Maximal and Minimal Scattered Context Rewriting

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Scattered Context Grammar

Scattered context grammar (SC grammar)

G = (V, T, P, S), where

V is a finite alphabet

T is a set of terminals, $T \subset V$

S is the start symbol, $S \in V - T$

P is a finite set of productions of the form

$$(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n),$$

where $A_1, \ldots, A_n \in V - T$, $x_1, \ldots, x_n \in V^*$

Propagating scattered context grammar (PSC grammar)

- each $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \in V^+$

SC Grammar—Derivation Step

Derivation step

For $(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n) \in P$ and

$$u = u_1 A_1 \ldots u_n A_n u_{n+1}$$
$$v = u_1 x_1 \ldots u_n x_n u_{n+1}$$

we write $u \Rightarrow v \; [(A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)]$

Generated language

$L(G) = \{x \in T^* : S \Rightarrow^* x\}$

Generative power

- $\mathscr{L}(SC) = \mathscr{L}(RE)$
- $\mathscr{L}(CF) \subset \mathscr{L}(PSC) \subseteq \mathscr{L}(CS)$

SC Grammar—Example

Example

SC grammar $G = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P, S)$ with

$$\begin{aligned} P = \{&(S) \to (ABC), \\ &(A, B, C) \to (aA, bB, cC), \\ &(A, B, C) \to (\varepsilon, \varepsilon, \varepsilon)\} \end{aligned}$$

$S \Rightarrow ABC \Rightarrow aAbBcC \Rightarrow aaAbbBccC \Rightarrow aabbcc$
$L(G) = \{a^n b^n c^n : n \geq 0\}$

Maximal and Minimal Derivation

Production length

- $\text{len}((A_1, \ldots, A_n) \rightarrow (x_1, \ldots, x_n)) = |A_1 \ldots A_n| = n$

Maximal derivation step

$u \ {}_{\max}\!\Rightarrow v\ [p]$, $p \in P$ if and only if

1. $u \Rightarrow v\ [p]$,
2. there is no $r \in P$, $\text{len}(r) > \text{len}(p)$, such that $u \Rightarrow w\ [r]$

Minimal derivation step

$u \ {}_{\min}\!\Rightarrow v\ [p]$, $p \in P$ if and only if

1. $u \Rightarrow v\ [p]$,
2. there is no $r \in P$, $\text{len}(r) < \text{len}(p)$, such that $u \Rightarrow w\ [r]$

Maximal and Minimal Languages

Maximal and minimal languages

- $L_{\max}(G) = \{x \in T^* : S \ {}_{\max}\Rightarrow^* x\}$
- $L_{\min}(G) = \{x \in T^* : S \ {}_{\min}\Rightarrow^* x\}$
- language families denoted by $\mathscr{L}(PSC, \max)$ and $\mathscr{L}(PSC, \min)$

Example

SC grammar $G = (\{A, B, C, S, a, b, c\}, \{a, b, c\}, P, S)$ with

$$\begin{aligned} P = \{&(S) \rightarrow (ABC), \\ &(A) \rightarrow (a), (B) \rightarrow (b), (C) \rightarrow (c), \\ &(A, B, C) \rightarrow (aA, bB, cC), \\ &(A, B, C) \rightarrow (\varepsilon, \varepsilon, \varepsilon)\} \end{aligned}$$

$L_{\max}(G) = \{a^n b^n c^n : n \geq 0\}$,
$L_{\min}(G) = \{abc\}$

Main Result I

Theorem
$\mathscr{L}(CS) = \mathscr{L}(PSC, \max)$

State Grammar

State grammar (ST grammar)

$G = (V, T, K, P, S, p_0)$, where

- **V** is a finite alphabet
- **T** is a set of terminals, $T \subset V$
- **K** is a finite set of states
- **S** is the start symbol, $S \in V - T$
- p_0 $p_0 \in K$
- **P** is a finite set of productions of the form

$$(A, p) \rightarrow (x, q),$$

  where $p, q \in K$, $A \in V - T$, $x \in V^+$

State Grammar—Derivation Step

Derivation step

For $(A, p) \rightarrow (x, q) \in P$,

$$u = (rAs, p),$$
$$v = (rxs, q),$$

where $r, s \in V^*$, and for every $(B, p) \rightarrow (y, t) \in P$, $B \notin \text{alph}(r)$, we write

$$u \Rightarrow v \; [(A, p) \rightarrow (x, q)]$$

Generated language

$L(G) = \{x \in T^* : (S, p_0) \Rightarrow^* (x, q) \text{ for some } q \in K\}$

Generative power

$\mathscr{L}(ST) = \mathscr{L}(CS)$

Basic Idea I

CS language representation

- let $L \in \mathscr{L}(CS)$
- let $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$ be a state grammar such that $L = L(\bar{G})$
- let $(x_1 B x_2 A x_3 C x_4, p)$ be a sentential form of $\bar{G}$
- let $(x_1 B x_2 A x_3 C x_4, p) \Rightarrow (x_1 B x_2 x x_3 C x_4, q)$ $[(A, p) \rightarrow (x, q)]$ in $\bar{G}$

Construction of maximal PSC grammar

- $(x_1 B x_2 A x_3 C x_4, p)$ corresponds to $x_1 B x_2 \langle A, p\rangle x_3 C x_4$ in G
- simulate $(A, p) \rightarrow (x, q)$ by
    1. $(B, \langle A, p\rangle, C) \rightarrow (X, X, X)$ for every $(B, p) \rightarrow (y, r) \in P$
    2. $(B, \langle A, p\rangle) \rightarrow (\langle B, q\rangle, x)$

$$x_1 B x_2 \langle A, p\rangle x_3 C x_4 \Rightarrow x_1 \langle B, q\rangle x_2 x x_3 C x_4$$

