Reduction of Scattered Context Generators of Sentences Preceded by Their Leftmost Parses

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Scattered Context Grammar

Scattered context grammar (SC grammar)

G = (V, T, P, S), where

V is a finite alphabet

T is a set of terminals, $T \subset V$

S is the start symbol, $S \in V - T$

P is a finite set of productions of the form

$$(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n),$$

where $A_1, \dots, A_n \in V - T$, $x_1, \dots, x_n \in V^*$

Propagating scattered context grammar (PSC grammar)

- each $(A_1, \dots, A_n) \rightarrow (x_1, \dots, x_n)$ satisfies $x_1, \dots, x_n \in V^+$

Derivation Step

Derivation step

For $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n) \in P$ and

$$u = u_1 A_1 \ldots u_n A_n u_{n+1}$$
$$v = u_1 x_1 \ldots u_n x_n u_{n+1}$$

we write $u \Rightarrow v \; [(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)]$

Leftmost derivation step

- each $A_i \notin \text{alph}(u_i)$ for all $1 \leq i \leq n$
- $\text{alph}(w)$ denotes the set of all symbols occuring in w

Generated Language

Generated language

$L(G) = \{x \in T^* : S \Rightarrow^* x\}$

Language generated in a leftmost way

- in addition, each step in every successful derivation is leftmost

Generative power

- $\mathscr{L}(SC) = \mathscr{L}(RE)$
- $\mathscr{L}(CF) \subset \mathscr{L}(PSC) \subseteq \mathscr{L}(CS)$

PSC Grammar—Example

Example

SC grammar $G_1 = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P_1, S)$ with

$$
\begin{aligned}
P_1 = \{&(S) \rightarrow (ABC),\\
&(A, B, C) \rightarrow (aA, bB, cC),\\
&(A, B, C) \rightarrow (\varepsilon, \varepsilon, \varepsilon)\}
\end{aligned}
$$

$S \Rightarrow ABC \Rightarrow aAbBcC \Rightarrow aaAbbBccC \Rightarrow aabbcc$
$L(G_1) = \{a^n b^n c^n \ : \ n \geq 0\}$

Example

PSC grammar $G_2 = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P_2, S)$ with

$$
\begin{aligned}
P_2 = \{&(S) \rightarrow (\varepsilon), (S) \rightarrow (ABC),\\
&(A, B, C) \rightarrow (aA, bB, cC),\\
&(A, B, C) \rightarrow (a, b, c)\}
\end{aligned}
$$

Production Labels I

- for every grammar G there is a set of production labels
- we denote them $lab(G)$
- each $p \in lab(G)$ uniquely identifies one production
- we write $p : (A_1, \ldots, A_n) \to (x_1, \ldots, x_n)$

Example

$$G = (\{S, A, B, C, a, b, c\}, \{a, b, c\}, P, S)$$

with

$$lab(G) = \{1, 2, 3\}$$

$$\begin{aligned} P = \{&1 : (S) \to (ABC), \\ &2 : (A, B, C) \to (aA, bB, cC), \\ &3 : (A, B, C) \to (\epsilon, \epsilon, \epsilon)\} \end{aligned}$$

$$L(G) = \{a^n b^n c^n \ : \ n \geq 0\}$$

Production Labels II

- to express that $x \Rightarrow y$ by $p : (A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)$, we write $x \Rightarrow y\ [p]$

Example

$S \Rightarrow ABC\ [1] \Rightarrow aAbBcC\ [2] \Rightarrow aaAbbBccC\ [2] \Rightarrow aabbcc\ [3]$ in G

- to express that $x \Rightarrow^* y$ by productions labeled with $p_1, \ldots, p_n$, we write $x \Rightarrow^* y\ [p_1 \ldots p_n]$
- $p_1 \ldots p_n \in lab(G)^*$

Example

$S \Rightarrow^* aabbcc\ [1223]$ in G
$1223 \in lab(G)^*$

Generator of its Sentences Preceded by Their Parses

Parse (Szilard word, control word)

If $S \Rightarrow^* x\ [\rho], x \in T^*, \rho \in lab(G)^*$, then x is a sentence generated by G according to parse ρ

