

Lindenmayer Systems

Jiří Techet Tomáš Masopust Alexander Meduna

Department of Information Systems
Faculty of Information Technology
Brno University of Technology
Božetěchova 2, Brno 61266, Czech Republic

Modern Formal Language Theory, 2007

0L System

0L System

An 0L system (0 stands for zero-sided context, i.e. context-free productions) is a triple

$$G = (T, P, w)$$

where

T is an alphabet

P is a finite set of productions of the form

$$a \rightarrow x$$

with $a \in T$ and $x \in T^*$

w is the start string (axiom), $w \in T^+$

D0L and P0L System

D0L System

If for each $a \in T$ there is exactly one production

$$a \rightarrow x \in P,$$

then G is a D0L system (D stands for Deterministic)

P0L System

If for each $a \rightarrow x \in P$,

$$x \neq \varepsilon,$$

then G is a P0L system (P stands for Propagating)

0L System – Generated Language

Direct Derivation

For some $n \geq 1$,

$$a_1 a_2 \dots a_n \Rightarrow x_1 x_2 \dots x_n$$

if for each $i = 1, \dots, n$,

$$a_i \rightarrow x_i \in P$$

Generated Language

For an L system $G = (T, P, w)$,

$$L(G) = \{y : w \Rightarrow^* y\}$$

0L System – Example I

Example

PD0L system $G = (\{a\}, \{a \rightarrow aa\}, a)$

$$L(G) = \{a^{2^n} : n \geq 0\}$$

Example

0L system $G = (\{a, b\}, \{a \rightarrow b, b \rightarrow ab\}, a)$

$$a \Rightarrow b \Rightarrow ab \Rightarrow bab \Rightarrow abbab \Rightarrow \dots$$

$$|L(G)| = \{i : i \geq 1, i \text{ is a Fibonacci number}\}$$

Every Fibonacci number f_n (for all $n \geq 0$) is defined as

- $f_0 = 0, f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

1

1

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#3

2	3
---	---

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#4

2	2	4
---	---	---

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#504

2	2	5/4
---	---	-----

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#60504

2	2	6/5	4
---	---	-----	---

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#7060504

2	2	7/6	5/4
---	---	-----	-----

(...) branch

8 branch position

0 oblique wall

vertical wall

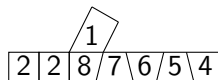
0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#8(1)07060504



(...) branch

8 branch position

0 oblique wall

vertical wall

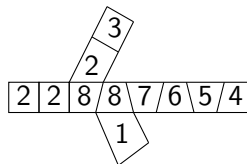
0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, 1)$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#8(2#3)08(1)07060504



(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Length Set

Length set of L

$$|L| = \{|x| : x \in L\}$$

Example

$$G = (\{a, b, c\}, \{a \rightarrow abcc, b \rightarrow bcc, c \rightarrow c\}, a)$$

$$a \Rightarrow abcc \Rightarrow abccbcccc \Rightarrow abccbccccbcccccc \dots$$

$$|L(G)| = \{i^2 : i \text{ is a natural number}\}$$

0L System – Closure Properties I

Theorem

$\mathcal{L}(0L)$ is not closed under union.

Proof

$\{a\} \in \mathcal{L}(0L)$ and $\{aa\} \in \mathcal{L}(0L)$, but

$$\{a, aa\} \notin \mathcal{L}(0L)$$



0L System – Closure Properties II

Theorem

$\mathcal{L}(0L)$ is not closed under positive closure $(+)$.

Basic Idea

Set

$$L = \{aa\} \cup \{b^{2^n} : n \geq 2\}$$

and prove that

1 $L \in \mathcal{L}(0L)$

2 $L^+ \notin \mathcal{L}(0L)$

Proof of $L \in \mathcal{L}(0L)$

Set $G = (\{a, b\}, P, aa)$ with $P = \{a \rightarrow bb, b \rightarrow bb\}$. Then,

$$L(G) = L = \{aa\} \cup \{b^{2^n} : n \geq 2\}$$

0L System – Closure Properties III

Proof of $L^+ \notin \mathcal{L}(0L)$

(Proof by contradiction.) Assume that there exists an 0L system

$$G = (\{a, b\}, P, w)$$

such that $L(G) = L^+$. As $\varepsilon \notin L^+$, G is propagating. Thus,

$$w = aa.$$

Consider $a^4 \in L^+$.

1 Assume $a^2 \Rightarrow a^4$.

a Let $\{a \rightarrow a, a \rightarrow aaa\} \subseteq P$. Then, $a^2 \Rightarrow b^4$ or $a^4 \Rightarrow b^4$. Thus,

$$a \rightarrow b^i \in P$$

for some $i \in \{1, 2, 3\}$. Hence, $aa \Rightarrow ab^i$ and $ab^i \notin L^+$ – a contradiction.

