \dots \underbrace{r_1^n r_2^n \dots r_k^n}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, r_k^0 is the first rule of the form $qx \rightarrow q_t$, for some $q \in Q$, $x \in \Sigma^*$, and

$$(r_i^j, r_i^{j+1}) \in R$$

for all $1 \leq i \leq k,\ 0 \leq j < n$.
The family of languages accepted by n-all-SFAs is denoted S_n.

Outline

First-Move Self-Regulating Finite Automata

Definition

For $n \geq 0$, an SFA M is an *n-turn first-move SFA*, n-first-SFA, if M accepts w as follows. There is $q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, such that

$$\mu = \underbrace{r_1^0 \dots r_k^0}_{k \text{ rules}} \underbrace{r_1^1 \dots r_k^1}_{k \text{ rules}} \dots \underbrace{r_1^n \dots r_k^n}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, r_k^0 is the first rule of the form $qx \rightarrow q_t$, for some $q \in Q$, $x \in \Sigma^*$, and

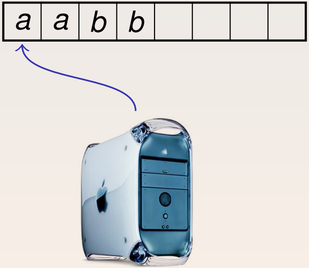
$$(r_1^j, r_1^{j+1}) \in R$$

for all $0 \leq j < n$.

The family of languages accepted by n-first-SFAs is denoted W_n.

First-Move SFA—Example

tape

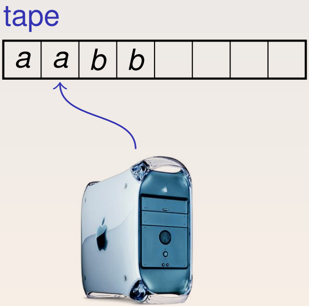


state: s, gen. rules: 3, –

Description

- δ contains
  - $(1.sa \rightarrow s, \{3\})$
  - $(2.sa \rightarrow t, -)$
  - $(3.tb \rightarrow f, -)$
  - $(4.fb \rightarrow f, -)$
- f is the final state
- t is the turn state
- $\mathscr{L}(M) = \{a^n b^n : n \geq 1\}$

First-Move SFA—Example



state: s, gen. rules: 3

Description

- δ contains
  - $(1.sa \rightarrow s, \{3\})$
  - $(2.sa \rightarrow t, -)$
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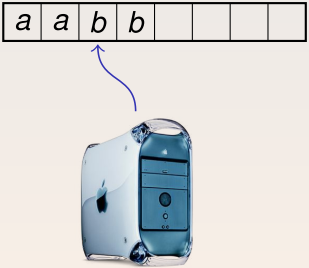
First-Move SFA—Example

tape



state: t, gen. rules: 3,-

Automaton makes a turn

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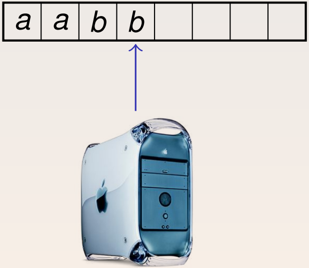
First-Move SFA—Example

tape



state: f, gen. rules: 3,-

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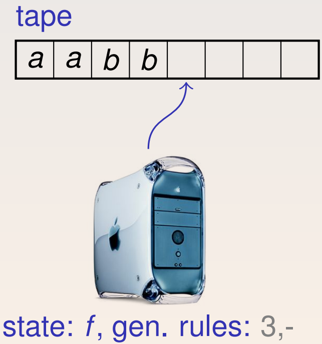
First-Move SFA—Example



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Outline

Parallel Right-Linear Grammars (PRLG)

Definition

For $n > 0$, an n-PRLG is an $(n+3)$-tuple

$$G = (N_1, \dots, N_n, T, S, P),$$

where

- N_i are mutually disjoint nonterminal alphabets, $1 \leq i \leq n$,
- T is a terminal alphabet,
- $S \notin N_1 \cup \dots \cup N_n$,
- P contains three kinds of rules:
  1. $S \to X_1 \dots X_n$ $X_i \in N_i$, $1 \leq i \leq n$,
  2. $X \to wY$ $X, Y \in N_i$ for some i, $1 \leq i \leq n$, $w \in T^*$, and
  3. $X \to w$ $X \in N$, $w \in T^*$.

PRLG—Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $S \rightarrow y \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\quad\quad\downarrow\quad\downarrow\dots\quad\downarrow$
   $y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n$
   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $X_i \rightarrow x_i \in P$, $1 \leq i \leq n$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

$R_n = \{\mathscr{L}(G) : G \text{ is an } n\text{-PRLG}\}$.

