Transducers and Translation Grammars

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Finite Transducers

Finite Transducer

A finite transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q is a finite set of states

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = I \cup O$, I and O are input and output alphabets

R is a finite set of rules of the form

$$pa \to qz$$

$p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $z \in O^*$

$s \in Q$ is the start state

$F \subseteq Q$ is a set of final states

Input Finite Automaton

Input Finite Automaton

Let $M = (Q, \Sigma, R, s, F)$ be a finite transducer, then

$$M_I = (Q, I, R_I, s, F)$$

where

- I is the input alphabet of M and
- $R_I = \{qa \to p \,:\, qa \to px \in R, x \in O^*\}$

is the input finite automaton

Finite Transducers – Computational Step

Configuration

$$\chi \in QI^*\{|\}O^*$$

Move
If

$$r : qa \rightarrow pz \in R,$$

$$\chi = qaw|y,$$

$$\chi' = pw|yz,$$

then

$$\chi \Rightarrow \chi' \; [r]$$

Finite Transducers – Translation

Translation of a Word

M translates x into y if

$$sx| \Rightarrow^* f|y \text{ where } f \in F$$

Translation Defined by M

$$T(M) = \{(x, y) \in I^* \times O^* : sx| \Rightarrow^* f|y, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Finite Transducers – Input and Output Language

Input Language

$$L_I(M) = \{x \in I^* \,:\, (x, y) \in T(M) \text{ for some } y \in O^*\}$$

Output Language

$$L_O(M) = \{y \in O^* \,:\, (x, y) \in T(M) \text{ for some } x \in I^*\}$$

Theorem

Both input and output languages are regular.

Finite Transducers – Example

Example

$$M = (\{s, q, f\}, \{!, a\}, R, s, \{f\})$$

where

$$R = \{1 : s! \rightarrow q, \qquad 3 : q! \rightarrow s, \\ 2 : sa \rightarrow fa, \qquad 4 : qa \rightarrow f!a\}$$

$$s!!!a| \Rightarrow q!!a|\ [1] \Rightarrow s!a|\ [3] \Rightarrow qa|\ [1] \Rightarrow f|!a\ [4]$$

$$T(M) = \{(!^i a, a) : i \geq 0, i = 2k, k \geq 0\} \\ \cup \{(!^i a, !a) : i \geq 1, i = 2k+1, k \geq 0\}$$

$$L_I(M) = \{!^i a : i \geq 0\} \qquad L_O(M) = \{a, !a\}$$

Finite Transducers – Determinism

Deterministic Finite Transducer

M is deterministic if each rule $r \in R$ with $\text{lhs}(r) = pa$ satisfies

$$\{r\} = \{r' \in R \,:\, pa = \text{lhs}(r') \text{ or } p = \text{lhs}(r')\}$$

Example

Deterministic finite transducer

$$M = (\{f\}, \{0\}, \{f \rightarrow f0\}, f, \{f\}),$$

then

$$T(M) = \{(\varepsilon, 0^i) \,:\, i \geq 0\}$$

Pushdown Transducers

Pushdown Transducer

A pushdown transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q, s, F have the same meaning as in the case of finite transducers

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = I \cup O \cup P_D$,
I, O, P_D are input, output and pushdown alphabets,
$S \in P_D$ is the start pushdown symbol

R is a finite set of rules of the form

$$Apa \rightarrow uqv$$

$A \in P_D$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u \in P_D^*$, $v \in O^*$

Input Pushdown Automaton

Input Pushdown Automaton

Let $M = (Q, \Sigma, R, s, F)$ be a pushdown transducer, then

$$M_I = (Q, I \cup P_D, R_I, s, F)$$

where

- I and P_D are the input and the pushdown alphabets of M
- $R_I = \{Aqa \rightarrow up \,:\, Aqa \rightarrow upv \in R, v \in O^*\}$

is the input pushdown automaton

Pushdown Transducers – Computational Step

Configuration

$$\chi \in P_D^* Q I^* \{|\} O^*$$

Move
If

$$r : Aqa \rightarrow upv \in R,$$

$$\chi = zAqaw|y,$$

$$\chi' = zupw|yv,$$

then

$$\chi \Rightarrow \chi' \; [r]$$

Pushdown Transducers – Translation

Translation of a Word
M translates x into y if

$$Ssx| \Rightarrow^* zf|y \text{ where } f \in F$$

Translation Defined by M

$$T(M) = \{(x, y) \in I^* \times O^* : Ssx| \Rightarrow^* zf|y, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Pushdown Transducers – Input and Output Language

