k-Limited Erasing Performed by Scattered Context Grammars

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Scattered Context Grammar

Scattered context grammar (SC grammar)

A SC grammar is a quadruple, $G = (V, T, P, S)$, where

V is a finite alphabet

T is a set of terminals, $T \subset V$

S is a starting symbol, $S \in (V - T)$

P is a finite set of productions of the form: $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)$; $A_1, \dots, A_n \in (V - T)$; $x_1, \dots, x_n \in V^*$

- $\text{len}((A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)) = n$
- $\text{pos}(a_1 \dots a_i \dots a_n, i) = a_i$

Propagating scattered context grammar (PSC grammar)

- every $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \in V^+$

Generated Language

Derivation step

For $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n) \in P$ and

$$u = u_1 A_1 \ldots u_n A_n u_{n+1}$$
$$v = u_1 x_1 \ldots u_n x_n u_{n+1}$$

we write $u \Rightarrow v\ [(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)]$

Generated language

$L(G) = \{x \in T^* : S \Rightarrow^* x\}$

Generative power

- $\mathcal{L}_{SC} = \mathcal{L}_{RE}$
- $\mathcal{L}_{CF} \subset \mathcal{L}_{PSC} \subseteq \mathcal{L}_{CS}$

PSC Grammar—Example

Example

$G_1 = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P_1, S)$ with

$$\begin{aligned} P_1 = \{&(S) \rightarrow (ABC), \\ &(A, B, C) \rightarrow (aA, bB, cC), \\ &(A, B, C) \rightarrow (\varepsilon, \varepsilon, \varepsilon)\} \end{aligned}$$

$S \Rightarrow ABC \Rightarrow aAbBcC \Rightarrow aaAbbBccC \Rightarrow aabbcc$

$L(G_1) = \{a^n b^n c^n : n \geq 0\}$

Example

$G_2 = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P_2, S)$ with

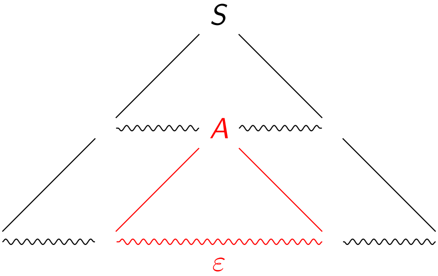
$$\begin{aligned} P_2 = \{&(S) \rightarrow (\varepsilon), (S) \rightarrow (ABC), \\ &(A, B, C) \rightarrow (aA, bB, cC), \\ &(A, B, C) \rightarrow (a, b, c)\} \end{aligned}$$

Symbols Erased During Derivation

Symbols erased during derivation

A symbol, A, is erased during a derivation if the frontier of the subtree rooted at A is ε.

- If the symbol A is erased, we write $\check{A}$;
- otherwise the symbol is not erased and we write $\hat{A}$.

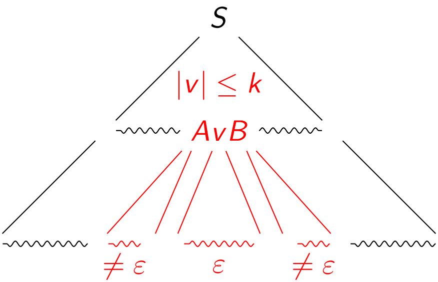


Nonterminals Erased in a k-Limited Way

Nonterminals erased in a k-limited way

For every $y \in L(G)$ there exists a derivation in which every sentential form x satisfies:

1. Every $x = uAvBw$, $\hat{A}$, $\hat{B}$, $\check{v}$, satisfies $|v| \leq k$.
2. Every $x = uAw$, $\hat{A}$, satisfies: if $\check{u}$ or $\check{w}$, then $|u| \leq k$ or $|w| \leq k$, respectively.



Results

Theorem

For every SC grammar, G, which erases its nonterminals in a k-limited way there exists a propagating SC grammar, $\bar{G}$, such that $L(G) = L(\bar{G})$.