PRLG—Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $S \rightarrow y \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$

   $\downarrow \quad \downarrow \dots \quad \downarrow$

   $y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n$

   $y_i \in T^*$, $x_i \in T^* N \cup T^*$, $X_i \in N_i$, $X_i \rightarrow x_i \in P$, $1 \leq i \leq n$.

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PRLG—Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

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if and only if

1. either $x = S$ and $S \rightarrow y \in P$,
2. $x = y_1 X_1 y_2 X_2 \ldots y_n X_n$

   $\downarrow \quad \downarrow \ldots \quad \downarrow$

   $y = y_1\, x_1\, y_2\, x_2 \ldots y_n\, x_n$

   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $X_i \rightarrow x_i \in P$, $1 \leq i \leq n$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

$R_n = \{\mathscr{L}(G) : G$ is an n-PRLG$\}$.

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \rightarrow AB$
- $A \rightarrow aA$
- $A \rightarrow a$
- $B \rightarrow bB$
- $B \rightarrow b$

Consider a derivation in G:

$$S$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \rightarrow AB$
- $A \rightarrow aA$
- $A \rightarrow a$
- $B \rightarrow bB$
- $B \rightarrow b$

Consider a derivation in G:

$$S \Rightarrow AB$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \rightarrow AB$
- $A \rightarrow aA$
- $A \rightarrow a$
- $B \rightarrow bB$
- $B \rightarrow b$

Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

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Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aaAbbB$$

PRLG—Example

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Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aaAbbB \Rightarrow a^3b^3$$

PRLG—Example

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Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $S \rightarrow AB$
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- $A \rightarrow a$
- $B \rightarrow bB$
- $B \rightarrow b$

Consider a derivation in G:

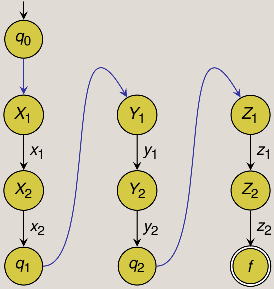
$$\mathscr{L}(G) = \{a^n b^n : n \geq 1\}$$

Lemma

Let G be a 3-PRLG. There is a 2-first-SFA M s. t. $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea.

$$S \Downarrow X_1 Y_1 Z_1 \Downarrow x_1 X_2 y_1 Y_2 z_1 Z_2 \Downarrow x_1 x_2 y_1 y_2 z_1 z_2$$



□

Lemma

Let M be a 2-first-SFA. There is a 3-PRLG G s. t. $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
$(q_ty_0 \to r_1), (r_1y_1 \to r_2), (r_2y_2 \to q_i),$
$(q_iz_0 \to s_1), (s_1z_1 \to s_2), (s_2z_2 \to q_f)$

be an acceptance of $x_0x_1x_2y_0y_1y_2z_0z_1z_2z_2$ in M. Then,

$$
\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
&\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\
&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
&\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\
&\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2
\end{aligned}
$$

□

Lemma

Let M be a 2-first-SFA. There is a 3-PRLG G s. t. $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea.

Let
$$\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$$
$$(q_ty_0 \to r_1), (r_1y_1 \to r_2), (r_2y_2 \to q_i),$$
$$(q_iz_0 \to s_1), (s_1z_1 \to s_2), (s_2z_2 \to q_f)$$

be an acceptance of $x_0x_1x_2y_0y_1y_2z_0z_1z_2z_2$ in M. Then,

$$\begin{aligned} S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\ &\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\ &\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\ &\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\ &\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2 \end{aligned}$$

□

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Let M be a 2-first-SFA. There is a 3-PRLG G s. t. $\mathscr{L}(G) = \mathscr{L}(M)$.

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$$
\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
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&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
&\Rightarrow x_0x_1x_2[q_t, 0, q_t]y_0y_1y_2[q_i, 1, q_i]z_0z_1z_2[q_f, 2, q_f] \\
&\Rightarrow x_0x_1x_2y_0y_1y_2z_0z_1z_2
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Proof idea.

