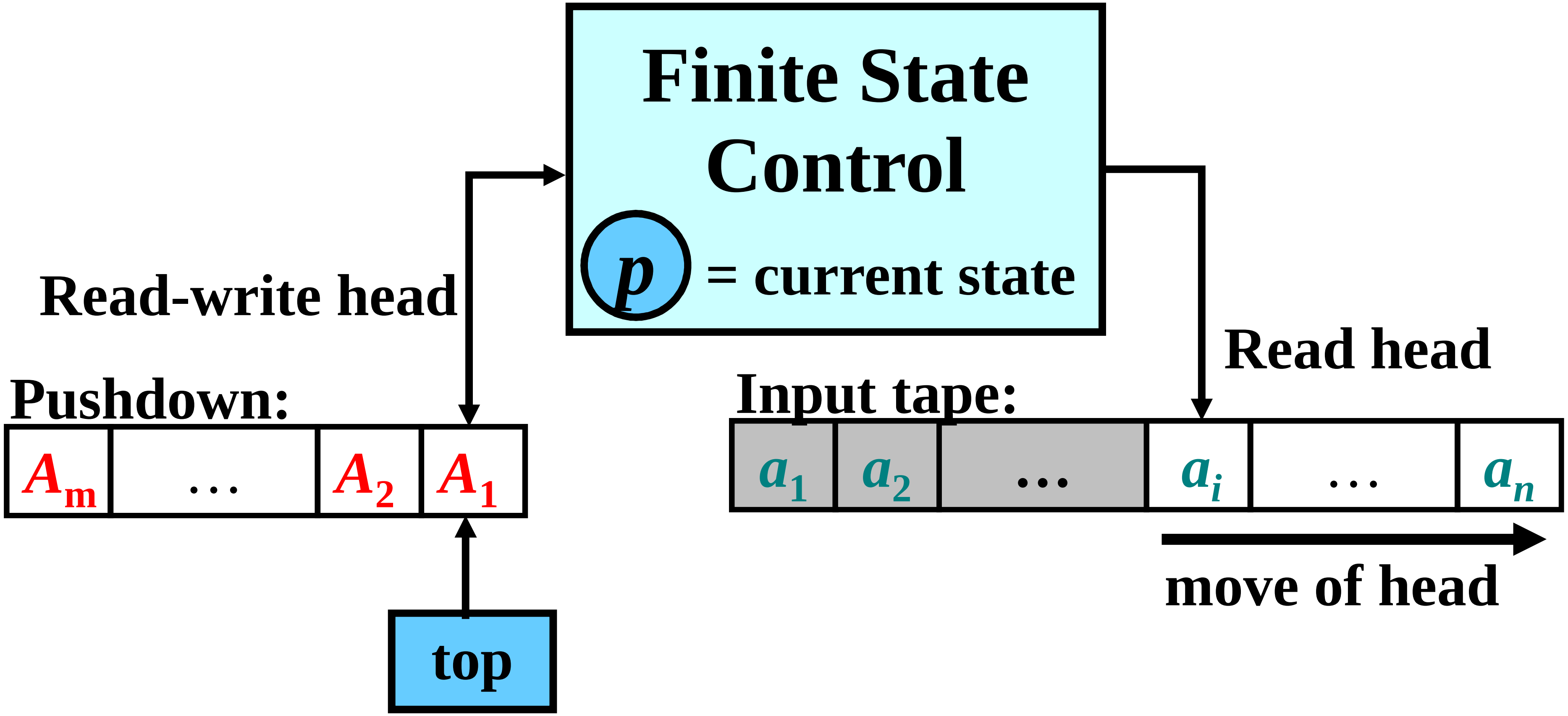
Deep Pushdown Automata

Pushdown Automaton (PDA)



Inspiration

- conversion of CFG to PDA that acts as a general top-down parser
    - if pd top = **input** symbol, **pop**
    - if pd top = **non-input** symbol, **expand**

PDA as a general Top-Down Parser

- Configuration:

$$(\textit{state}, \textit{input}, \textit{pd})$$

- **Pop:**

$$(q, ax, a\alpha) \;{}_{p}\!\Rightarrow (q, x, \alpha)$$

- **Expansion:**

$$(q, x, A\alpha) \;{}_{e}\!\Rightarrow (p, x, \beta\alpha)$$

by rule $qA \to p\beta$

Final state

- Acceptance: $(s, x, S) \Rightarrow^{*} (f, \varepsilon, \varepsilon)$

Deep Pushdown Automata: Fundamental Modification

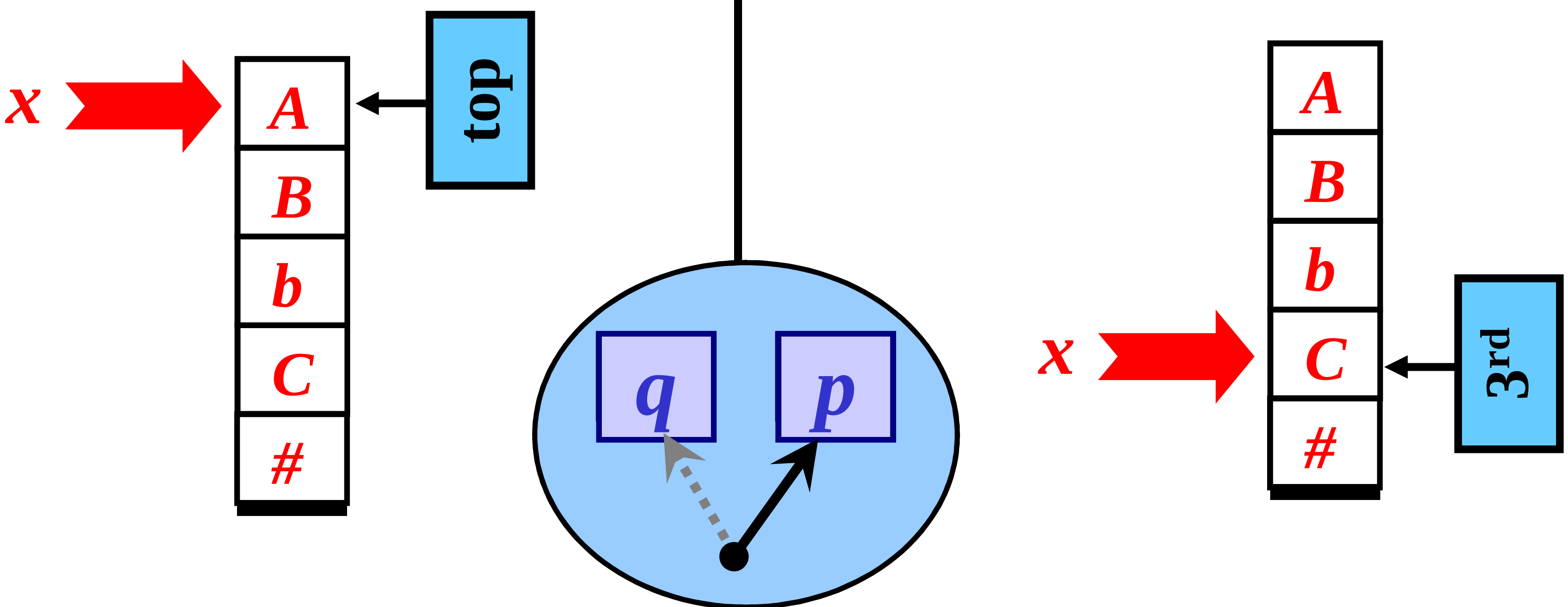
- expansion may be performed **deeper in pd**

Standard Expansion

$qA \rightarrow px$

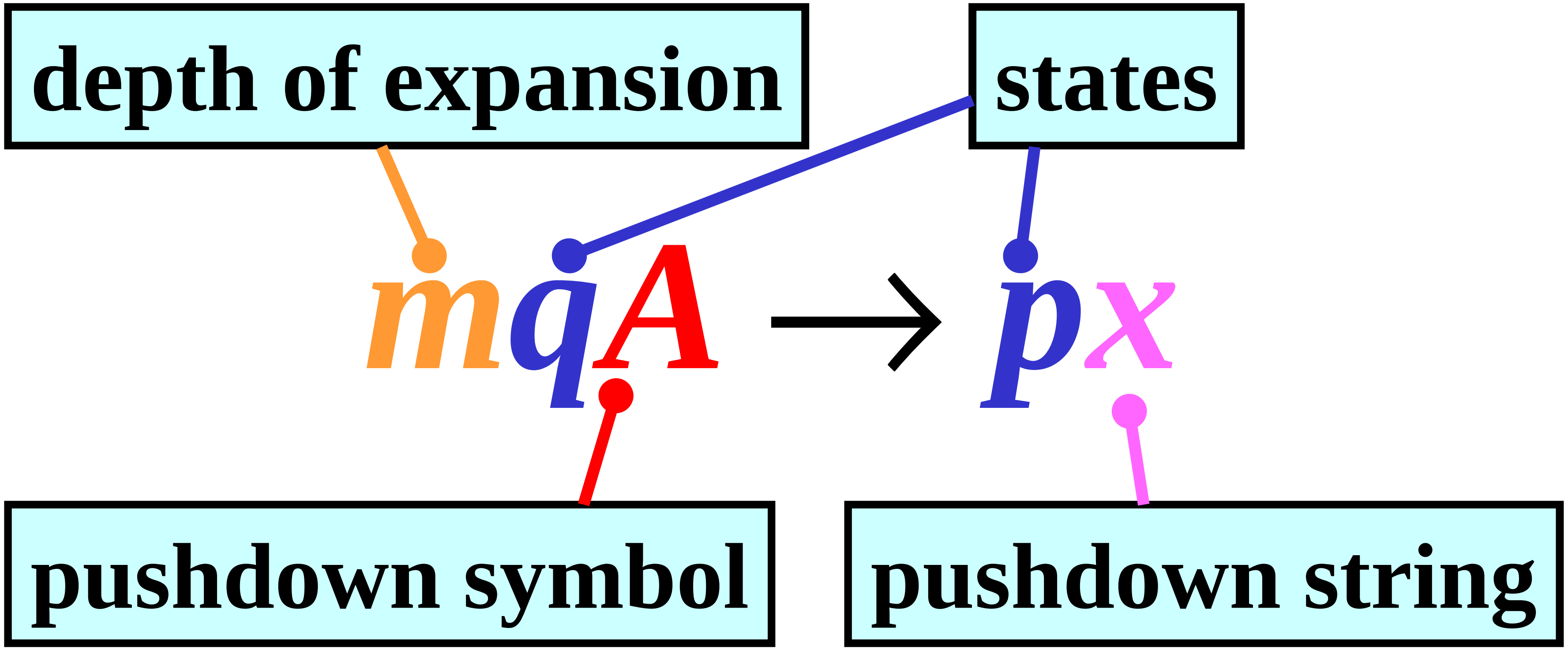
Deep Expansion

$3qC \rightarrow px$



Deep Pushdown Automata

- Same as Top-Down Parser except *deep expansions*
- *Expansion of depth m*:
  - the *m*th topmost non-input pd symbol is replaced with a string by rule



