Matrix Grammars

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Regulated Rewriting

Example

$$
\begin{aligned}
1 &: S \rightarrow AB \\
2 &: A \rightarrow aA \\
3 &: B \rightarrow bBc \\
4 &: A \rightarrow a \\
5 &: B \rightarrow bc
\end{aligned}
$$

$$
\begin{tikzcd}
1 \arrow[r] \arrow[rrr, bend left] & 2 \arrow[r] & 3 \arrow[r] \arrow[l, bend left] & 4 \arrow[r] & 5
\end{tikzcd}
$$

Example of derivation

$$
\begin{aligned}
S \Rightarrow AB\ [1] &\Rightarrow aAB\ [2] \Rightarrow aAbBc\ [3] \\
&\Rightarrow aaAbBc\ [2] \Rightarrow aaAbbBcc\ [3] \\
&\Rightarrow aaabbBcc\ [4] \Rightarrow aaabbbccc\ [5]
\end{aligned}
$$

Generated language

$$L(G) = \{a^n b^n c^n \,:\, n \geq 1\}$$

Matrix Grammar

Matrix Grammar

A matrix grammar is a pair

$$H = (G, M),$$

where

- $G = (N, T, P, S)$ is a context-free grammar
- M is a finite language over P ($M \subseteq P^*$)

Notation

- Let $N = \{A_1, \ldots, A_m\}$ for some $m \geq 1$
- For some $m_i = p_{i_1} \ldots p_{i_j} \ldots p_{i_{k_i}} \in M$,

$$p_{i_j} : A_{i_j} \rightarrow x_{i_j}$$

Generated Language

Derivation Step

For $x, y \in (N \cup T)^*$, $m \in M$,

$$x \Rightarrow y\ [m]$$

in H if there are $x_0, \ldots, x_n$ such that $x = x_0$, $x_n = y$, and

1. $x_0 \Rightarrow x_1\ [p_1] \Rightarrow x_2\ [p_2] \Rightarrow\ \ldots\ \Rightarrow x_n\ [p_n]$ in G, and
2. $m = p_1 \ldots p_n$

Generated Language

$$L(H) = \{x \in T^*\ :\ S \Rightarrow^* x\}$$

Example I

Example

$$H = (G, M),$$

where

- $G = (N, T, P, S)$, where

$$\begin{aligned} N &= \{S, A, B\} \\ T &= \{a, b, c\} \\ P &= \{1 : S \rightarrow AB, \\ &\quad\ 2 : A \rightarrow aA, \\ &\quad\ 3 : B \rightarrow bBc, \\ &\quad\ 4 : A \rightarrow a, \\ &\quad\ 5 : B \rightarrow bc\} \end{aligned}$$

- $M = \{1, 23, 45\}$

Example II

Example

1. In G,

$$\begin{aligned} aAbBc &\Rightarrow aaAbBc \quad [2] \\ &\Rightarrow aaAbbBcc \ [3] \end{aligned}$$

   As $23 \in M$,

$$aAbBc \Rightarrow aaAbbBcc \ [23]$$

   in H

2. In G,

$$\begin{aligned} aAbBc &\Rightarrow aaAbBc \ [2] \\ &\Rightarrow aaAbbcc \ [5] \end{aligned}$$

   As $25 \notin M$

$$aAbBc \not\Rightarrow aaAbbcc \ [25]$$

   in H

Example III

Example

$$
\begin{aligned}
S &\Rightarrow AB && [1]\\
&\Rightarrow aAbBc && [23]\\
&\Rightarrow aaAbbBcc && [23]\\
&\Rightarrow aaabbbccc && [45]\\
&\text{in } H
\end{aligned}
\qquad
\begin{aligned}
S &\Rightarrow AB && [1]\\
&\Rightarrow aAB && [2]\\
&\Rightarrow aAbBc && [3]\\
&\Rightarrow aaAbBc && [2]\\
&\Rightarrow aaAbbBcc && [3]\\
&\Rightarrow aaabbBcc && [4]\\
&\Rightarrow aaabbbccc && [5]\\
&\text{in } G
\end{aligned}
$$

Example IV

Example

By using $23 \in M$ n-times, $n \geq 0$

$$\begin{aligned} S &\Rightarrow AB & [1] \\ &\Rightarrow aAbBc & [23] \\ &\Rightarrow aaAbbBcc & [23] \\ &\ \ \vdots & \\ &\Rightarrow a^n A b^n B c^n & [23] \\ &\Rightarrow a^{n+1} b^{n+1} c^{n+1} & [45] \end{aligned}$$

Generated language

$$L(H) = \{a^m b^m c^m : m \geq 1\}$$

Example V

Example

Claim A
If $AB \Rightarrow^n x$, where $n \geq 1$, then $x \in \{a^n b^n c^n, a^n A b^n B c^n\}$.

Proof by Induction on $n \geq 1$

- Basis: $n = 1$.

$$AB \Rightarrow aAbBc$$
$$AB \Rightarrow abc$$

- Induction Hypothesis:
  Assume Claim A holds for all $n = 1, \ldots, k$, where k is a positive integer.

Example VI

Example

Proof by Induction on $n \geq 1$

- Induction Step:

$$S \Rightarrow^{k+1} x$$

can be rewritten as

$$S \Rightarrow^{k} y \Rightarrow x$$

By Induction Hypothesis, $y \in \{a^k A b^k B c^k, a^k b^k c^k\}$. As $y \Rightarrow x$, $y = a^k A b^k B c^k$,

$$y \Rightarrow x \text{ [23] and } x = a^{k+1} A b^{k+1} B c^{k+1}$$
$$y \Rightarrow x \text{ [45] and } x = a^{k+1} b^{k+1} c^{k+1}$$

so $x \in \{a^{k+1} A b^{k+1} B c^{k+1}, a^{k+1} b^{k+1} c^{k+1}\}$ $\square$

Example VII

Example

Claim B
$L(H) \subseteq \{a^k b^k c^k \,:\, k \geq 1\}$.

