Lindenmayer Systems

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0L System

0L System

An 0L system (0 stands for zero-sided context, i.e. context-free productions) is a triple

$$G = (T, P, w)$$

where

T is an alphabet

P is a finite set of productions of the form

$$a \rightarrow x$$

with $a \in T$ and $x \in T^*$

w is the start string (axiom), $w \in T^+$

D0L and P0L System

D0L System

If for each $a \in T$ there is exactly one production

$$a \rightarrow x \in P,$$

then G is a D0L system (D stands for Deterministic)

P0L System

If for each $a \rightarrow x \in P$,

$$x \neq \varepsilon,$$

then G is a P0L system (P stands for Propagating)

0L System – Generated Language

Direct Derivation

For some $n \geq 1$,

$$a_1 a_2 \dots a_n \Rightarrow x_1 x_2 \dots x_n$$

if for each $i = 1, \dots, n$,

$$a_i \rightarrow x_i \in P$$

Generated Language

For an L system $G = (T, P, w)$,

$$L(G) = \{y : w \Rightarrow^* y\}$$

0L System – Length Set

Length set of L

$$|L| = \{|x| \, : \, x \in L\}$$

Example

$G = (\{a, b, c\}, \{a \rightarrow abcc, b \rightarrow bcc, c \rightarrow c\}, a)$

$$a \Rightarrow abcc \Rightarrow abccbcccc \Rightarrow abccbccccbcccccc \ldots$$

$$|L(G)| = \{i^2 \, : \, i \text{ is a natural number}\}$$

0L System – Example I

Example

PD0L system $G = (\{a\}, \{a \rightarrow aa\}, a)$

$$L(G) = \{a^{2^n} : n \geq 0\}$$

Example

0L system $G = (\{a, b\}, \{a \rightarrow b, b \rightarrow ab\}, a)$

$$a \Rightarrow b \Rightarrow ab \Rightarrow bab \Rightarrow abbab \Rightarrow \dots$$

$$|L(G)| = \{i : i \geq 1, i \text{ is a Fibonacci number}\}$$

Every Fibonacci number f_n (for all $n \geq 0$) is defined as

- $f_0 = 0, f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	(	)	#	0
2#3	2	2#4	504	6	7	8(1)	8	(	)	#	0

1 $\qquad\qquad$ $\boxed{1}$

(. . .) branch

8 branch position

0 oblique wall

vertical wall

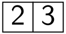
0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	(	)	#	0
2#3	2	2#4	504	6	7	8(1)	8	(	)	#	0

2#3



(...) branch

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2#2#4

2	2	4

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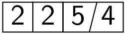
2#2#504



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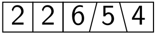
1	2	3	4	5	6	7	8	(	)	#	0
2#3	2	2#4	504	6	7	8(1)	8	(	)	#	0

2#2#60504



(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

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1	2	3	4	5	6	7	8	(	)	#	0
2#3	2	2#4	504	6	7	8(1)	8	(	)	#	0

2#2#7060504 2 | 2 | 7 / 6 \ 5 / 4

(. . .) branch

8 branch position

0 oblique wall

vertical wall

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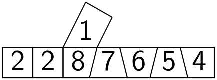
$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	(	)	#	0
2#3	2	2#4	504	6	7	8(1)	8	(	)	#	0

2#2#8(1)07060504



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0L System – Example II

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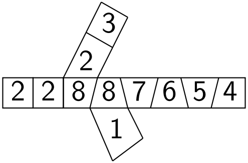
2#3	2	2#4	504	6	7	8(1)	8	(	)	#	0

2#2#8(2#3)08(1)07060504



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0 oblique wall

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0L System – Closure Properties I

Theorem
$\mathscr{L}(0L)$ *is not closed under union.*

Proof
$\{a\} \in \mathscr{L}(0L)$ and $\{aa\} \in \mathscr{L}(0L)$, but

$$\{a, aa\} \notin \mathscr{L}(0L)$$

□

0L System – Closure Properties II

Theorem

$\mathscr{L}(0L)$ is not closed under positive closure $(+)$.