Example

aabbcc is a sentence generated according to parse 1223 in G

Proper generator of its sentences preceded by their parses

G is a proper generator of its sentences preceded by their parses iff
$L(G) = \{x : x = \rho y, y \in (T - lab(G))^*, \rho \in lab(G)^*, S \Rightarrow^* x\ [\rho]\}$

Proper leftmost generator of its sentences preceded by their parses

- in addition, G generates $L(G)$ in a leftmost way

Proper Generator of its Sentences Preceded by Their Parses—Example

Example

$$G = (\{S, A, B, C, a, b, c, 1, 2, 3, \$\}, \{a, b, c, 1, 2, 3\}, P, S)$$

with

$$lab(G) = \{1, 2, 3\},$$

$$\begin{aligned} P = \{1 &: (S) \rightarrow (1\$ABC) \\ 2 &: (\$, A, B, C) \rightarrow (2\$, aA, bB, cC) \\ 3 &: (\$, A, B, C) \rightarrow (3, \epsilon, \epsilon, \epsilon)\} \end{aligned}$$

$S \Rightarrow\ 1\$ABC\ [1] \Rightarrow 12\$aAbBcC\ [2] \Rightarrow 122\$aaAbbBccC\ [2] \Rightarrow$
$1223aabbcc\ [3]$
$S \Rightarrow^* 1223aabbcc\ [1223]$

$L(G) = \{\rho a^n b^n c^n\ :\ n \geq 0, S \Rightarrow^* \rho a^n b^n c^n\ [\rho], \rho = 12^n 3\}$

G is a proper leftmost generator of its sentences preceded by their parses

Theorem 1

Production Length

Let $G = (V, T, P, S)$ be a SC grammar. Set

- $\text{len}((A_1, \dots, A_n) \to (x_1, \dots, x_n)) = |A_1 \dots A_n| = n$
- $\text{len}_{\text{max}}(P) = \text{len}(p)$, where $p \in P$ is a production satisfying $\text{len}(p) \geq \text{len}(r)$ for all $r \in P$

Left Quotient

$L_2 \backslash L_1 = \{y : xy \in L_1, \text{ for some } x \in L_2\}$

Theorem

For every recursively enumerable language L there exists a PSC grammar $G = (V, T, P, S)$ such that G is a proper leftmost generator of its sentences preceded by their parses, G contains no more than six nonterminals, $\text{len}_{\text{max}}(P) = 4$, *and* $L = \text{lab}(G)^+ \backslash L(G) \cap \text{alph}(L)^*$.

Extended Post Correspondence

Extended Post Correspondence (EPC)

$$E = (\{(u_1, v_1), \ldots, (u_r, v_r)\}, (z_{a_1}, \ldots, z_{a_n})),$$

where $T = \{a_1, \ldots, a_n\}$, $u_i, v_i, z_{a_j} \in \{0,1\}^*$ for each $1 \leq i \leq r$ and $1 \leq j \leq n$

Language of EPC

$$L(E) = \{b_1 \ldots b_k \in T^* \,:\, v_{s_1} \ldots v_{s_l} = u_{s_1} \ldots u_{s_l} z_{b_1} \ldots z_{b_k} \text{ for some } s_1, \ldots, s_l \in \{1, \ldots, r\}, l \geq 1, k \geq 0\}$$

Example

$E = (\{(01, 011), (1, 10), (1, 11)\}, (1_a, 01_b))$, $T = \{a, b\}$
$0111011 = 011101_b 1_a$
$ba \in L(E)$

Power of EPC

Theorem

Every recursively enumerable language can be represented by an EPC.

Representation of a RE language

- let $L \subseteq T^*$, where $T = \{a_1, \dots, a_n\}$, be a RE language
- L is represented by an EPC

$$E = (D, (z_{a_1}, \dots, z_{a_n})),$$

where

$$D = \{(u_1, v_1), \dots, (u_r, v_r)\},$$

$u_i, v_i, z_{a_j} \in \{0, 1\}^*$ for each $1 \leq i \leq r$, $1 \leq j \leq n$

Construction

Basic Idea

0. Represent the RE language L by an EPC
1. Generate strings $z_{a_{j_1}} \dots z_{a_{j_n}}$ and $a_{j_1} \dots a_{j_n}$
2. Generate strings $u_{s_1} \dots u_{s_l}$ and $v_{s_1} \dots v_{s_l}$
3. Verify if $u_{s_1} \dots u_{s_l} z_{a_{j_1}} \dots z_{a_{j_n}} = v_{s_1} \dots v_{s_l}$
4. Replace all auxiliary symbols with labels

