Transducers and Translation Grammars

Jiří Techet Tomáš Masopust Alexander Meduna

Department of Information Systems
Faculty of Information Technology
Brno University of Technology
Božetěchova 2, Brno 61266, Czech Republic

Modern Formal Language Theory, 2007

Finite Transducers

Finite Transducer

A finite transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q is a finite set of states

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = I \cup O$,
I and O are input and output alphabets

R is a finite set of rules of the form

$$pa \rightarrow qz$$

$p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $z \in O^*$

$s \in Q$ is the start state

$F \subseteq Q$ is a set of final states

Input Finite Automaton

Input Finite Automaton

Let $M = (Q, \Sigma, R, s, F)$ be a finite transducer, then

$$M_I = (Q, I, R_I, s, F)$$

where

- I is the input alphabet of M and
- $R_I = \{qa \to p \,:\, qa \to px \in R, x \in O^*\}$

is the input finite automaton

Finite Transducers – Computational Step

Configuration

$$\chi \in QI^*\{|\}O^*$$

Move

If

$$r : qa \rightarrow pz \in R,$$

$$\chi = qaw|y,$$

$$\chi' = pw|yz,$$

then

$$\chi \Rightarrow \chi' \ [r]$$

Finite Transducers – Translation

Translation of a Word

M translates x into y if

$$sx| \Rightarrow^* f|y \text{ where } f \in F$$

Translation Defined by M

$$T(M) = \{(x, y) \in I^* \times O^* : sx| \Rightarrow^* f|y, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Finite Transducers – Input and Output Language

Input Language

$$L_I(M) = \{x \in I^* : (x, y) \in T(M) \text{ for some } y \in O^*\}$$

Output Language

$$L_O(M) = \{y \in O^* : (x, y) \in T(M) \text{ for some } x \in I^*\}$$

Theorem

Both input and output languages are regular.

Finite Transducers – Example

Example

$$M = (\{s, q, f\}, \{!, a\}, R, s, \{f\})$$

where

$$R = \{1 : s! \rightarrow q, \quad 3 : q! \rightarrow s,$$
$$2 : sa \rightarrow fa, \quad 4 : qa \rightarrow f!a\}$$

$$s!!!a| \Rightarrow q!!a|\ [1] \Rightarrow s!a|\ [3] \Rightarrow qa|\ [1] \Rightarrow f|!a\ [4]$$

$$T(M) = \{(!^i a, a) \ : \ i \geq 0, i = 2k, k \geq 0\}$$
$$\cup \{(!^i a, !a) \ : \ i \geq 1, i = 2k+1, k \geq 0\}$$

$$L_I(M) = \{!^i a \ : \ i \geq 0\} \qquad L_O(M) = \{a, !a\}$$

Finite Transducers – Determinism

Deterministic Finite Transducer

M is deterministic if each rule $r \in R$ with $\text{lhs}(r) = pa$ satisfies

$$\{r\} = \{r' \in R \,:\, pa = \text{lhs}(r') \text{ or } p = \text{lhs}(r')\}$$

Example

Deterministic finite transducer

$$M = (\{f\}, \{0\}, \{f \to f0\}, f, \{f\}),$$

then

$$T(M) = \{(\varepsilon, 0^i) \,:\, i \geq 0\}$$

Pushdown Transducers

Pushdown Transducer

A pushdown transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q, s, F have the same meaning as in the case of finite transducers

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = I \cup O \cup P_D$,
I, O, P_D are input, output and pushdown alphabets,
$S \in P_D$ is the start pushdown symbol

R is a finite set of rules of the form

$$Apa \rightarrow uqv$$

$A \in P_D$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u \in P_D^*$, $v \in O^*$

Input Pushdown Automaton

Input Pushdown Automaton

Let $M = (Q, \Sigma, R, s, F)$ be a pushdown transducer, then

$$M_I = (Q, I \cup P_D, R_I, s, F)$$

where

- I and P_D are the input and the pushdown alphabets of M
- $R_I = \{Aqa \rightarrow up : Aqa \rightarrow upv \in R, v \in O^*\}$

is the input pushdown automaton

Pushdown Transducers – Computational Step

Configuration

$$\chi \in P_D^* Q I^* \{|\} O^*$$

Move

If

$$r : Aqa \rightarrow upv \in R,$$

$$\chi = zAqaw|y,$$

$$\chi' = zupw|yv,$$

then

$$\chi \Rightarrow \chi' \; [r]$$

Pushdown Transducers – Translation

Translation of a Word

M translates x into y if

$$Ssx| \Rightarrow^* zf|y \text{ where } f \in F$$

Translation Defined by M

$$T(M) = \{(x, y) \in I^* \times O^* : Ssx| \Rightarrow^* zf|y, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Pushdown Transducers – Input and Output Language