Expansion of Depth *m*

- ***Expansion of depth m***:

$$(q, w, uAz) \ {}_e\!\Rightarrow (p, w, uvz)$$

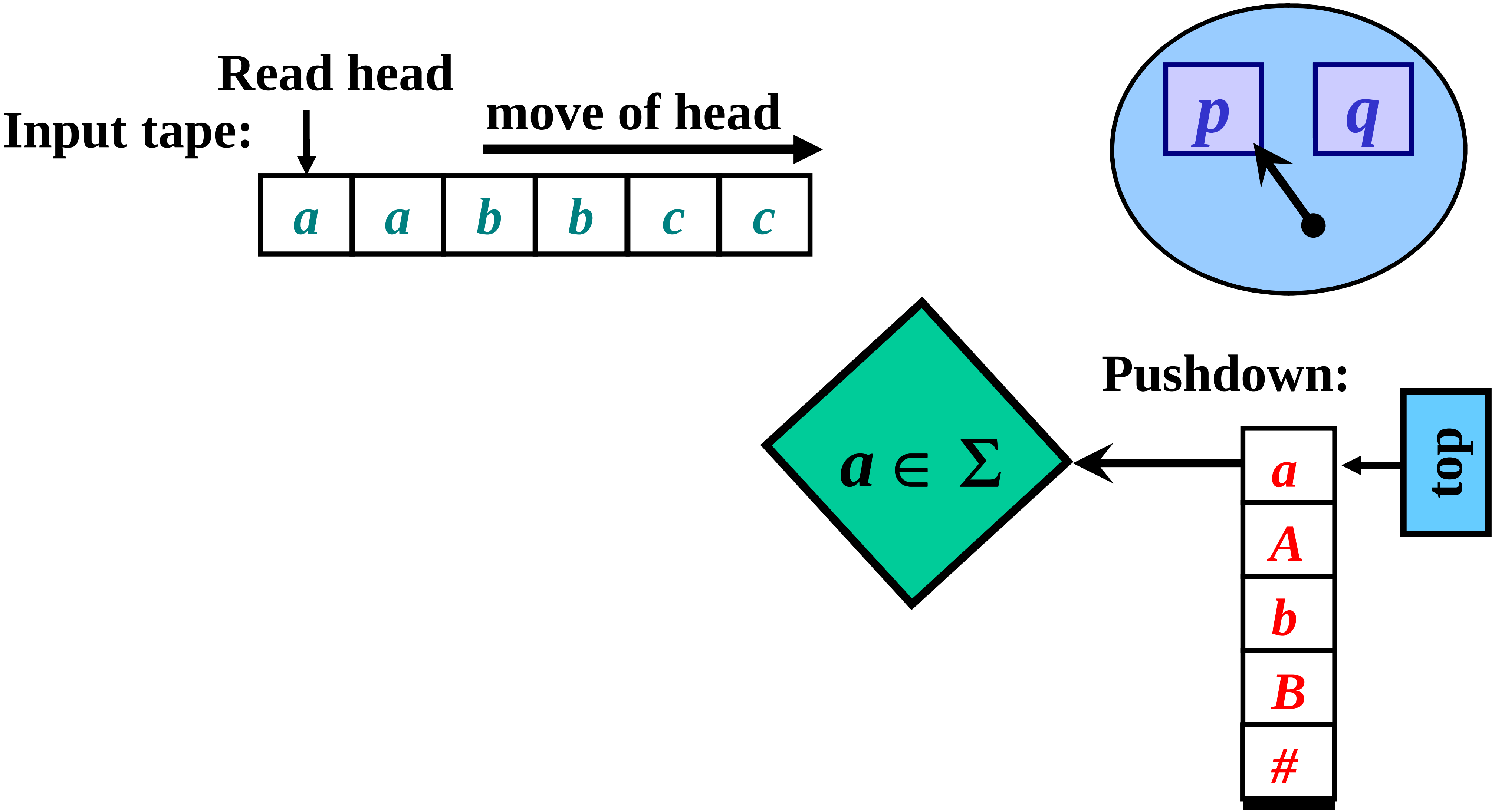
by ***rule of depth m***

$$[mqA \rightarrow pv],$$

where u contains $m - 1$ non-input symbols

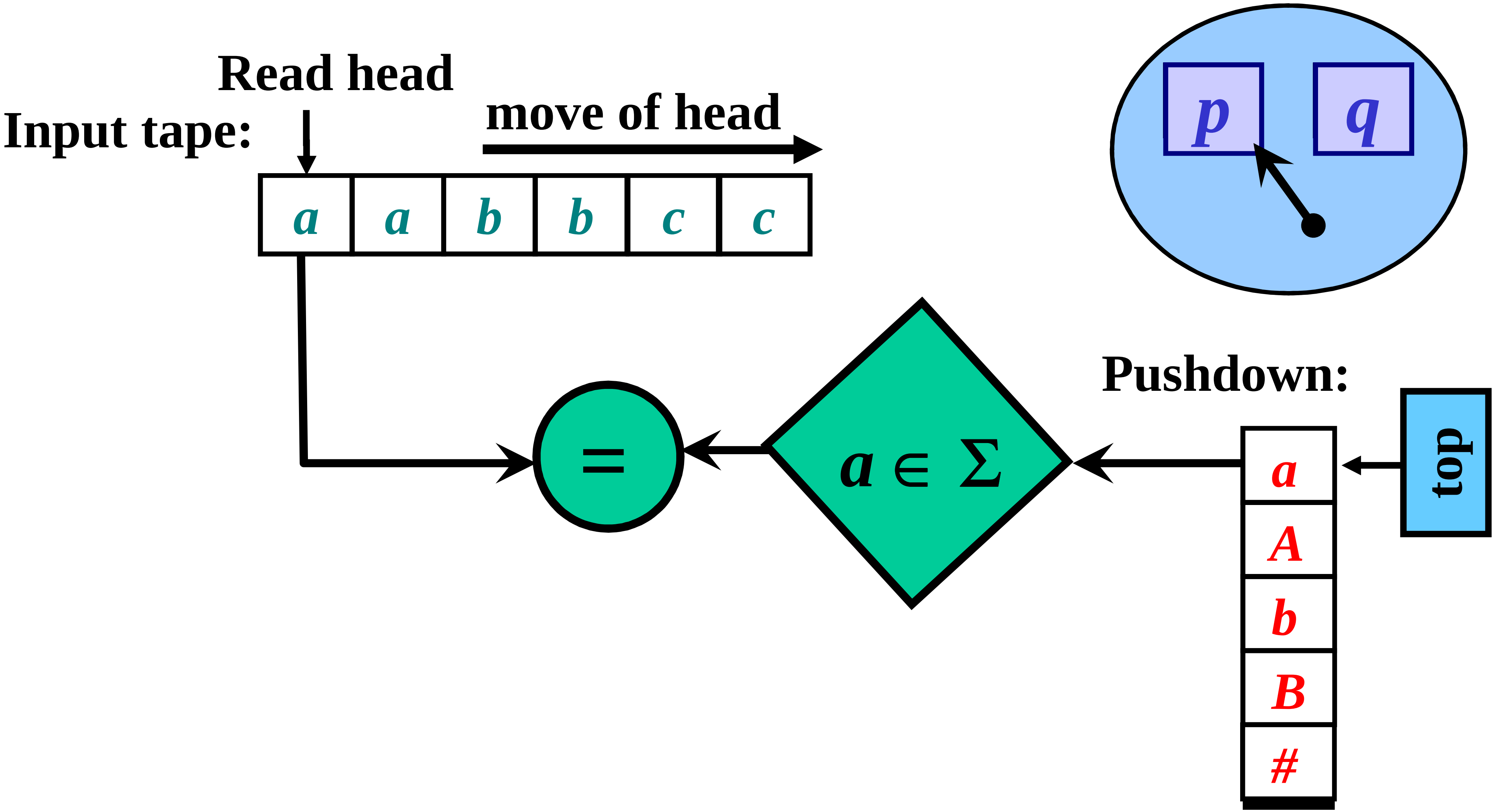