$+$ after using a production, add its label to a sentential form

Define the PSC grammar

$$G = (\{S, A, B, 0, 1, \#\} \cup T \cup \mathrm{lab}(G), T \cup \mathrm{lab}(G), P, S),$$

where

$$\begin{aligned}\mathrm{lab}(G) = {} & \{\lfloor 1 \rfloor, \lfloor 3 \rfloor, \lfloor 3_0 \rfloor, \lfloor 3_1 \rfloor, \lfloor 4 \rfloor, \lfloor 4_0 \rfloor, \lfloor 4_1 \rfloor, \lfloor 4_2 \rfloor\} \\ & \cup \{\lfloor 1_a \rfloor : a \in T\} \cup \{\lfloor 2_{u_i v_i} \rfloor, \lfloor 2_{0 u_i v_i} \rfloor : (u_i, v_i) \in D\}\end{aligned}$$

Construction of P, Step 1

Step 1

For each $a \in T$, add

1. $\lfloor 1 \rfloor : (S) \rightarrow (\lfloor 1 \rfloor AA)$
2. $\lfloor 1_a \rfloor : (A, A) \rightarrow (\lfloor 1_a \rfloor Az_a, Aa)$

Example

$$\begin{aligned} S &\Rightarrow \lfloor 1 \rfloor AA \\ &\Rightarrow \lfloor 1 \rfloor \lfloor 1_{a_{j_n}} \rfloor Az_{a_{j_n}} Aa_{j_n} \\ &\quad \vdots \\ &\Rightarrow \lfloor 1 \rfloor \lfloor 1_{a_{j_n}} \rfloor \dots \lfloor 1_{a_{j_1}} \rfloor Az_{a_{j_1}} \dots z_{a_{j_n}} Aa_{j_1} \dots a_{j_n} \end{aligned}$$

Construction of P, Step 2

Step 2

For each $(u_i, v_i) \in D$, $1 \leq i \leq r$, add

1. $\lfloor 2_{u_i v_i} \rfloor : (A, A) \rightarrow (\lfloor 2_{u_i v_i} \rfloor B u_i, B v_i)$
2. $\lfloor 2_{0 u_i v_i} \rfloor : (B, B) \rightarrow (\lfloor 2_{0 u_i v_i} \rfloor B u_i, B v_i)$

Example

$$
\begin{aligned}
& \lfloor 1 \rfloor \lfloor 1_{a_{j_n}} \rfloor \ldots \lfloor 1_{a_{j_1}} \rfloor A z_{a_{j_1}} \ldots z_{a_{j_n}} A a_{j_1} \ldots a_{j_n} \\
\Rightarrow & \lfloor 1 \rfloor \lfloor 1_{a_{j_n}} \rfloor \ldots \lfloor 1_{a_{j_1}} \rfloor \lfloor 2_{u_{s_l} v_{s_l}} \rfloor B u_{s_l} z_{a_{j_1}} \ldots z_{a_{j_n}} B v_{s_l} a_{j_1} \ldots a_{j_n} \\
\Rightarrow & \lfloor 1 \rfloor \lfloor 1_{a_{j_n}} \rfloor \ldots \lfloor 1_{a_{j_1}} \rfloor \lfloor 2_{u_{s_l} v_{s_l}} \rfloor \lfloor 2_{0 u_{s_{l-1}} v_{s_{l-1}}} \rfloor \\
& B u_{s_{l-1}} u_{s_l} z_{a_{j_1}} \ldots z_{a_{j_n}} B v_{s_{l-1}} v_{s_l} a_{j_1} \ldots a_{j_n} \\
& \vdots \\
\Rightarrow & \lfloor 1 \rfloor \lfloor 1_{a_{j_n}} \rfloor \ldots \lfloor 1_{a_{j_1}} \rfloor \lfloor 2_{u_{s_l} v_{s_l}} \rfloor \lfloor 2_{0 u_{s_{l-1}} v_{s_{l-1}}} \rfloor \ldots \lfloor 2_{0 u_{s_1} v_{s_1}} \rfloor \\
& B u_{s_1} \ldots u_{s_l} z_{a_{j_1}} \ldots z_{a_{j_n}} B v_{s_1} \ldots v_{s_l} a_{j_1} \ldots a_{j_n}
\end{aligned}
$$