Proof
$L(H) = \{x \in T^* \,:\, S \Rightarrow^* x\}$. Every $S \Rightarrow^* x$ with $x \in T^*$ has the form

$$S \Rightarrow AB \Rightarrow^* x$$

From Claim A,

$$L(H) \subseteq \{a^k b^k c^k \,:\, k \geq 1\}$$

$\square$

Example VIII

Example

Claim C

For every $a^n b^n c^n$, where $n \geq 1$, $S \Rightarrow^* a^n b^n c^n$.

Proof by Induction on $n \geq 1$

- Basis: $n = 1$, $abc = x$.

$$\begin{aligned} S &\Rightarrow AB \ [1] \\ &\Rightarrow abc \ [45] \end{aligned}$$

- Induction Hypothesis:
  Assume Claim C holds for all $n = 1, \ldots, k$, where k is a positive integer.

Example IX

Example

Proof by Induction on $n \geq 1$

- Induction Step:

$$x = a^{k+1}b^{k+1}c^{k+1}$$

Consider $a^k b^k c^k$. By Induction Hypothesis, $S \Rightarrow^* a^k b^k c^k$. Express this derivation as

$$\begin{aligned} S &\Rightarrow^* a^{k-1}Ab^{k-1}Bc^{k-1} \\ &\Rightarrow a^k b^k c^k \qquad [45] \end{aligned}$$

Then,

$$\begin{aligned} S &\Rightarrow^* a^{k-1}Ab^{k-1}Bc^{k-1} \\ &\Rightarrow a^k A b^k B c^k \qquad [23] \\ &\Rightarrow a^{k+1}b^{k+1}c^{k+1} = x \end{aligned}$$

□

Example X

Example

From Claim B,

$$L(H) \subseteq \{a^k b^k c^k \,:\, k \geq 1\}$$

From Claim C,

$$\{a^k b^k c^k \,:\, k \geq 1\} \subseteq L(H)$$

Thus,

$$L(H) = \{a^k b^k c^k \,:\, k \geq 1\}$$

Matrix Grammar with Appearance Checking

Matrix Grammar with Appearance Checking

A matrix grammar with appearance checking is a pair

$$H = (G, M)$$

where

- $G = (N, T, P, S)$ is a context-free grammar
- M is a finite language over $P \times \{-, +\}$

Derivation Step

Derivation Step

For $x, y \in (N \cup T)^*$, $m = (p_1, q_1) \dots (p_n q_n) \in M$, $p_i \in P$, $q_i \in \{-, +\}$, $i = 1, \dots, n$,

$$x \Rightarrow y \ [m]$$

in H if there are $x_0, \dots, x_n$ such that $x = x_0$, $y = x_n$, and for $i = 1, \dots, n$

- either $x_{i-1} \Rightarrow x_i \ [p_i]$ in G
- or $q_i = +$, $x_{i-1} = x_i$, and p_i is not applicable to x_{i-1}

Example I

Example

$$
\begin{array}{ll}
1: S \rightarrow a & 6: \; A' \rightarrow A', C \rightarrow B \\
2: S \rightarrow aa & 7: \; A' \rightarrow A, \underline{C \rightarrow X} \\
3: S \rightarrow AB & 8: \; A' \rightarrow A'', \underline{C \rightarrow X} \\
4: A \rightarrow A, B \rightarrow CC & 9: \; A'' \rightarrow A'', B \rightarrow a \\
5: A \rightarrow A'C, \underline{B \rightarrow X} & 10: A'' \rightarrow a
\end{array}
$$

$$
\begin{tikzcd}
3 \arrow[r] & 4 \arrow[r] \arrow[loop above] & 5 \arrow[r] & 6 \arrow[r] \arrow[loop above] \arrow[rr, bend left] & 7 \arrow[lll, bend left] & 8 \arrow[r] & 9 \arrow[r] \arrow[loop above] & 10
\end{tikzcd}
$$

Notes:

<u>Underlined</u> productions are in appearance checking mode $(+)$

X is a "block" symbol

Example II

Example

Derivation example

$$\begin{aligned}
&\ldots\ ABBB \Rightarrow ABCCB\ [4] \Rightarrow ACCCCB\ [4] \Rightarrow ACCCCCC\ [4] \\
&\Rightarrow A'C^7\ [5] \Rightarrow A'CCBCCCC\ [6] \Rightarrow^6 A'B^7\ [6\ldots6] \Rightarrow AB^7\ [7\ldots7] \\
&\Rightarrow \ldots \\
&\Rightarrow A''BBBBBBB\ [8] \Rightarrow^7 A''aaaaaaa\ [9] \Rightarrow aaaaaaaa\ [10]
\end{aligned}$$

The generated language is $L(G) = \{a^{2^i} : i \geq 0\}$

- for $i \geq 2$, the derivation can be expressed as

$$\begin{aligned}
S\ {}_3\!\Rightarrow AB\ {}_{4,5}\!\Rightarrow^* A'C^3\ {}_{6,7}\!\Rightarrow^* AB^3 \Rightarrow^* AC^7 \\
\vdots \\
\Rightarrow A'B^{2^i-1}\ {}_8\!\Rightarrow A''B^{2^i-1}\ {}_{9,10}\!\Rightarrow^* a^{2^i}
\end{aligned}$$

- for $i = 1, 2$, $S \Rightarrow a$ [1] and $S \Rightarrow aa$ [2]

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