Definitions

1. let $\Delta(t)$ be the set of all permutations of $\{1, \dots, t\}$
2. let permute$(n, m) =$
   $\{(i_1, \dots, i_{n+m}) \in \Delta(n+m) : 1 \leq i_k < i_l \leq n \text{ implies } k < l\}$
3. let reorder$((x_1, \dots, x_n), (i_1, \dots, i_n)) = (x_{i_1}, \dots, x_{i_n})$
   for $x_1, \dots, x_n \in V^*$, $(i_1, \dots, i_n) \in \Delta(n)$
4. let $\Pi((x_1, \dots, x_n), i) = x_i$

Example

1. $\Delta(3) = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$
2. permute$(2, 1) = \{(1,2,3), (1,3,2), (3,1,2)\}$
3. reorder$((x_1, x_2, x_3), (3,1,2)) = (x_3, x_1, x_2)$
4. $\Pi((x_3, x_1, x_2), 3) = x_2$

Construction I

- let $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$
- set $Y = \{\langle A, q\rangle : A \in \bar{V} - T, q \in K\}$
- set $Z = \{\bar{a} : a \in T\}$
- define the homomorphism α, form $\bar{V}^*$ to $((\bar{V} - T) \cup Z)^*$ as
  - $\alpha(A) = A$ for all $A \in \bar{V} - T$ and
  - $\alpha(a) = \bar{a}$ for all $a \in T$
- set $V = \bar{V} \cup Y \cup Z \cup \{S, X\}$
- define the propagating scattered context grammar $G = (V, T, P, S)$ with P defined as follows:

Step 1

For every $x \in L(\bar{G})$, $|x| \leq 2$, add
$(S) \rightarrow (x)$

Construction II

Step 2

For every

$$(x, q) \in \{(x, q) : (\bar{S}, p_0) \Rightarrow^{+}_{\bar{G}} (x, q) \text{ for some } q \in K \\ \text{and } 3 \leq |x| \leq \min\{3, \max\{|\Pi(\text{rhs}(p), 1)| : p \in \bar{P}\}\}\},$$

where

1. $x \in T^*$, add
   $(S) \rightarrow (x)$
2. $x = x_1 A x_2$, $A \in \bar{V} - T$, $x_1, x_2 \in \bar{V}^*$, add
   $(S) \rightarrow (\alpha(x_1)\langle A, q\rangle\alpha(x_2))$

Construction III

Step 3

For every $(A, p) \rightarrow (x, q)$, $(B, p) \rightarrow (y, r) \in \bar{P}$, $C \in \bar{V}$, $\Gamma_{21} \in \text{permute}(2, 1)$,

$$z = \text{reorder}((B, \langle A, p \rangle, \alpha(C)), \Gamma_{21}),$$

add
$z \rightarrow (X, X, X)$

Construction IV

Step 4

For every $(A, p) \to (x, q) \in \bar{P}$, $B \in \bar{V} - T$, $C \in \bar{V}$, $\Gamma_{11} \in \mathrm{permute}(1, 1)$,

$$y = \mathrm{reorder}((\langle A, p \rangle, \alpha(C)), \Gamma_{11}),$$

add

1. $(B, \langle A, p \rangle) \to (\langle B, q \rangle, \alpha(x))$,
2. $(\langle A, p \rangle, B) \to (\alpha(x), \langle B, q \rangle)$
3. If $x = vBw$, $v, w \in \bar{V}^*$, for every

   $$z = \mathrm{reorder}((\alpha(v)\langle B, q \rangle \alpha(w), \alpha(C)), \Gamma_{11}),$$

   add $y \to z$
4. For every $u = \mathrm{reorder}((\alpha(x), \alpha(C)), \Gamma_{11})$,
   add $y \to u$

Construction V

Step 5

For every $a \in T$, add
$(\bar{a}) \rightarrow (a)$

Main Result II

Theorem
$\mathscr{L}(CS) = \mathscr{L}(PSC, \min)$

Basic Idea II

CS language representation

- let $L \in \mathscr{L}(CS)$
- let $\bar{G} = (\bar{V}, T, K, \bar{P}, \bar{S}, p_0)$ be a state grammar such that $L = L(\bar{G})$
- let $(x_1 B x_2 A x_3 C x_4, p)$ be a sentential form of $\bar{G}$
- let $(x_1 B x_2 A x_3 C x_4, p) \Rightarrow (x_1 B x_2 x x_3 C x_4, q)$ $[(A, p) \to (x, q)]$ in $\bar{G}$

Construction of minimal *PSC* grammar

- $(x_1 B x_2 A x_3 C x_4, p)$ corresponds to $x_1 B x_2 \langle A, p \rangle x_3 C x_4$ in G
- simulate $(A, p) \to (x, q)$ by
  1. $(B, \langle A, p \rangle) \to (X, X)$ for every $(B, p) \to (y, r) \in P$
  2. $(B, \langle A, p \rangle, C) \to (\langle B, q \rangle, x, C)$

$$x_1 B x_2 \langle A, p \rangle x_3 C x_4 \Rightarrow x_1 \langle B, q \rangle x_2 x x_3 C x_4$$

Summary

$$\mathscr{L}(CF) \subset \mathscr{L}(PSC) \subseteq \mathscr{L}(CS)$$

By restricting derivations to

- the longest applicable productions, we obtain

$$\mathscr{L}(PSC, \mathrm{max}) = \mathscr{L}(CS)$$

- the shortest applicable productions, we obtain

$$\mathscr{L}(PSC, \mathrm{min}) = \mathscr{L}(CS)$$