

Turing Machines and Two-Pushdown Automata

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Turing Machines

Turing Machine

A Turing machine is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q is a finite set of states

Σ is a tape alphabet, $\Sigma \cap Q = \emptyset$,

$I \subset \Sigma$ is an input alphabet,

$\sqcup \in \Sigma - I$ is the blank symbol

$R \subseteq Q\Sigma \times Q\Sigma$ is a finite set of rules,

$R = R_s \cup R_r \cup R_l$ (stationary, right, and left moves)

$s \in Q$ is the start state

$F \subseteq Q$ is a set of final states

Turing Machines – Notation

Stationary move

$(qX, pY) \in R_s$ is symbolically written as

$$qX \rightarrow_s pY$$

Right move

$(qX, pY) \in R_r$ is symbolically written as

$$qX \rightarrow_r pY$$

Left move

$(qX, pY) \in R_l$ is symbolically written as

$$qX \rightarrow_l pY$$

Turing Machines – Computational Step

Configuration

$$\chi \in \Sigma^* Q \Sigma^* \{\sqcup\}$$

Move

If at least one of the following holds,

Stationary move $\chi = x p U y$, $\chi' = x q V y$, and $r : p U \rightarrow_s q V \in R$,

Right move $\chi = x p U y$, $\chi' = x V q y'$, and $r : p U \rightarrow_r q V \in R$,
 $y' = y$ if $y \neq \varepsilon$, and $y' = \sqcup$ if $y = \varepsilon$

Left move $\chi = x X p U y$, $\chi' = x q X V y$, and $r : p U \rightarrow_l q V \in R$,
for some $X \in \Sigma$

then

$$\chi \Rightarrow \chi' [r]$$

Turing Machines – Accepted Language

Accepted Word

Turing machine M accepts $w \in I^*$ if

$$sw \sqcup \Rightarrow^* uf v$$

for some configuration ufv with $f \in F$

■ \Rightarrow^* denotes the reflexive and transitive closure of \Rightarrow

Accepted Language

The set of all words M accepts is the language of M , denoted by $L(M)$, thus

$$L(M) = \{w \in I^* : sw \sqcup \Rightarrow^* uf v, f \in F\}$$

Turing Machines – Example

Example

$$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{a, b, \sqcup\}, R, q_0, \{q_4\})$$

where

$$R = \{ \begin{array}{ll} 1 : q_0 a \rightarrow_r q_1 \sqcup, & 5 : q_1 \sqcup \rightarrow_l q_2 \sqcup, \\ 2 : q_1 a \rightarrow_r q_1 a, & 6 : q_2 b \rightarrow_l q_3 \sqcup, \\ 3 : q_1 b \rightarrow_r q_1 b, & 7 : q_3 a \rightarrow_l q_3 a, \\ 4 : q_3 \sqcup \rightarrow_r q_0 \sqcup, & 8 : q_3 b \rightarrow_l q_3 b, \quad 9 : q_0 \sqcup \rightarrow_s q_4 \sqcup \end{array} \}$$

$$\begin{aligned} q_0 a a b b \sqcup &\Rightarrow \sqcup q_1 a b b \sqcup [1] \Rightarrow \sqcup a q_1 b b \sqcup [2] \Rightarrow \sqcup a b q_1 b \sqcup [3] \\ &\Rightarrow \sqcup a b b q_1 \sqcup [3] \Rightarrow \sqcup a b q_2 b \sqcup [5] \Rightarrow \sqcup a q_3 b \sqcup \sqcup [6] \Rightarrow \sqcup q_3 a b \sqcup [8] \\ &\Rightarrow q_3 \sqcup a b \sqcup [7] \Rightarrow \sqcup q_0 a b \sqcup [4] \Rightarrow \sqcup \sqcup q_1 b \sqcup [1] \Rightarrow \sqcup b q_1 \sqcup [3] \\ &\Rightarrow \sqcup q_2 b \sqcup [5] \Rightarrow q_3 \sqcup \sqcup \sqcup [6] \Rightarrow \sqcup q_0 \sqcup [4] \Rightarrow \sqcup q_4 \sqcup [9] \end{aligned}$$

$$L(M) = \{a^n b^n : n \geq 0\}$$

Church's Thesis

Church's Thesis

For every algorithm that exists there is an equivalent Turing Machine.