0L System – Closure Properties IV

Proof of $L^+ \notin \mathcal{L}(0L)$

1 b Assume $a^2 \Rightarrow a^4$ and $a \rightarrow aa \in P$.

■ If $a^2 \Rightarrow b^4$, then

$$a \rightarrow b^i$$

for some $i \in \{1, 2, 3\}$. Thus, $a^2 b^i \in L(G)$ – a contradiction.

■ If $a^4 \Rightarrow b^4$,

$$a \rightarrow b \in P.$$

Thus, $aab \in L(G)$ – a contradiction.

c Assume $a^2 \Rightarrow b^4 \Rightarrow a^4$. Then,

$$\{a \rightarrow bb, b \rightarrow a\} \subseteq P.$$

Consider any $x \in L(G)$ with $|x| = 6$. Then, $x \in \{a^2 b^4, b^4 a^2, a^6\}$.

0L System – Closure Properties V

Proof of $L^+ \notin \mathcal{L}(0L)$

1 c $a^4 \not\Rightarrow x$.

A $a^4 \Rightarrow b^4 a^2$.

If $a \rightarrow a^i \in P$, $i \in \{1, 2\}$, then $aa \Rightarrow a^i b^2 \in L(G)$ – a contradiction.

If $b \rightarrow b \in P$, then $bbba \in L(G)$ – a contradiction.

B $a^4 \Rightarrow a^2 b^4$ – analogy.

C If $a^4 \Rightarrow a^6$, then $bba^i \in L(G)$ – a contradiction.

d $b^4 \not\Rightarrow x$

A $b^4 \Rightarrow b^4 a^2$. Then $b \rightarrow b^i \in P$ for some $i \geq 1$. Then, $b^4 \Rightarrow b^{3i} a$ – a contradiction.

B $b^4 \Rightarrow a^2 b^4$ – analogy.

C ...

e $a^2 \not\Rightarrow x$.

\vdots



OL System – Closure Properties VI

Theorem

$\mathcal{L}(OL)$ is not closed under

- *homomorphism*
- *inverse homomorphism*
- *intersection and intersection with a regular set*
- *concatenation*
- *complementation*

Theorem

$\mathcal{L}(OL)$ is closed under reversal.

OL System – Other Properties

Theorem

If $L \in \mathcal{L}(OL)$, $L \subseteq \{a\}^$, then $L^* \in \mathcal{L}(OL)$.*

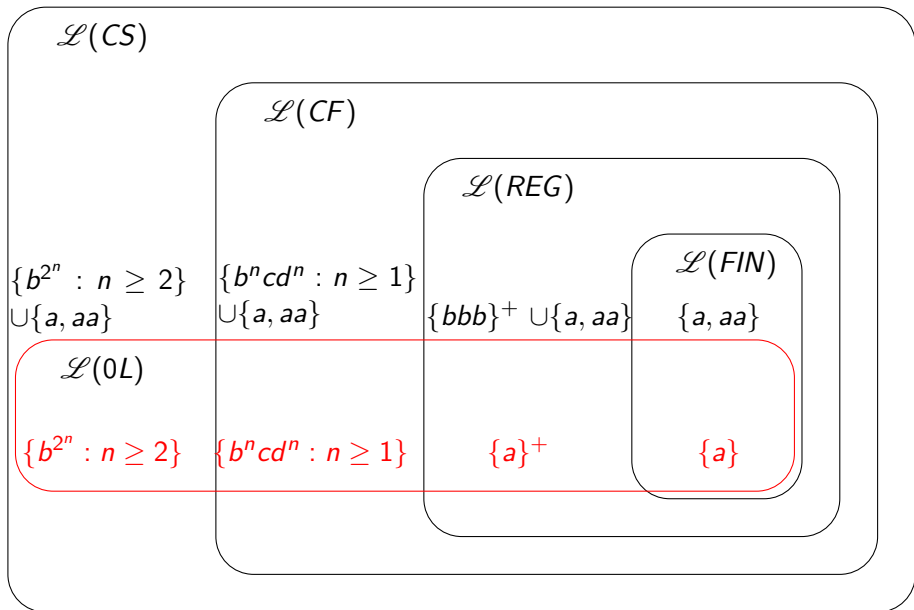
Theorem

If L is finite, then $L^ \in \mathcal{L}(OL)$.*

Theorem

If $L \in \mathcal{L}(OL)$, $L \subseteq \{a\}^$, $\varepsilon \in L$, then L is regular.*