Basic Idea

Set

$$L = \{aa\} \cup \{b^{2^n} : n \geq 2\}$$

and prove that

1. $L \in \mathscr{L}(0L)$
2. $L^+ \notin \mathscr{L}(0L)$

Proof of $L \in \mathscr{L}(0L)$

Set $G = (\{a, b\}, P, aa)$ with $P = \{a \rightarrow bb, b \rightarrow bb\}$. Then,

$$L(G) = L = \{aa\} \cup \{b^{2^n} : n \geq 2\}$$

0L System – Closure Properties III

Proof of $L^+ \notin \mathscr{L}(0L)$

(Proof by contradiction.) Assume that there exists an 0L system

$$G = (\{a, b\}, P, w)$$

such that $L(G) = L^+$. As $\varepsilon \notin L^+$, G is propagating. Thus,

$$w = aa.$$

Consider $a^4 \in L^+$.

1. Assume $a^2 \Rightarrow a^4$.
   - **a** Let $\{a \to a, a \to aaa\} \subseteq P$. Then, $a^2 \Rightarrow b^4$ or $a^4 \Rightarrow b^4$. Thus,

     $$a \to b^i \in P$$

     for some $i \in \{1, 2, 3\}$. Hence, $aa \Rightarrow ab^i$ and $ab^i \notin L^+$ – a contradiction.

0L System – Closure Properties IV

Proof of $L^+ \notin \mathscr{L}(0L)$

1. **b** Assume $a^2 \Rightarrow a^4$ and $a \rightarrow aa \in P$.
   - If $a^2 \Rightarrow b^4$, then
     $$a \rightarrow b^i$$
     for some $i \in \{1, 2, 3\}$. Thus, $a^2 b^i \in L(G)$ – a contradiction.
   - If $a^4 \Rightarrow b^4$,
     $$a \rightarrow b \in P.$$
     Thus, $aab \in L(G)$ – a contradiction.

   c Assume $a^2 \Rightarrow b^4 \Rightarrow a^4$. Then,
   $$\{a \rightarrow bb, b \rightarrow a\} \subseteq P.$$
   Consider any $x \in L(G)$ with $|x| = 6$. Then, $x \in \{a^2 b^4, b^4 a^2, a^6\}$.

0L System – Closure Properties V

Proof of $L^+ \notin \mathscr{L}(0L)$

1 **c** $a^4 \not\Rightarrow x$.

- **A** $a^4 \Rightarrow b^4a^2$.
  If $a \rightarrow a^i \in P$, $i \in \{1, 2\}$, then $aa \Rightarrow a^ib^2 \in L(G)$ – a contradiction.
  If $b \rightarrow b \in P$, then $bbba \in L(G)$ – a contradiction.
- **B** $a^4 \Rightarrow a^2b^4$ – analogy.
- **C** If $a^4 \Rightarrow a^6$, then $bba^i \in L(G)$ – a contradiction.

d $b^4 \not\Rightarrow x$

- **A** $b^4 \Rightarrow b^4a^2$. Then $b \rightarrow b^i \in P$ for some $i \geq 1$. Then, $b^4 \Rightarrow b^{3i}a$ – a contradiction.
- **B** $b^4 \Rightarrow a^2b^4$ – analogy.
- **C** ...

e $a^2 \not\Rightarrow x$.