Recursively Enumerable Language

A language L is recursively enumerable if there is a Turing machine M such that $L(M) = L$.

Recursive Language

A language L is recursive if there is a Turing machine M that always halts such that $L(M) = L$.

Deterministic Turing Machine

Deterministic Turing Machine

Turing machine M is deterministic if every rule $r \in R$ satisfies

$$\text{lhs}(r) \notin \{\text{lhs}(r') : r' \in R - \{r\}\}$$

Theorem

A language L is recursively enumerable if there is a deterministic Turing machine M such that $L(M) = L$.

Theorem

A language L is recursive if there is a deterministic Turing machine M that always halts such that $L(M) = L$.

Linear Bounded Automata

Linear Bounded Automaton

A linear bounded automaton is a Turing machine M that never extends its tape.

Consequence

With an input word w , M uses no more than the first $|w|$ tape squares.

Theorem

A language L is context-sensitive if and only if there is a linear bounded automaton M such that $L(M) = L$.

Open Problem

Are deterministic linear bounded automata as powerful as linear bounded automata?

Two-Pushdown Automata

Two-Pushdown Automaton

A two-pushdown automaton is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q , s , F have the same meaning as in the definition of Turing machine

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = \{|\} \cup I \cup P_D$, where

$|$ is a special symbol, $| \notin I \cup P_D$,

I is an input alphabet, P_D is a pushdown alphabet, $S \in P_D$ is a start pushdown symbol

R is a finite set of rules of the form

$$A|Bpa \rightarrow u|vq$$

where $A, B \in P_D$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u, v \in P_D^*$

Two-Pushdown Automata – Computational Step

Configuration

$$\chi \in P_D^* \{|\} P_D^* Q I^*$$

Move

If

$$r : A|Bpa \rightarrow u|vq \in R,$$

$$\chi = yA|xBpaz,$$

$$\chi' = yu|xvqz,$$

then

$$\chi \Rightarrow \chi' [r]$$

Two-Pushdown Automata – Accepted Language

Accepted Language by Final State

$$L_f(M) = \{w \in I^* : S|Ssw \Rightarrow^* x|yf, f \in F\}$$

Accepted Language by Empty Pushdown

$$L_e(M) = \{w \in I^* : S|Ssw \Rightarrow^* |q, q \in Q\}$$

Accepted Language by Final State and Empty Pushdown

$$L_{fe}(M) = \{w \in I^* : S|Ssw \Rightarrow^* |f, f \in F\}$$

■ \Rightarrow^* denotes the reflexive and transitive closure of \Rightarrow

Two-Pushdown Automata – Example

Example

$$M = (\{s, p, q, f\}, \{S, a, b, c, |\}, R, s, \{f\}),$$

where

$$R = \{ \begin{array}{ll} 1 : S|Ssa \rightarrow S|Sas, & 4 : b|aqb \rightarrow bb|q, \\ 2 : S|asa \rightarrow S|aas, & 5 : b|Sqc \rightarrow |Sp, \\ 3 : S|asb \rightarrow Sb|q, & 6 : b|Spc \rightarrow |Sp, \quad 7 : S|Sp \rightarrow |f \end{array} \}$$

Then,

$$\begin{aligned} S|Ssaabbcc &\Rightarrow S|Sasabbcc [1] \Rightarrow S|Saasbbcc [2] \Rightarrow Sb|Saqbcc [3] \\ &\Rightarrow Sbb|Sqcc [4] \Rightarrow Sb|Spc [5] \Rightarrow S|Sp [6] \Rightarrow |f [7] \end{aligned}$$

$$L_f(M) = L_e(M) = L_{fe}(M) = \{a^n b^n c^n : n \geq 1\}$$

Two-Pushdown Automata – Results

Determinism

M is deterministic if each $r \in R$ with $\text{lhs}(r) = A|Bpq$ satisfies

$$\{r\} = \{r' \in R : A|Bpa = \text{lhs}(r') \text{ or } A|Bp = \text{lhs}(r')\}$$

Theorem

All acceptance modes (f , e , fe) are equivalent.

Theorem

The following models are equivalent:

- *Turing machines*
- *deterministic Turing machines*
- *two-pushdown automata*
- *deterministic two-pushdown automata*

Bibliography



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