Basic Idea—Demonstration

Example

$G_3 = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P_3, S)$ with

$$P_3 = \{(S) \to (ABC), (A, B, C) \to (\varepsilon, \varepsilon, \varepsilon), (A) \to (aAA), (B) \to (bBB), (C) \to (cCC)\}$$

$$S \Rightarrow ABC \Rightarrow^3 aAAbBBcCC \Rightarrow aAbBcC \Rightarrow^3 aaAAbbBBccCC \Rightarrow^2 aabbcc$$

		$\langle S\rangle$		$(S) \to (ABC)$
$\Rightarrow$	$\langle A\rangle$	$\langle B\rangle$	$\langle C\rangle$	$(A) \to (aAA), \dots, (C) \to (cCC)$
$\Rightarrow^3$	$\langle aAA\rangle$	$\langle bBB\rangle$	$\langle cCC\rangle$	$(A, B, C) \to (\varepsilon, \varepsilon, \varepsilon)$
$\Rightarrow$	$\langle aA\rangle$	$\langle bB\rangle$	$\langle cC\rangle$	$(A) \to (aAA), \dots, (C) \to (cCC)$
$\Rightarrow^3$	$\langle a\rangle\langle aAA\rangle$	$\langle b\rangle\langle bBB\rangle$	$\langle c\rangle\langle cCC\rangle$	$(A, B, C) \to (\varepsilon, \varepsilon, \varepsilon)$
$\Rightarrow$	$\langle a\rangle\langle aA\rangle$	$\langle b\rangle\langle bB\rangle$	$\langle c\rangle\langle cC\rangle$	$(A, B, C) \to (\varepsilon, \varepsilon, \varepsilon)$
$\Rightarrow$	$\langle a\rangle\langle a\rangle$	$\langle b\rangle\langle b\rangle$	$\langle c\rangle\langle c\rangle$	
$\Rightarrow^6$	aa	bb	cc	

Basic Idea

Let $G = (V, T, P, S)$ be a grammar which erases its nonterminals in a k-limited way. Every application of $p = (A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)$ is simulated by a PSC grammar $\bar{G}$ as follows:

1. Every CF component of a SC production is simulated independently.

$$\langle z_{11} \lfloor p, 1 \rfloor z_{12} \rangle \langle z_{21} B_1 A_2 z_{22} A_3 z_{23} \rangle \ldots \langle z_{n1} A_n z_{n2} \rangle$$
$$\Rightarrow \langle z_{11} x_1 \lfloor p, 1 \rfloor' z_{12} \rangle \langle z_{21} B_1 A_2 z_{22} A_3 z_{23} \rangle \ldots \langle z_{n1} A_n z_{n2} \rangle$$

2. Transition to the next component of the SC production is performed.

$$\langle z_{11} x_1 \lfloor p, 1 \rfloor' z_{12} \rangle \langle z_{21} B_1 A_2 z_{22} A_3 z_{23} \rangle \ldots \langle z_{n1} A_n z_{n2} \rangle$$
$$\Rightarrow \langle z_{11} x_1 z_{12} \rangle \langle z_{21} B_1 \lfloor p, 2 \rfloor z_{22} A_3 z_{23} \rangle \ldots \langle z_{n1} A_n z_{n2} \rangle$$

3. After the last component is simulated, transition to the following SC production, $q = (B_1, \ldots, B_m) \rightarrow (y_1, \ldots, y_m)$, is performed.

$$\langle z_{11} x_1 z_{12} \rangle \langle z_{21} B_1 x_2 z_{22} x_3 z_{23} \rangle \ldots \langle z_{n1} x_n \lfloor p, n \rfloor' z_{n2} \rangle$$
$$\Rightarrow \langle z_{11} x_1 z_{12} \rangle \langle z_{21} \lfloor q, 1 \rfloor x_2 z_{22} x_3 z_{23} \rangle \ldots \langle z_{n1} x_n z_{n2} \rangle$$

Finally, we replace symbols of the form $\langle a \rangle$ with a, where $a \in T$.

Construction of Symbols

- $\Psi = \{\lfloor p, i \rfloor : p \in P, 1 \leq i \leq \text{len}(p)\}$
- $\Psi' = \{\lfloor p, i \rfloor' : \lfloor p, i \rfloor \in \Psi\}$
- $\bar{N}_1 = \{\langle x \rangle : x \in (V - T)^* \cup (V - T)^* T (V - T)^*, |x| \leq 2k + 1\}$

For every $\langle x \rangle \in \bar{N}_1$ and $\lfloor p, i \rfloor \in \Psi$, define
$\text{lhs-replace}(\langle x \rangle, \lfloor p, i \rfloor) = \{\langle x_1 \lfloor p, i \rfloor x_2 \rangle : x_1, x_2 \in V^*, x_1 \,\text{lhs}(\lfloor p, i \rfloor) x_2 = x\}$

- $\bar{N}_2 = \{\langle x \rangle : \langle x \rangle = \text{lhs-replace}(\langle y \rangle, \lfloor p, i \rfloor), \langle y \rangle \in \bar{N}_1, \lfloor p, i \rfloor \in \Psi\}$