Pop: Illustration

Move: (p, *aabbcc*, *aAbB#*) $_p\Rightarrow$ (p, *abbcc*, *AbB#*)



Pop: Illustration

Move: (p, $aabbcc$, $aAbB\#$) $_p\Rightarrow$ (p, $abbcc$, $AbB\#$)



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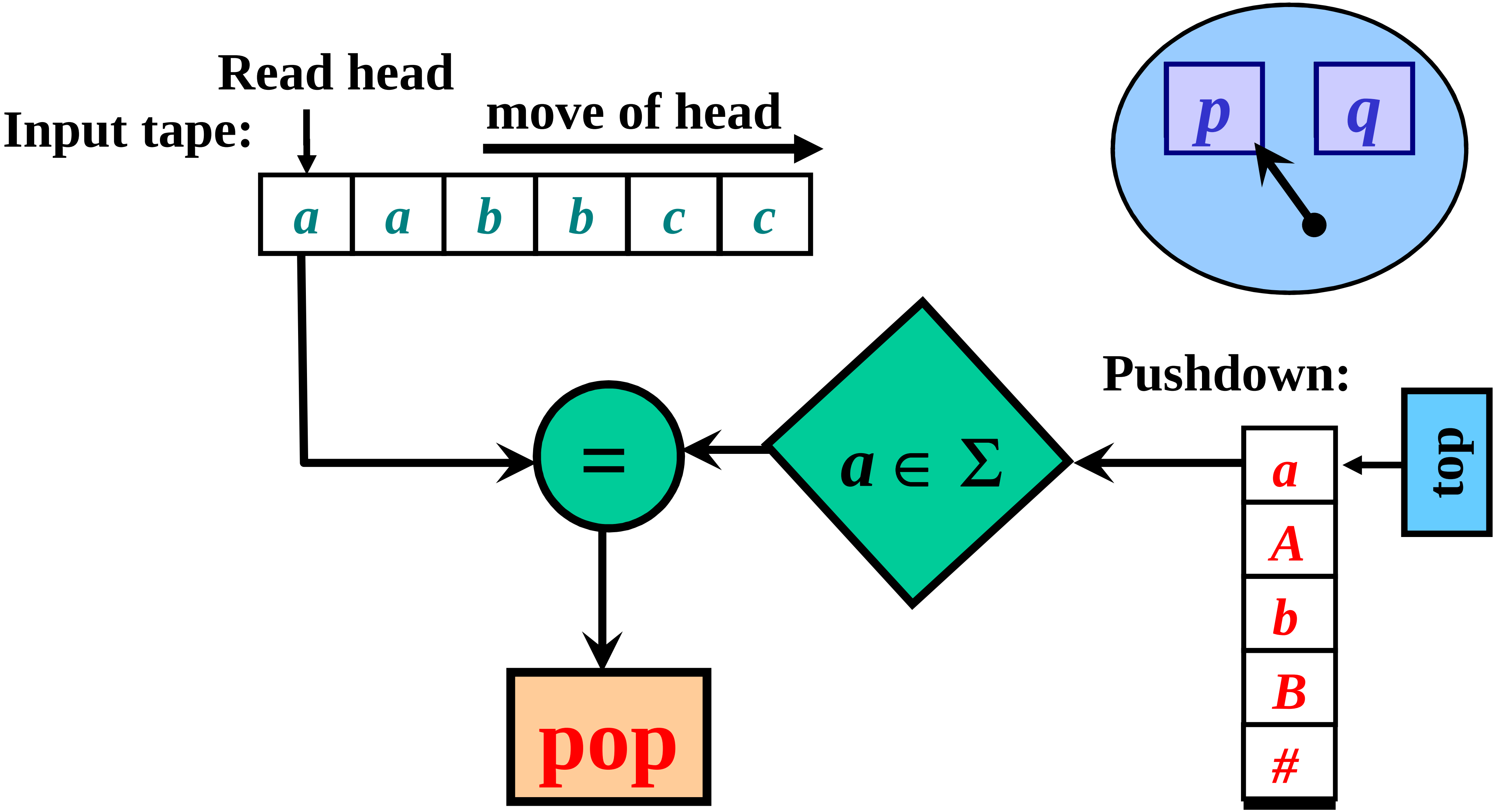