Construction of P, Step 3

Step 3 ($u_{s_1} \ldots u_{s_l} z_{a_{j_1}} \ldots z_{a_{j_n}} \stackrel{?}{=} v_{s_1} \ldots v_{s_l}$)

Add

1. $\lfloor 3 \rfloor : (B, B) \rightarrow (\lfloor 3 \rfloor A, B)$
2. $\lfloor 3_0 \rfloor : (A, 0, B, 0) \rightarrow (\lfloor 3_0 \rfloor, A, \#, B)$
3. $\lfloor 3_1 \rfloor : (A, 1, B, 1) \rightarrow (\lfloor 3_1 \rfloor, A, \#, B)$

Example

$$
\begin{aligned}
& \ldots B u_{s_1} \ldots u_{s_l} z_{a_{j_1}} \ldots z_{a_{j_n}} B v_{s_1} \ldots v_{s_l} a_{j_1} \ldots a_{j_n} \\
\Rightarrow & \ldots \lfloor 3 \rfloor A01 \ldots 01B01 \ldots 01 a_{j_1} \ldots a_{j_n} \\
\Rightarrow & \ldots \lfloor 3 \rfloor \lfloor 3_0 \rfloor A1 \ldots 01\#B1 \ldots 01 a_{j_1} \ldots a_{j_n} \\
& \vdots \\
\Rightarrow & \ldots \lfloor 3 \rfloor \lfloor 3_0 \rfloor \ldots \lfloor 3_1 \rfloor A\# \ldots \#B a_{j_1} \ldots a_{j_n}
\end{aligned}
$$

Construction of P, Step 4

Step 4

Add

1. $\lfloor 4 \rfloor : (A, B) \to (\lfloor 4 \rfloor B, A)$
2. $\lfloor 4_0 \rfloor : (B, \#) \to (\lfloor 4_0 \rfloor, B)$
3. $\lfloor 4_1 \rfloor : (B, A) \to (\lfloor 4_1 \rfloor, B)$
4. $\lfloor 4_2 \rfloor : (B) \to (\lfloor 4_2 \rfloor)$

Example

$$
\begin{aligned}
& \ldots A\# \ldots \# B a_{j_1} \ldots a_{j_n} \Rightarrow \ldots \lfloor 4 \rfloor B\# \ldots \# A a_{j_1} \ldots a_{j_n} \\
\Rightarrow\ & \ldots \lfloor 4 \rfloor \lfloor 4_0 \rfloor B\# \ldots \# A a_{j_1} \ldots a_{j_n} \Rightarrow \ldots \\
\ldots \Rightarrow\ & \ldots \lfloor 4 \rfloor \lfloor 4_0 \rfloor \ldots \lfloor 4_0 \rfloor B A a_{j_1} \ldots a_{j_n} \\
\Rightarrow\ & \ldots \lfloor 4 \rfloor \lfloor 4_0 \rfloor \ldots \lfloor 4_0 \rfloor \lfloor 4_1 \rfloor B a_{j_1} \ldots a_{j_n} \\
\Rightarrow\ & \ldots \lfloor 4 \rfloor \lfloor 4_0 \rfloor \ldots \lfloor 4_0 \rfloor \lfloor 4_1 \rfloor \lfloor 4_2 \rfloor a_{j_1} \ldots a_{j_n}
\end{aligned}
$$

Theorem 2

Theorem

For every recursively enumerable language L there exists a PSC grammar $G = (V, T, P, S)$ such that G is a proper leftmost generator of its sentences preceded by their parses, G contains no more than nine nonterminals, $\mathrm{len}_{\mathrm{max}}(P) = 2$, and $L = \mathrm{lab}(G)^{+} \backslash L(G) \cap \mathrm{alph}(L)^{*}$.