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$(q_t y_0 \rightarrow r_1), (r_1 y_1 \rightarrow r_2), (r_2 y_2 \rightarrow q_i),$
$(q_i z_0 \rightarrow s_1), (s_1 z_1 \rightarrow s_2), (s_2 z_2 \rightarrow q_f)$

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$$
\begin{aligned}
S &\Rightarrow [q_0 x_0, q_1, 0, q_t][q_t y_0, r_1, 1, q_i][q_i z_0, s_1, 2, q_f] \\
&\Rightarrow x_0 [q_1, 0, q_t] y_0 [r_1, 1, q_i] z_0 [s_1, 2, q_f] \\
&\Rightarrow x_0 x_1 [q_2, 0, q_t] y_0 y_1 [r_2, 1, q_i] z_0 z_1 [s_2, 2, q_{i_2}] \\
&\Rightarrow x_0 x_1 x_2 [q_t, 0, q_t] y_0 y_1 y_2 [q_i, 1, q_i] z_0 z_1 z_2 [q_f, 2, q_f] \\
&\Rightarrow x_0 x_1 x_2 y_0 y_1 y_2 z_0 z_1 z_2
\end{aligned}
$$

□

PRLG vs. first-SFA

Lemma

Let G be an n-PRLG. There is an $(n-1)$-first-SFA M such that $\mathscr{L}(G) = \mathscr{L}(M)$.

Lemma

Let M be an n-first-SFA. There is an $(n+1)$-PRLG G such that $\mathscr{L}(G) = \mathscr{L}(M)$.

Theorem

For all $n \geq 0$, $W_n = R_{n+1}$.

Language Families of First-Move SFAs

Corollary

1. $REG = W_0 \subset W_1 \subset W_2 \subset \cdots \subset CS$.
2. $W_1 \subset CF$.
3. $W_2 \not\subseteq CF$.
4. $CF \not\subseteq W_n$ *for any* $n \geq 0$.
5. *For all* $n \geq 0$, W_n *is closed under union, finite substitution, homomorphism, intersection with a regular language and right quotient with a regular language.*
6. *For all* $n \geq 1$, W_n *is not closed under intersection,complement, and inverse homomorphism.*

Outline

Right-Linear Simple Matrix Grammar (RLSMG)

Definition

For $n > 0$, an n-RLSMG is an $(n+3)$-tuple

$$G = (N_1, \dots, N_n, T, S, P),$$

where

- N_i are mutually disjoint nonterminal alphabets, $1 \le i \le n$,
- T is a terminal alphabet,
- $S \notin N_1 \cup \dots \cup N_n$,
- P contains three kinds of matrix rules:
  1. $[S \to X_1 \dots X_n]$ $\quad X_i \in N_i, 1 \le i \le n$,
  2. $[X_1 \to w_1 Y_1, \dots, X_n \to w_n Y_n]$ $\quad w_i \in T^*, X_i, Y_i \in N_i, 1 \le i \le n$,
  3. $[X_1 \to w_1, \dots, X_n \to w_n]$ $\quad X_i \in N_i, w_i \in T^*, 1 \le i \le n$.

RLSMG–Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $[S \rightarrow y] \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\downarrow \quad \downarrow \dots \quad \downarrow$
   $y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n$
   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $1 \leq i \leq n$,
   $[X_1 \rightarrow x_1, \dots, X_n \rightarrow x_n] \in P$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ is defined as usual.

$R_{[n]} = \{\mathscr{L}(G) : G$ is an n-RLSMG$\}$.

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RLSMG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P containes rules

- $[S \rightarrow AB]$
- $[A \rightarrow aA, B \rightarrow aB]$
- $[A \rightarrow bA, B \rightarrow bB]$
- $[A \rightarrow \varepsilon, B \rightarrow \varepsilon]$

Consider a derivation in G:

$$S$$

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Consider a derivation in G:

$$S \Rightarrow AB$$

RLSMG—Example

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Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAaB$$

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RLSMG—Example

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- $[A \rightarrow aA, B \rightarrow aB]$
- $[A \rightarrow bA, B \rightarrow bB]$
- $[A \rightarrow \varepsilon, B \rightarrow \varepsilon]$

Consider a derivation in G:

$$\mathscr{L}(G) = \{ww : w \in \{a, b\}^*\}$$

RLSMG vs. all-SFA

Lemma

Let G be an n-RLSMG. There is an $(n-1)$-all-SFA M such that $\mathscr{L}(G) = \mathscr{L}(M)$.

Lemma

Let M be an n-all-SFA. There is an $(n+1)$-RLSMG G such that $\mathscr{L}(G) = \mathscr{L}(M)$.

Theorem

For all $n \geq 0$, $S_n = R_{[n+1]}$.