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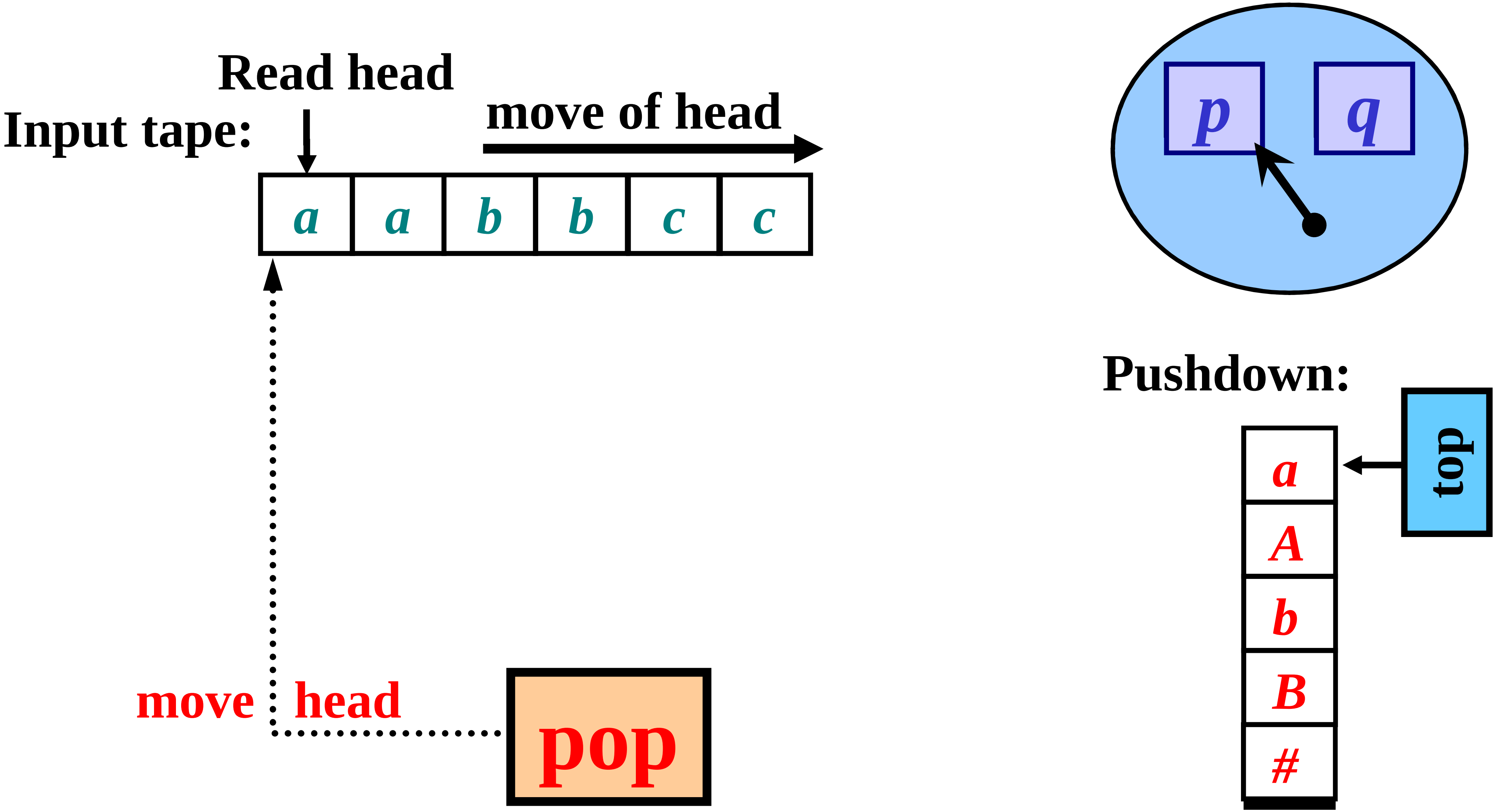