0L Systems in Chomsky Hierarchy



E0L System

E0L System

An E0L system is a quadruple

$$G = (V, T, P, w)$$

where

V is a total alphabet

T is a terminal alphabet, $T \subseteq V$

P is a finite set of productions of the form

$$a \rightarrow x$$

with $a \in V$ and $x \in V^*$

w is the axiom, $w \in V^+$

E0L System – Generated Language

■ $\Rightarrow, \Rightarrow^*$ – by analogy with 0L systems

Generated Language

For an E0L system $G = (V, T, P, w)$,

$$L(G) = \{y \in T^* : w \Rightarrow^* y\}$$

Example

E0L system

$$G = (\{S, a, b\}, \{a, b\}, \{S \rightarrow a, S \rightarrow b, a \rightarrow aa, b \rightarrow bb\}, S)$$

$$L(G) = \{a^{2^n} : n \geq 0\} \cup \{b^{2^n} : n \geq 0\}$$

$$L(G) \in \mathcal{L}(E0L) - \mathcal{L}(0L)$$

E0L System – Generative Power

Example

EOL system

$$G = (\{A, a, b\}, \{a, b\}, \{A \rightarrow A, A \rightarrow a, a \rightarrow aa, b \rightarrow b\})$$

$$L(G) = \{a^{2^n}ba^{2^m} : n, m \geq 0\}$$

Theorem

$$\mathcal{L}(CF) \subset \mathcal{L}(EOL).$$

Proof

Homework

E0L System – Closure Properties

Theorem

$\mathcal{L}(E0L)$ is closed under

- *union*
- *concatenation*
- *positive closure*
- *intersection with a regular set*

Theorem

$\mathcal{L}(E0L)$ is not closed under inverse homomorphism.

T0L System

T0L System

A T0L system (T stands for Tables) is an $(n + 2)$ -tuple

$$G = (T, P_1, P_2, \dots, P_n, w)$$

where

- $n \geq 1$
- for all $i = 1, \dots, n$, $G_i = (T, P_i, w)$ is an 0L system

Direct Derivation

For $u, v \in T^*$,

$$u \Rightarrow v \text{ in } G$$

if $u \Rightarrow v$ in $G_i = (T, P_i, w)$ for some $i \in \{1, \dots, n\}$

- \Rightarrow^* , $L(G)$ – by analogy with 0L systems

ET0L System

ET0L System

An ET0L system is an $(n + 3)$ -tuple

$$G = (V, T, P_1, P_2, \dots, P_n, w)$$

where

- $n \geq 1$
- for all $i = 1, \dots, n$, $G_i = (V, T, P_i, w)$ is an E0L system

Direct Derivation

For $u, v \in V^*$,

$$u \Rightarrow v \text{ in } G$$

if $u \Rightarrow v$ in $G_i = (V, T, P_i, w)$ for some $i \in \{1, \dots, n\}$

- \Rightarrow^* , $L(G)$ – by analogy with E0L systems

Two-Table ETOL System

Theorem

For every ETOL system H , there exists an equivalent ETOL system of the form $G = (V, T, P_1, P_2, w)$.

Proof

Let

$$H = (W, T, R_1, \dots, R_n, w)$$

be an n -table ETOL system. Define the two-table ETOL system

$$G = (V, T, P_1, P_2, w)$$

with

- 1** $V = W \cup \{\langle a, i \rangle : a \in W, i = 1, \dots, n\}$
- 2** $P_1 = \{a \rightarrow \langle a, 1 \rangle : a \in W\} \cup \{\langle a, j \rangle \rightarrow \langle a, j+1 \rangle : 1 \leq j \leq n-1\}$
- 3** $P_2 = \{\langle a, k \rangle \rightarrow x : 1 \leq k \leq n, a \rightarrow x \in R_k\}$

Generative Power of E0L and ET0L Systems

Theorem

$$\mathcal{L}(CF) \subset \mathcal{L}(E0L) \subset \mathcal{L}(ET0L) \subset \mathcal{L}(CS).$$

Proof – Basic Idea

1 $\mathcal{L}(E0L) \subset \mathcal{L}(ET0L)$ can be proved by showing that

- $\{\#w\#w\#w : w \in \{a, b\}^*\}$ or
- $\{a^i b^j a^i : j \geq i \geq 1\}$

can be generated by an ET0L system and cannot be generated by any E0L system

2 $\mathcal{L}(ET0L) \subset \mathcal{L}(CS)$ can be proved by showing that

- $\{(ab^n)^m : m \geq n \geq 1\}$ or
- $\{a^{2^{2^n}} : n \geq 0\}$

are context-sensitive languages which cannot be generated by any ET0L system

Bibliography I



A. Lindenmayer.

Mathematical models for cellular interactions in development, parts I–II.

Journal of Theoretical Biology, 18:280–315, 1968.



P. Prusinkiewicz and A. Lindenmayer.

The Algorithmic Beauty of Plants.

Springer-Verlag, 1990.



G. Rozenberg.

T0L systems and languages.

Information and Control, 23(4):357–381, 1973.





G. Rozenberg and A. Salomaa.

The Mathematical Theory of L Systems.

Academic Press, 1980.

Bibliography II

 G. Rozenberg and A. Salomaa.
Handbook of Formal Languages, volume 1–3.
Springer, Berlin, 1997.

 A. Salomaa.
Formal Languages.
Academic Press, New York, 1973.