Turing Machines and Two-Pushdown Automata

Jiří Techet Tomáš Masopust Alexander Meduna

Department of Information Systems
Faculty of Information Technology
Brno University of Technology
Božetěchova 2, Brno 61266, Czech Republic

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Turing Machine

A Turing machine is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q is a finite set of states

Σ is a tape alphabet, $\Sigma \cap Q = \emptyset$,
$I \subset \Sigma$ is an input alphabet,
$\sqcup \in \Sigma - I$ is the blank symbol

$R \subseteq Q\Sigma \times Q\Sigma$ is a finite set of rules,
$R = R_s \cup R_r \cup R_l$ (stationary, right, and left moves)

$s \in Q$ is the start state

$F \subseteq Q$ is a set of final states

Turing Machines – Notation

Stationary move

$(qX, pY) \in R_s$ is symbolically written as

$$qX \rightarrow_s pY$$

Right move

$(qX, pY) \in R_r$ is symbolically written as

$$qX \rightarrow_r pY$$

Left move

$(qX, pY) \in R_l$ is symbolically written as

$$qX \rightarrow_l pY$$

Turing Machines – Computational Step

Configuration

$$\chi \in \Sigma^* Q \Sigma^* \{\sqcup\}$$

Move

If at least one of the following holds,

Stationary move $\chi = xpUy$, $\chi' = xqVy$, and $r : pU \rightarrow_s qV \in R$,

Right move $\chi = xpUy$, $\chi' = xVqy'$, and $r : pU \rightarrow_r qV \in R$,
$y' = y$ if $y \neq \varepsilon$, and $y' = \sqcup$ if $y = \varepsilon$

Left move $\chi = xXpUy$, $\chi' = xqXVy$, and $r : pU \rightarrow_l qV \in R$,
for some $X \in \Sigma$

then

$$\chi \Rightarrow \chi' \; [r]$$

Turing Machines – Accepted Language

Accepted Word

Turing machine M accepts $w \in I^*$ if

$$sw\sqcup \Rightarrow^* ufv$$

for some configuration ufv with $f \in F$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Accepted Language

The set of all words M accepts is the language of M, denoted by $L(M)$, thus

$$L(M) = \{w \in I^* : sw\sqcup \Rightarrow^* ufv, f \in F\}$$

Example

$$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{a, b, \sqcup\}, R, q_0, \{q_4\})$$

where

$$R = \{1 : q_0 a \rightarrow_r q_1 \sqcup,\quad 2 : q_1 a \rightarrow_r q_1 a,\quad 3 : q_1 b \rightarrow_r q_1 b,\quad 4 : q_3 \sqcup \rightarrow_r q_0 \sqcup,$$
$$5 : q_1 \sqcup \rightarrow_l q_2 \sqcup,\quad 6 : q_2 b \rightarrow_l q_3 \sqcup,\quad 7 : q_3 a \rightarrow_l q_3 a,\quad 8 : q_3 b \rightarrow_l q_3 b,\quad 9 : q_0 \sqcup \rightarrow_s q_4 \sqcup\}$$

$$q_0 aabb \sqcup \Rightarrow \sqcup q_1 abb \sqcup\ [1] \Rightarrow \sqcup a q_1 bb \sqcup\ [2] \Rightarrow \sqcup ab q_1 b \sqcup\ [3]$$
$$\Rightarrow \sqcup abb q_1 \sqcup\ [3] \Rightarrow \sqcup ab q_2 b \sqcup\ [5] \Rightarrow \sqcup a q_3 b \sqcup \sqcup\ [6] \Rightarrow \sqcup q_3 ab \sqcup\ [8]$$
$$\Rightarrow q_3 \sqcup ab \sqcup\ [7] \Rightarrow \sqcup q_0 ab \sqcup\ [4] \Rightarrow \sqcup \sqcup q_1 b \sqcup\ [1] \Rightarrow \sqcup b q_1 \sqcup\ [3]$$
$$\Rightarrow \sqcup q_2 b \sqcup\ [5] \Rightarrow q_3 \sqcup \sqcup \sqcup\ [6] \Rightarrow \sqcup q_0 \sqcup\ [4] \Rightarrow \sqcup q_4 \sqcup\ [9]$$

$$L(M) = \{a^n b^n : n \geq 0\}$$

Church's Thesis

Church's Thesis

For every algorithm that exits there is an equivalent Turing Machine.