Input Language

$$L_I(M) = \{x \in I^* \ : \ (x, y) \in T(M) \text{ for some } y \in O^*\}$$

Output Language

$$L_O(M) = \{y \in O^* \ : \ (x, y) \in T(M) \text{ for some } x \in I^*\}$$

Pushdown Transducers – Example

Example

$$M = (\{s, q, f\}, \{S, A, +, *, a\}, R, s, \{f\})$$

where

$$R = \{1 : Ss \to SAq, \quad 2 : Aqa \to qa, \quad 3 : Aq+ \to +AAq, \quad 4 : Aq* \to *AAq, \quad 5 : +q \to q+, \quad 6 : *q \to q*, \quad 7 : Sq \to f\}$$

$$Ss + *aaa| \Rightarrow SAq + *aaa|\ [1] \Rightarrow S + AAq * aaa|\ [3]$$
$$\Rightarrow S + A * AAqaaa|\ [4] \Rightarrow S + A * Aqaa|a\ [2] \Rightarrow S + A * qa|aa\ [2]$$
$$\Rightarrow S + Aqa|aa*\ [6] \Rightarrow S + q|aa * a\ [2] \Rightarrow Sq|aa * a+\ [5]$$
$$\Rightarrow f|aa * a+\ [7]$$

$T(M) = \{(pre, post) : pre =$ prefix expression, $post$=postfix expression$\}$

Pushdown Transducers – Determinism

Deterministic Pushdown Transducer

M is deterministic if each rule $r \in R$ with $\text{lhs}(r) = Apa$ satisfies

$$\{r\} = \{r' \in R \,:\, Apa = \text{lhs}(r') \text{ or } Ap = \text{lhs}(r')\}$$

Extended Pushdown Transducer

$M = (Q, \Sigma, R, s, F)$ is extended if R is a finite set of productions of the form

$$zpa \rightarrow uqv$$

$z \in P_D^*$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u \in P_D^*$, $v \in O^*$

Translation Grammars

Translation Grammar

A translation grammar is a quadruple

$$G = (N, T, P, S)$$

where

N, S are defined as usual, $S \in N$

T is a terminal alphabet, $T \cap N = \emptyset$, $T = I \cup O$

I and O are input and output alphabets

P is a finite set of productions of the form

$$A \twoheadrightarrow u_0 B_1 u_1 \dots B_n u_n | v_0 B_1 v_1 \dots B_n v_n$$

$|$ is a special symbol, $B_i \in N$, $u_j \in I^*$, $v_j \in O^*$,
$i = 1, \dots, n$, $j = 0, \dots, n$

Translation Grammars – Direct Derivation

Notation

For

$$p = A \twoheadrightarrow u_0 B_1 u_1 \dots B_n u_n | v_0 B_1 v_1 \dots B_n v_n,$$

- $\text{lhs}(p) = A$
- $\text{irhs}(p) = u_0 B_1 u_1 \dots B_n u_n$
- $\text{orhs}(p) = v_0 B_1 v_1 \dots B_n v_n$

Direct Derivation

For $G = (N, T, P, S)$, $p \in P$,$x, y, u, v \in (N \cup T)^*$, then

$$x\,\text{lhs}(p)y|u\,\text{lhs}(p)v \Rightarrow x\,\text{irhs}(p)y|u\,\text{orhs}(p)v$$

where the same number of nonterminals occurs in x and u.

Translation Grammars – Translation

Translation Defined by G

$$T(G) = \{u|v \,:\, S|S \Rightarrow^* u|v, u \in I^*, v \in O^*\}$$

Input Grammar

$$G_I = (N, I, P_I, S)$$

where $P_I = \{A \rightarrow x \,:\, A \twoheadrightarrow x|y \in P\}$

Output Grammar

$$G_O = (N, O, P_O, S)$$

where $P_O = \{A \rightarrow y \,:\, A \twoheadrightarrow x|y \in P\}$

Translation Grammars – Example

Example

Let the translation grammar G be defined by the following productions:

$$
\begin{aligned}
\langle expr \rangle &\twoheadrightarrow \langle expr \rangle + \langle term \rangle | \langle expr \rangle \langle term \rangle + \\
\langle expr \rangle &\twoheadrightarrow \langle term \rangle | \langle term \rangle \\
\langle term \rangle &\twoheadrightarrow \langle term \rangle * \langle factor \rangle | \langle term \rangle \langle factor \rangle * \\
\langle term \rangle &\twoheadrightarrow \langle factor \rangle | \langle factor \rangle \\
\langle factor \rangle &\twoheadrightarrow (\langle expr \rangle) | \langle expr \rangle \\
\langle factor \rangle &\twoheadrightarrow a | a
\end{aligned}
$$

$$\langle expr \rangle | \langle expr \rangle \Rightarrow^* (a + a) * a | aa + a*$$

G translates an infix expression with $+$ and $*$ to the corresponding postfix expression.

Bibliography

J. Evey.
Application of pushdown store machines.
In *Proceedings 1963 Fall Joint Computer Conference*, pages 215–227, Montvale, NJ: AFIPS Press, 1963.

A. Meduna.
Automata and Languages: Theory and Applications.
Springer, London, 2000.

J. Sheperdson.
The reduction of two-way automata to one-way automata.
IBM Journal of Research and Development, 3:198–200, 1959.