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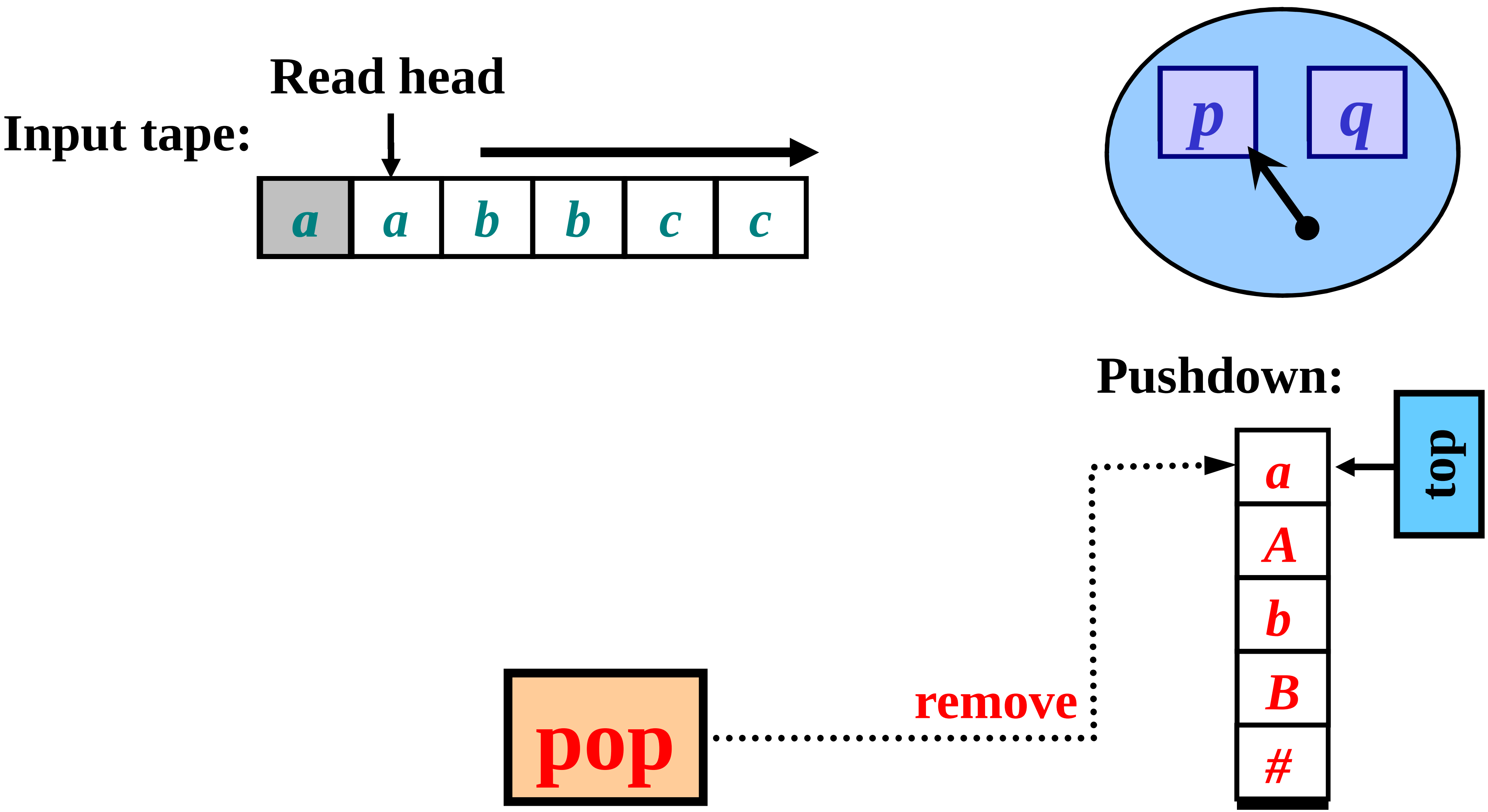