⋮

□

0L System – Closure Properties VI

Theorem

$\mathscr{L}(0L)$ *is not closed under*

- *homomorphism*
- *inverse homomorphism*
- *intersection and intersection with a regular set*
- *concatenation*
- *complementation*

Theorem

$\mathscr{L}(0L)$ *is closed under reversal.*

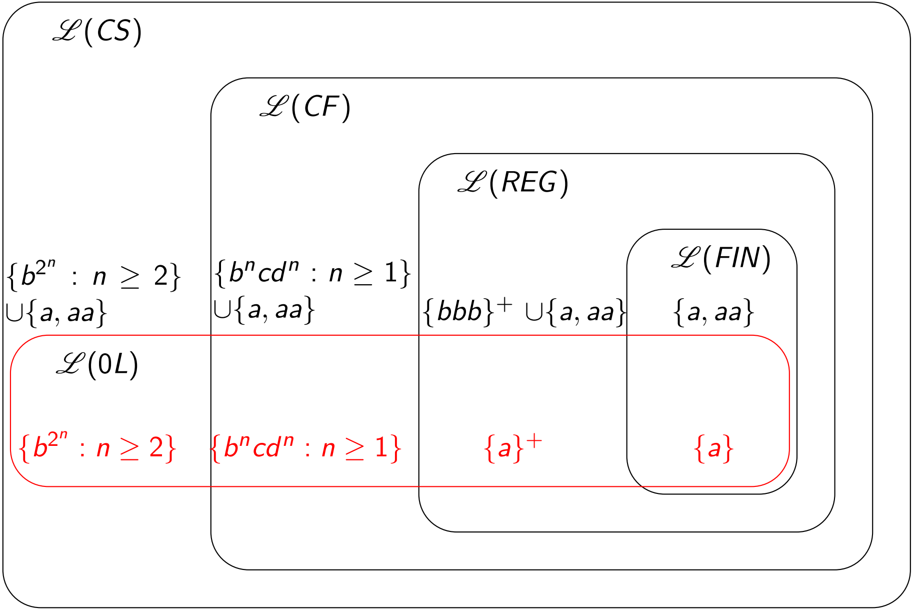
0L System – Other Properties

Theorem
If $L \in \mathscr{L}(0L)$, $L \subseteq \{a\}^*$, then $L^* \in \mathscr{L}(0L)$.

Theorem
If L is finite, then $L^* \in \mathscr{L}(0L)$.

Theorem
If $L \in \mathscr{L}(0L)$, $L \subseteq \{a\}^*$, $\varepsilon \in L$, then L is regular.

0L Systems in Chomsky Hierarchy



E0L System

E0L System

An E0L system is a quadruple

$$G = (V, T, P, w)$$

where

V is a total alphabet

T is a terminal alphabet, $T \subseteq V$

P is a finite set of productions of the form

$$a \rightarrow x$$

with $a \in V$ and $x \in V^*$

w is the axiom, $w \in V^+$

E0L System – Generated Language

- $\Rightarrow, \Rightarrow^*$ – by analogy with 0L systems

Generated Language

For an E0L system $G = (V, T, P, w)$,

$$L(G) = \{y \in T^* : w \Rightarrow^* y\}$$

Example

E0L system

$$G = (\{S, a, b\}, \{a, b\}, \{S \rightarrow a, S \rightarrow b, a \rightarrow aa, b \rightarrow bb\}, S)$$

$$L(G) = \{a^{2^n} : n \geq 0\} \cup \{b^{2^n} : n \geq 0\}$$

$$L(G) \in \mathscr{L}(E0L) - \mathscr{L}(0L)$$

E0L System – Generative Power

Example

E0L system

$$G = (\{A, a, b\}, \{a, b\}, \{A \rightarrow A, A \rightarrow a, a \rightarrow aa, b \rightarrow b\}, AbA)$$

$$L(G) = \{a^{2^n} b a^{2^m} \ : \ n, m \geq 0\}$$

Theorem

$\mathscr{L}(CF) \subset \mathscr{L}(E0L)$.

Proof

Homework

E0L System – Closure Properties

Theorem

$\mathscr{L}(E0L)$ is closed under

- *union*
- *concatenation*
- *positive closure*
- *intersection with a regular set*

Theorem

$\mathscr{L}(E0L)$ is not closed under inverse homomorphism.