Proof

Define the PSC grammar

$$G' = (\{S, A, B, C, 0, 1, \$_0, \$_1, \#\} \cup T \cup \mathrm{lab}(G'), T \cup \mathrm{lab}(G'), P', S),$$

where

$$\begin{aligned}\mathrm{lab}(G') = {} & (\mathrm{lab}(G) - \{\lfloor 3_0 \rfloor, \lfloor 3_1 \rfloor\}) \\ & \cup \{\lfloor 3_{01} \rfloor, \lfloor 3_{02} \rfloor, \lfloor 3_{03} \rfloor, \lfloor 3_{04} \rfloor, \lfloor 3_{11} \rfloor, \lfloor 3_{12} \rfloor, \lfloor 3_{13} \rfloor, \lfloor 3_{14} \rfloor\}\end{aligned}$$

Construction of P', Step 3

- Steps 1, 2, and 4 are the same as in the proof of Theorem 1

Step 3 ($u_{s_1} \dots u_{s_l} z_{a_{j_1}} \dots z_{a_{j_n}} \stackrel{?}{=} v_{s_1} \dots v_{s_l}$)

Add

1. $\lfloor 3 \rfloor : (B, B) \rightarrow (\lfloor 3 \rfloor A, B)$
2. **a** $\lfloor 3_{01} \rfloor : (B, 0) \rightarrow (\#, \$_0)$
   b $\lfloor 3_{02} \rfloor : (A, \$_0) \rightarrow (C, \$_0)$
   c $\lfloor 3_{03} \rfloor : (C, 0) \rightarrow (\lfloor 3_{01} \rfloor \lfloor 3_{02} \rfloor \lfloor 3_{03} \rfloor, \$_0)$
   d $\lfloor 3_{04} \rfloor : (\$_0, \$_0) \rightarrow (\lfloor 3_{04} \rfloor A, B)$
3. **a** $\lfloor 3_{11} \rfloor : (B, 1) \rightarrow (\#, \$_1)$
   b $\lfloor 3_{12} \rfloor : (A, \$_1) \rightarrow (C, \$_1)$
   c $\lfloor 3_{13} \rfloor : (C, 1) \rightarrow (\lfloor 3_{11} \rfloor \lfloor 3_{12} \rfloor \lfloor 3_{13} \rfloor, \$_1)$
   d $\lfloor 3_{14} \rfloor : (\$_1, \$_1) \rightarrow (\lfloor 3_{14} \rfloor A, B)$

Construction of P', Step 3—Example

Example

$$\begin{aligned} & \dots \lfloor 3 \rfloor A01 \dots 01B01 \dots 01a_{j_1} \dots a_{j_n} \\ \Rightarrow\ & \dots \lfloor 3 \rfloor \lfloor 3_0 \rfloor A1 \dots 01\#B1 \dots 01a_{j_1} \dots a_{j_n} \end{aligned}$$

in G is simulated by

$$\begin{aligned} & \dots \lfloor 3 \rfloor A01 \dots 01B01 \dots 01a_{j_1} \dots a_{j_n} \\ \Rightarrow\ & \dots \lfloor 3 \rfloor A01 \dots 01\#\$_0 1 \dots 01a_{j_1} \dots a_{j_n} \\ \Rightarrow\ & \dots \lfloor 3 \rfloor C01 \dots 01\#\$_0 1 \dots 01a_{j_1} \dots a_{j_n} \\ \Rightarrow\ & \dots \lfloor 3 \rfloor \lfloor 3_{01} \rfloor \lfloor 3_{02} \rfloor \lfloor 3_{03} \rfloor \$_0 1 \dots 01\#\$_0 1 \dots 01a_{j_1} \dots a_{j_n} \\ \Rightarrow\ & \dots \lfloor 3 \rfloor \lfloor 3_{01} \rfloor \lfloor 3_{02} \rfloor \lfloor 3_{03} \rfloor \lfloor 3_{04} \rfloor A1 \dots 01\#B1 \dots 01a_{j_1} \dots a_{j_n} \end{aligned}$$

in G'

Conclusion

We have proved that

- for every RE language there is a PSC grammar which generates its sentences preceded by their parses
- this grammar generates its language in a leftmost way
- the total number of nonterminals and production length can be reduced

Future investigation

- which other grammars can be used for this type of generation?
- is it possible to generate sentences together with other useful information?