Recursively Enumerable Language

A language L is recursively enumerable if there is a Turing machine M such that $L(M) = L$.

Recursive Language

A language L is recursive if there is a Turing machine M that always halts such that $L(M) = L$.

Deterministic Turing Machine

Deterministic Turing Machine

Turing machine M is deterministic if every rule $r \in R$ satisfies

$$\text{lhs}(r) \notin \{\text{lhs}(r') : r' \in R - \{r\}\}$$

Theorem

A language L is recursively enumerable if there is a deterministic Turing machine M such that $L(M) = L$.

Theorem

A language L is recursive if there is a deterministic Turing machine M that always halts such that $L(M) = L$.

Linear Bounded Automata

Linear Bounded Automaton

A linear bounded automaton is a Turing machine M that never extends its tape.

Consequence

With an input word w, M uses no more than the first $|w|$ tape squares.

Theorem

A language L is context-sensitive if and only if there is a linear bounded automaton M such that $L(M) = L$.

Open Problem

Are deterministic linear bounded automata as powerful as linear bounded automata?

Two-Pushdown Automata

Two-Pushdown Automaton

A two-pushdown automaton is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q, s, F have the same meaning as in the definition of Turing machine

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = \{|\} \cup I \cup P_D$, where
$|$ is a special symbol, $| \notin I \cup P_D$,
I is an input alphabet, P_D is a pushdown alphabet, $S \in P_D$ is a start pushdown symbol

R is a finite set of rules of the form

$$A|Bpa \rightarrow u|vq$$

where $A, B \in P_D$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u, v \in P_D^*$

Two-Pushdown Automata – Computational Step

Configuration

$$\chi \in P_D^* \{|\} P_D^* Q I^*$$

Move

If

$$r : A|Bpa \to u|vq \in R,$$

$$\chi = yA|xBpaz,$$

$$\chi' = yu|xvqz,$$

then

$$\chi \Rightarrow \chi' \; [r]$$

Two-Pushdown Automata – Accepted Language

Accepted Language by Final State

$$L_f(M) = \{w \in I^* : S|Ssw \Rightarrow^* x|yf, f \in F\}$$

Accepted Language by Empty Pushdown

$$L_e(M) = \{w \in I^* : S|Ssw \Rightarrow^* |q, q \in Q\}$$

Accepted Language by Final State and Empty Pushdown

$$L_{fe}(M) = \{w \in I^* : S|Ssw \Rightarrow^* |f, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Two-Pushdown Automata – Example

Example

$$M = (\{s, p, q, f\}, \{S, a, b, c, |\}, R, s, \{f\}),$$

where

$$\begin{aligned} R = \{ 1 &: S|Ssa \to S|Sas, & 4 &: b|aqb \to bb|q, & & \\ 2 &: S|asa \to S|aas, & 5 &: b|Sqc \to |Sp, & & \\ 3 &: S|asb \to Sb|q, & 6 &: b|Spc \to |Sp, & 7 &: S|Sp \to |f \} \end{aligned}$$

Then,

$$\begin{aligned} S|Ssaabbcc &\Rightarrow S|Sasabbcc\ [1] \Rightarrow S|Saasbbcc\ [2] \Rightarrow Sb|Saqbcc\ [3] \\ &\Rightarrow Sbb|Sqcc\ [4] \Rightarrow Sb|Spc\ [5] \Rightarrow S|Sp\ [6] \Rightarrow |f\ [7] \end{aligned}$$

$$L_f(M) = L_e(M) = L_{fe}(M) = \{a^n b^n c^n \ : \ n \geq 1\}$$

Two-Pushdown Automata – Results

Determinism

M is deterministic if each $r \in R$ with $\text{lhs}(r) = A|Bpq$ satisfies

$$\{r\} = \{r' \in R : A|Bpa = \text{lhs}(r') \text{ or } A|Bp = \text{lhs}(r')\}$$

Theorem

All acceptance modes (f, e, fe) are equivalent.

Theorem

The following models are equivalent:

- *Turing machines*
- *deterministic Turing machines*
- *two-pushdown automata*
- *deterministic two-pushdown automata*

Bibliography

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