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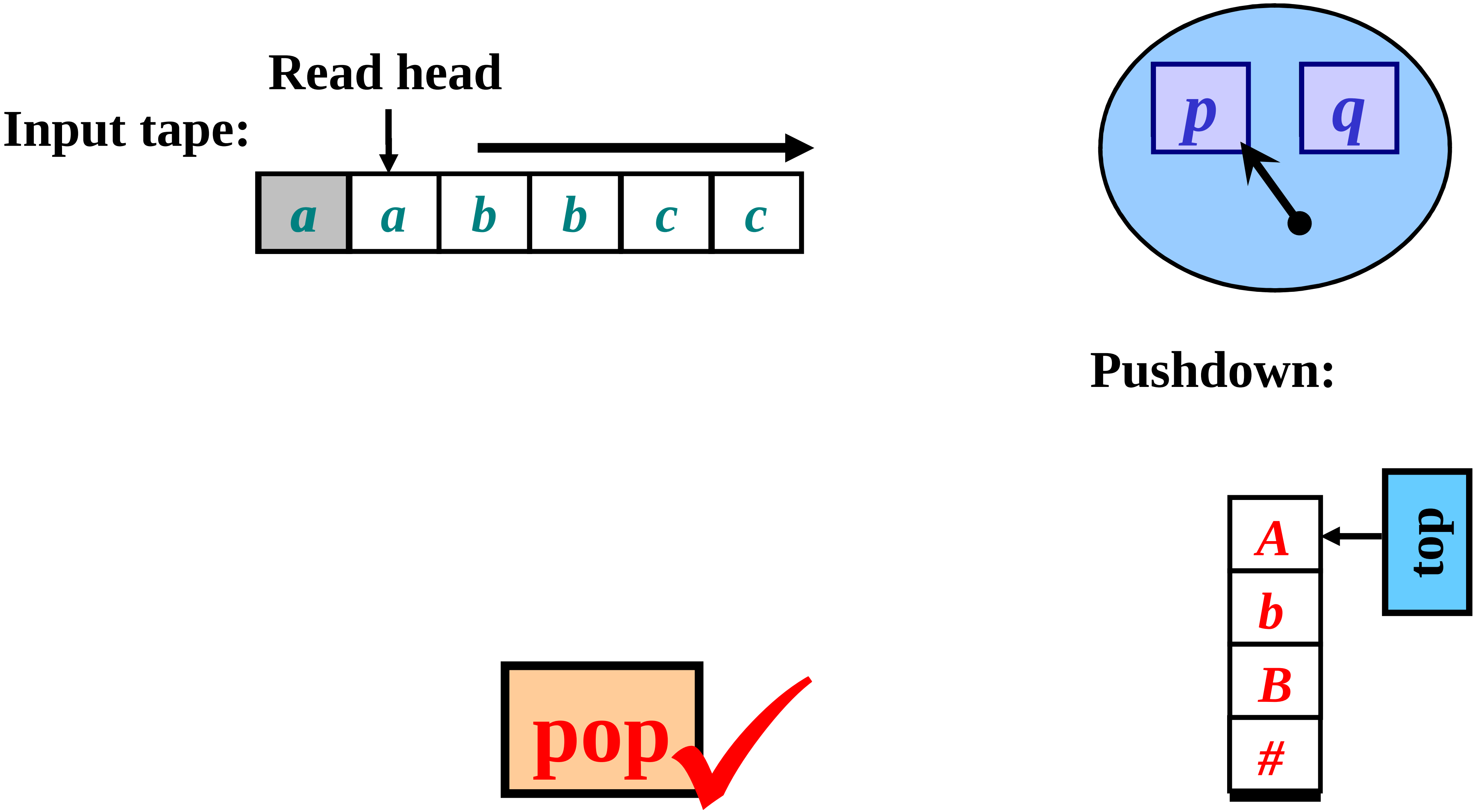
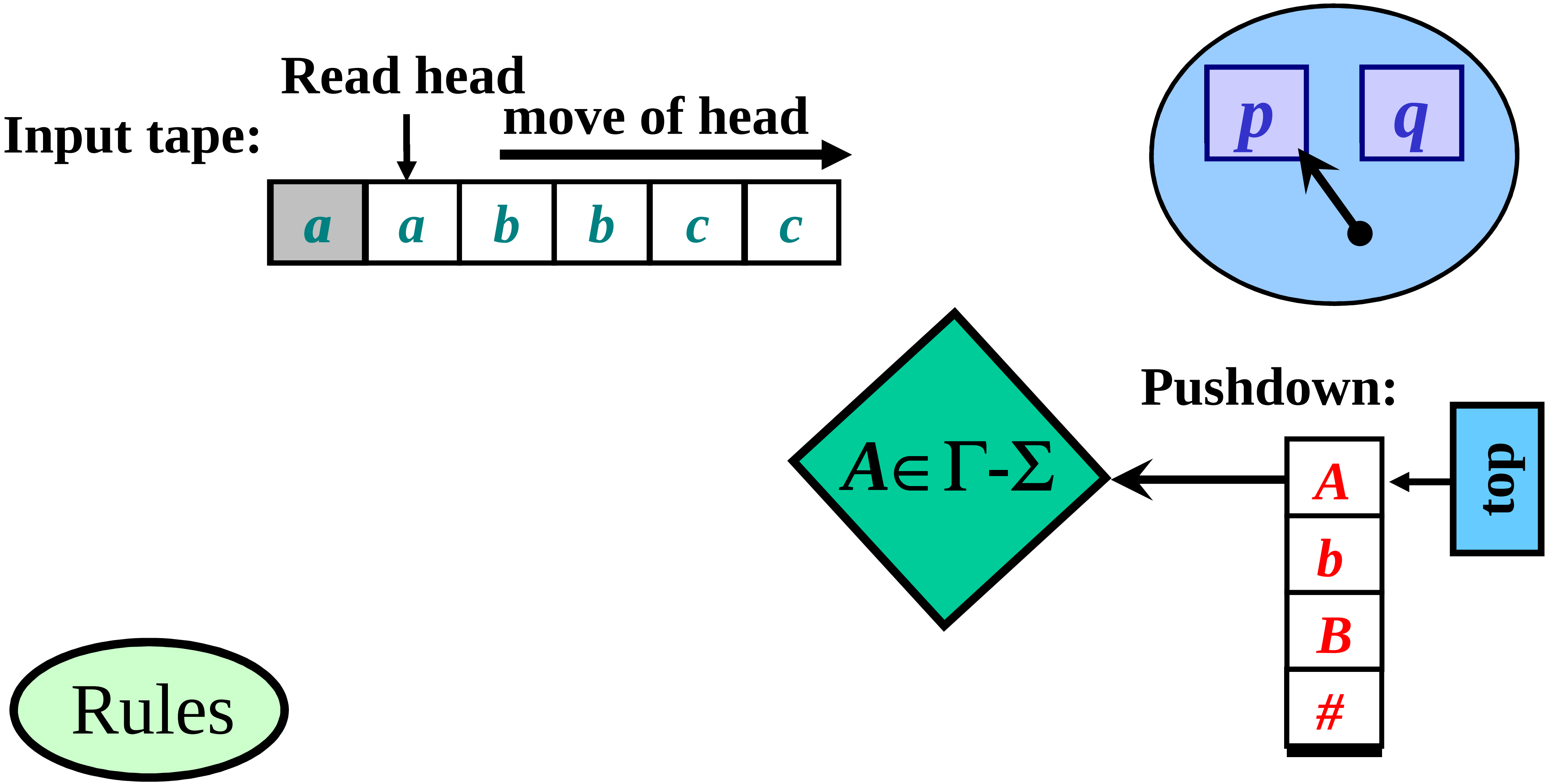