Language Families of All-Move SFAs

Corollary

1. $REG = S_0 \subset S_1 \subset S_2 \subset \cdots \subset CS$.
2. $S_1 \not\subseteq CF$.
3. $CF \not\subseteq S_n$, for $n \geq 0$.
4. For all $n \geq 0$, S_n is closed under union, finite substitution, homomorphism, intersection with a regular language, right quotient with a regular language, and inverse homomorphism.
5. For all $n \geq 0$, S_n is a full trio.
6. For all $n \geq 1$, S_n is not closed under intersection, and complement.

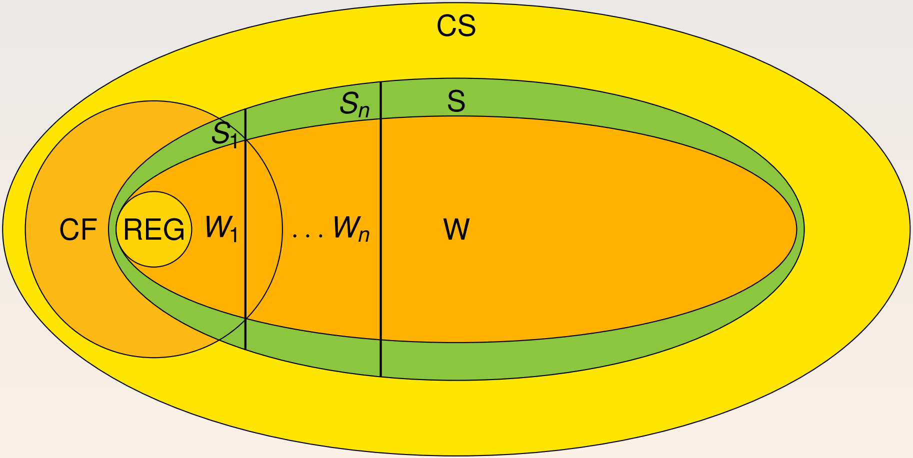
Outline

Comparison

Theorem

1. $W_0 = S_0 = REG$.
2. For all $n > 0$, $W_n \subset S_n$.
3. $W_n \not\subseteq S_{n-1}$, $n \geq 1$.
4. $S_n - W \neq \emptyset$, $n \geq 1$, where $W = \bigcup_{m=1}^{\infty} W_m$.

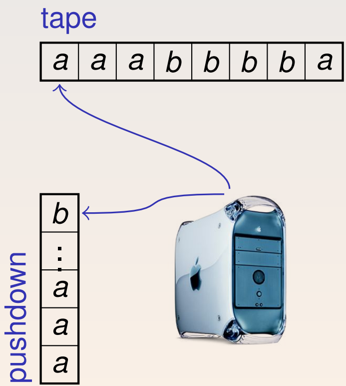
Comparison



W_n is the family of languages accepted by n-first-SFAs
S_n is the family of languages accepted by n-all-SFAs

Outline

Pushdown Automata—Concept



Characteristics

- Finite state control
- Input cannot be modified
- Head only moves forward
- Potentially infinite pushdown store
- Pushdown top can be modified
- Pushdown-head reads the top symbol

Pushdown Automata—Definition

Definition

A pushdown automaton is a quintuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F),$$

where

- Q is a finite set of states,
- Σ is an input alphabet,
- Γ is a pushdown alphabet,
- δ is a finite set of rules,
- $q_0 \in Q$ is an initial state,
- Z_0 is an initial pushdown symbol,
- $F \subseteq Q$ is a set of final states.

Pushdown Automata—Language

Definition

A configuration is a member of $\Gamma^* Q \Sigma^*$.
If

$$qwy \in \Gamma^* Q \Sigma^* \text{ and } r.Aqw \rightarrow \gamma p \in \delta,$$

then

$$xAqwy \Rightarrow x\gamma py\,[r]$$

The language of M is the set

$$\mathscr{L}(M) = \{w \in \Sigma^* : Z_0 q_0 w \Rightarrow^* f,\ f \in F\}.$$

Self-Regulating Pushdown Automata

Definition

Let

$$N = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

be a pushdown automaton.
A self-regulating pushdown automaton, SPDA, is a triple

$$M = (N, q_t, R),$$

where

1. $q_t \in Q$ is a turn state, and
2. $R \subseteq \Psi \times \Psi$ is a finite relation, where Ψ is an alphabet of rule labels.