For every $\langle x \rangle \in \bar{N}_1$ and $\lfloor p, i \rfloor' \in \Psi'$, define
$\text{insert}(\langle x \rangle, \lfloor p, i \rfloor') = \{\langle x_1 \lfloor p, i \rfloor' x_2 \rangle : x_1, x_2 \in V^*, x_1 x_2 = x\}$

- $\bar{N}_2' = \{\langle x \rangle : \langle x \rangle = \text{insert}(\langle y \rangle, \lfloor p, i \rfloor'), \langle y \rangle \in \bar{N}_1, \lfloor p, i \rfloor' \in \Psi'\}$

Define the PSC grammar,

$$\bar{G} = (T \cup \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_2' \cup \{\bar{S}\}, T, \bar{P}, \bar{S})$$

Construction of Productions I

For every $x = \langle x_1 \rangle \langle x_2 \rangle \dots \langle x_n \rangle \in (\bar{N}_1 \cup \bar{N}_2 \cup \bar{N}'_2)^*$ for some $n \geq 1$, define

$$\text{join}(x) = x_1 x_2 \dots x_n$$

For every $x \in \bar{N}_1 \cup \bar{N}_2 \cup \bar{N}'_2$, define

$$\text{split}(x) = \{ y \, : \, x = \text{join}(y) \}$$

1. **Initialization**
   For every $p = (S) \rightarrow (x) \in P$, add
   $(\bar{S}) \rightarrow (\langle \lfloor p, 1 \rfloor \rangle)$ to $\bar{P}$
2. **Termination**
   For every $a \in T$, add
   $(\langle a \rangle) \rightarrow (a)$ to $\bar{P}$

Construction of Productions II

3. Simulation of one SC production's CF component

   For every
   - $\langle x_1 \lfloor p, i \rfloor x_2 \rangle \in \text{lhs-replace}(\langle x \rangle, \lfloor p, i \rfloor)$, $\langle x \rangle \in \bar{N}_1$, $\lfloor p, i \rfloor \in \Psi$, $x_1, x_2 \in V^*$
   - $Y \in \text{split}(x_1 \, \text{rhs}(\lfloor p, i \rfloor) \lfloor p, i \rfloor' x_2)$

   add $(\langle x_1 \lfloor p, i \rfloor x_2 \rangle) \rightarrow (Y)$ to $\bar{P}$

4. Transition to the next SC production's CF component

   For every
   - $\langle x \rangle \in \bar{N}_1$
   - $X \in \text{insert}(\langle x \rangle, \lfloor p, i \rfloor')$, where $p \in P$, $i < \text{len}(p)$
   - $\langle y \rangle \in \bar{N}_1$
   - $Y \in \text{lhs-replace}(\langle y \rangle, \lfloor p, i + 1 \rfloor)$, where $q \in P$

   add
   1. $(X, \langle y \rangle) \rightarrow (\langle x \rangle, Y)$ to $\bar{P}$
   2. If
      - $\langle x \rangle = \langle y \rangle$
      - $\text{pos}(X, l) = \lfloor p, i \rfloor'$, $\text{pos}(Y, m) = \lfloor p, i + 1 \rfloor'$, $l < m$

      add $(X) \rightarrow (Y)$ to $\bar{P}$

Construction of Productions III

5. Transition to the next SC production

For every

- $\langle x \rangle \in \bar{N}_1$
- $X \in \text{insert}(\langle x \rangle, \lfloor p, n \rfloor')$, where $p \in P$, $\text{len}(p) = n$
- $\langle y \rangle \in \bar{N}_1$
- $Y \in \text{lhs-replace}(\langle y \rangle, \lfloor q, 1 \rfloor)$, where $q \in P$

add

1. $(X, \langle y \rangle) \rightarrow (\langle x \rangle, Y)$ to $\bar{P}$
2. $(\langle y \rangle, X) \rightarrow (Y, \langle x \rangle)$ to $\bar{P}$
3. If $\langle x \rangle = \langle y \rangle$, add
   $(X) \rightarrow (Y)$ to $\bar{P}$
4. Finishing the simulation
   $(X) \rightarrow (\langle x \rangle)$ to $\bar{P}$

Summary and Future Investigation

Summary

- In general, in SC grammars ε-productions cannot be removed
- This removal is, however, possible under some conditions

Future investigation

- There are modifications of SC grammars which contain ε-productions and characterize all CS languages
- Is it possible to convert them to equivalent grammars which delete their nonterminals in a k-limited way?