T0L System

T0L System

A T0L system (T stands for Tables) is an $(n+2)$-tuple

$$G = (T, P_1, P_2, \ldots, P_n, w)$$

where

- $n \geq 1$
- for all $i = 1, \ldots, n$, $G_i = (T, P_i, w)$ is an 0L system

Direct Derivation

For $u, v \in T^*$,

$$u \Rightarrow v \text{ in } G$$

if $u \Rightarrow v$ in $G_i = (T, P_i, w)$ for some $i \in \{1, \ldots, n\}$

- $\Rightarrow^*$, $L(G)$ – by analogy with 0L systems

ET0L System

ET0L System

An ET0L system is an $(n + 3)$-tuple

$$G = (V, T, P_1, P_2, \dots, P_n, w)$$

where

- $n \geq 1$
- for all $i = 1, \dots, n$, $G_i = (V, T, P_i, w)$ is an E0L system

Direct Derivation

For $u, v \in V^*$,

$$u \Rightarrow v \text{ in } G$$

if $u \Rightarrow v$ in $G_i = (V, T, P_i, w)$ for some $i \in \{1, \dots, n\}$

- $\Rightarrow^*$, $L(G)$ – by analogy with E0L systems

Two-Table ET0L System

Theorem

For every ET0L system H, there exists an equivalent ET0L system of the form $G = (V, T, P_1, P_2, w)$*.*

Proof

Let

$$H = (W, T, R_1, \ldots, R_n, w)$$

be an n-table ET0L system. Define the two-table ET0L system

$$G = (V, T, P_1, P_2, w)$$

with

1. $V = W \cup \{\langle a, i \rangle \,:\, a \in W, i = 1, \ldots, n\}$
2. $P_1 = \{a \rightarrow \langle a, 1 \rangle \,:\, a \in W\} \cup \{\langle a, j \rangle \rightarrow \langle a, j+1 \rangle \,:\, 1 \leq j \leq n-1\}$
3. $P_2 = \{\langle a, k \rangle \rightarrow x \,:\, 1 \leq k \leq n, a \rightarrow x \in R_k\}$

Generative Power of E0L and ET0L Systems

Theorem

$\mathscr{L}(CF) \subset \mathscr{L}(E0L) \subset \mathscr{L}(ET0L) \subset \mathscr{L}(CS)$.

Proof – Basic Idea

1. $\mathscr{L}(E0L) \subset \mathscr{L}(ET0L)$ can be proved by showing that
   - $\{\#w\#w\#w \,:\, w \in \{a,b\}^*\}$ or
   - $\{a^i b^j a^i \,:\, j \geq i \geq 1\}$

   can be generated by an ET0L system and cannot be generated by any E0L system
2. $\mathscr{L}(ET0L) \subset \mathscr{L}(CS)$ can be proved by showing that
   - $\{(ab^n)^m \,:\, m \geq n \geq 1\}$ or
   - $\{a^{2^{2^n}} \,:\, n \geq 0\}$

   are context-sensitive languages which cannot be generated by any ET0L system

Bibliography I

A. Lindenmayer.
Mathematical models for cellular interactions in development, parts I–II.
Journal of Theoretical Biology, 18:280–315, 1968.

P. Prusinkiewicz and A. Lindenmayer.
The Algorithmic Beauty of Plants.
Springer-Verlag, 1990.

G. Rozenberg.
T0L systems and languages.
Information and Control, 23(4):357–381, 1973.

G. Rozenberg and A. Salomaa.
The Mathematical Theory of L Systems.
Academic Press, 1980.

Bibliography II

G. Rozenberg and A. Salomaa.
Handbook of Formal Languages, volume 1–3.
Springer, Berlin, 1997.

A. Salomaa.
Formal Languages.
Academic Press, New York, 1973.