Input Language

$$L_I(M) = \{x \in I^* \,:\, (x, y) \in T(M) \text{ for some } y \in O^*\}$$

Output Language

$$L_O(M) = \{y \in O^* \,:\, (x, y) \in T(M) \text{ for some } x \in I^*\}$$

Pushdown Transducers – Example

Example

$$M = (\{s, q, f\}, \{S, A, +, *, a\}, R, s, \{f\})$$

where

$$R = \{1 : Ss \to SAq, \quad 2 : Aqa \to qa, \quad 3 : Aq+ \to +AAq, \quad 4 : Aq* \to *AAq, \quad 5 : +q \to q+, \quad 6 : *q \to q*, \quad 7 : Sq \to f\}$$

$$\begin{aligned} & Ss + *aaa| \Rightarrow SAq + *aaa| \ [1] \Rightarrow S + AAq * aaa| \ [3] \\ \Rightarrow\ & S + A * AAqaaa| \ [4] \Rightarrow S + A * Aqaa|a \ [2] \Rightarrow S + A * qa|aa \ [2] \\ \Rightarrow\ & S + Aqa|aa* \ [6] \Rightarrow S + q|aa * a \ [2] \Rightarrow Sq|aa * a+ \ [5] \\ \Rightarrow\ & f|aa * a+ \ [7] \end{aligned}$$

$T(M) = \{(pre, post) : pre = \text{prefix expression}, post = \text{postfix expression}\}$

Pushdown Transducers – Determinism

Deterministic Pushdown Transducer
M is deterministic if each rule $r \in R$ with $\text{lhs}(r) = Apa$ satisfies

$$\{r\} = \{r' \in R \, : \, Apa = \text{lhs}(r') \text{ or } Ap = \text{lhs}(r')\}$$

Extended Pushdown Transducer
$M = (Q, \Sigma, R, s, F)$ is extended if R is a finite set of productions of the form

$$zpa \rightarrow uqv$$

$z \in P_D^*$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u \in P_D^*$, $v \in O^*$

Translation Grammars

Translation Grammar

A translation grammar is a quadruple

$$G = (N, T, P, S)$$

where

N, S are defined as usual, $S \in N$

T is a terminal alphabet, $T \cap N = \emptyset$, $T = I \cup O$
I and O are input and output alphabets

P is a finite set of productions of the form

$$A \twoheadrightarrow u_0 B_1 u_1 \dots B_n u_n | v_0 B_1 v_1 \dots B_n v_n$$

$|$ is a special symbol, $A \in N$, $B_i \in N$, $u_j \in I^*$, $v_j \in O^*$,
$i = 1, \dots, n$, $j = 0, \dots, n$

Translation Grammars – Direct Derivation

Notation

For

$$p = A \twoheadrightarrow u_0 B_1 u_1 \dots B_n u_n | v_0 B_1 v_1 \dots B_n v_n,$$

- $\text{lhs}(p) = A$
- $\text{irhs}(p) = u_0 B_1 u_1 \dots B_n u_n$
- $\text{orhs}(p) = v_0 B_1 v_1 \dots B_n v_n$

Direct Derivation

For $G = (N, T, P, S)$, $p \in P$,$x, y, u, v \in (N \cup T)^*$, then

$$x \, \text{lhs}(p) y | u \, \text{lhs}(p) v \Rightarrow x \, \text{irhs}(p) y | u \, \text{orhs}(p) v$$

where the same number of nonterminals occurs in x and u.

Translation Grammars – Translation

Translation Defined by G

$$T(G) = \{u|v \,:\, S|S \Rightarrow^* u|v, u \in I^*, v \in O^*\}$$

Input Grammar

$$G_I = (N, I, P_I, S)$$

where $P_I = \{A \to x \,:\, A \twoheadrightarrow x|y \in P\}$

Output Grammar

$$G_O = (N, O, P_O, S)$$

where $P_O = \{A \to y \,:\, A \twoheadrightarrow x|y \in P\}$

Translation Grammars – Example

Example

Let the translation grammar G be defined by the following productions:

$$
\begin{array}{l}
\langle expr \rangle \twoheadrightarrow \langle expr \rangle + \langle term \rangle | \langle expr \rangle \langle term \rangle + \\
\langle expr \rangle \twoheadrightarrow \langle term \rangle | \langle term \rangle \\
\langle term \rangle \twoheadrightarrow \langle term \rangle * \langle factor \rangle | \langle term \rangle \langle factor \rangle * \\
\langle term \rangle \twoheadrightarrow \langle factor \rangle | \langle factor \rangle \\
\langle factor \rangle \twoheadrightarrow (\langle expr \rangle) | \langle expr \rangle \\
\langle factor \rangle \twoheadrightarrow a | a
\end{array}
$$

$$\langle expr \rangle | \langle expr \rangle \Rightarrow^* (a + a) * a | aa + a*$$

G translates an infix expression with $+$ and $*$ to the corresponding postfix expression.

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