All-Move Self-Regulating Pushdown Automata

Definition

For $n \geq 0$, an SPDA M is n-turn all-move SPDA, n-all-SPDA, if M accepts w as follows. There is $Z_0 q_0 w \Rightarrow^* f[\mu]$, $f \in F$, such that

$$\mu = \underbrace{r_1^0 r_2^0 \dots r_k^0}_{k \text{ rules}} \underbrace{r_1^1 r_2^1 \dots r_k^1}_{k \text{ rules}} \dots \underbrace{r_1^n r_2^n \dots r_k^n}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, r_k^0 is the first rule of the form $Zqx \rightarrow \gamma q_t$, for some $Z \in \Gamma$, $q \in Q$, $x \in \Sigma^*$, $\gamma \in \Gamma^*$, and

$$(r_i^j, r_i^{j+1}) \in R$$

for all $1 \leq i \leq k$, $0 \leq j < n$.
The family of languages accepted by n-all-SPDAs is denoted $\mathscr{L}(n\text{-all-SPDA})$.

First-Move Self-Regulating Pushdown Automata

Definition

For $n \geq 0$, an SPDA M is n-turn first-move SPDA, n-first-SPDA, if M accepts w as follows. There is $Z_0 q_0 w \Rightarrow^* f[\mu]$, $f \in F$, such that

$$\mu = \underbrace{r_1^0 r_2^0 \dots r_k^0}_{k \text{ rules}} \underbrace{r_1^1 r_2^1 \dots r_k^1}_{k \text{ rules}} \dots \underbrace{r_1^n r_2^n \dots r_k^n}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, r_k^0 is the first rule of the form $Zqx \rightarrow \gamma q_t$, for some $Z \in \Gamma$, $q \in Q$, $x \in \Sigma^*$, $\gamma \in \Gamma^*$, and

$$(r_1^j, r_1^{j+1}) \in R$$

for all $0 \leq j < n$.
The family of languages accepted by n-first-SPDAs is denoted $\mathscr{L}(n\text{-first-SPDA})$.

All-Move Self-Regulating Pushdown Automata

Theorem

$\mathscr{L}(0\text{-}\mathit{all}\text{-}\mathit{SPDA}) = \mathit{CF}$

Proof.

This is clear. □

All-Move Self-Regulating Pushdown Automata

Theorem

$\mathscr{L}(1\text{-}all\text{-}SPDA) = RE$.

Proof Idea.

$L \in RE$, then there are CFGs G and H such that

$$L = h(\mathscr{L}(G) \cap \mathscr{L}(H)).$$

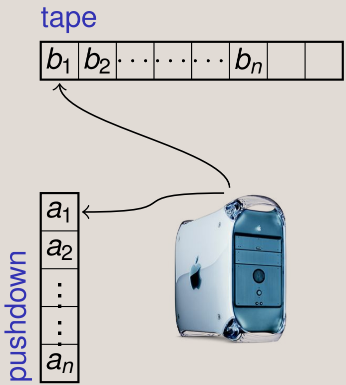
1-all-SPDA M simulates, on the pushdown,

1. a derivation in G. If a is on the top, M reads $h(a)$ from the input tape.
2. a derivation in H generating the same string (according to the ralation of M). M reads no input.

□

All-Move Self-Regulating Pushdown Automata

Proof Idea.



Characteristics

- $b_i = h(a_i)$
- check if $a_1 a_2 \ldots a_n \in \mathscr{L}(G)$
- if so, $h(a_1 a_2 \ldots a_n) \in h(\mathscr{L}(G))$
- check if $a_1 a_2 \ldots a_n \in \mathscr{L}(H)$
- if so,
  $a_1 a_2 \ldots a_n \in \mathscr{L}(G) \cap \mathscr{L}(H)$

□

All-Move Self-Regulating Pushdown Automata

Proof Idea—Construction.

$$M = (\{q_0, q, q_t, p, f\}, \Delta, \Sigma \cup N_G \cup N_H \cup \{Z\}, \delta, q_0, Z, \{f\}, R)$$

$Z \notin \Sigma \cup N_G \cup N_H$, with R and δ made as

1. add $(Zq_0 \rightarrow ZS_Gq, Zq_t \rightarrow ZS_Hp)$ to R
2. add $(Aq \rightarrow B_n \dots B_1 aq, Cp \rightarrow D_m \dots D_1 ap)$ to R if
   $A \rightarrow aB_1 \dots B_n \in P_G$ and
   $C \rightarrow aD_1 \dots D_m \in P_H$
3. add $(aqh(a) \rightarrow q, ap \rightarrow p)$ to R
4. add $(Zq \rightarrow Zq_t, Zp \rightarrow f)$ to R

□

Open Problems

	Clearly, $\mathscr{L}$(0-first-SPDA) $= CF$.
1	What is $\mathscr{L}$(n-first-SPDA), for $n \geq 1$?
2	Determinism.
3	Closure properties under other operations.

Related Publications

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