Deep Expansion: Illustration

Move: (p, *abbcc*, *AbB#*) $_e\Rightarrow$ (q, *abbcc*, *AbBc#*) [$2pB \rightarrow qBc$]



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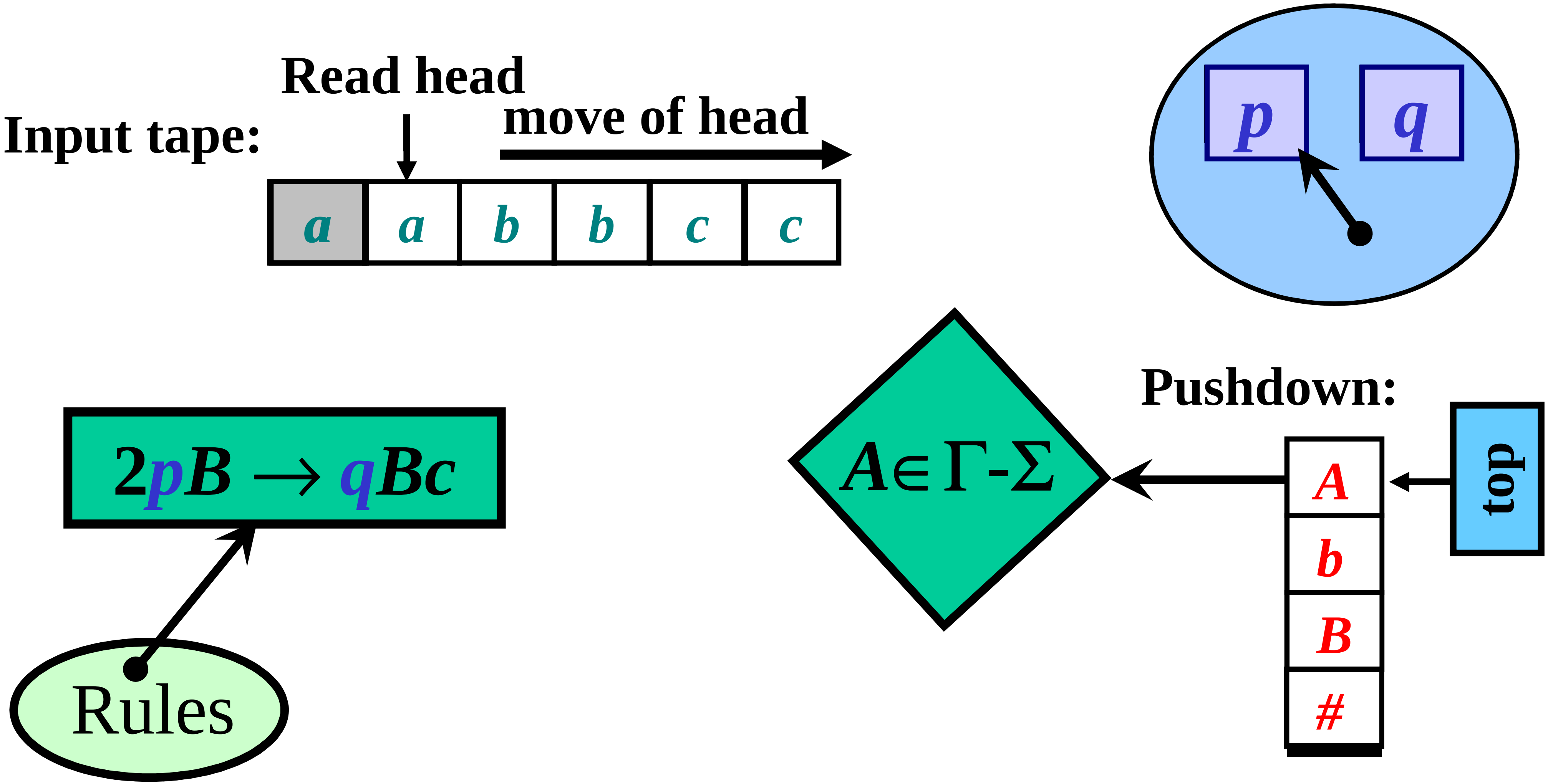
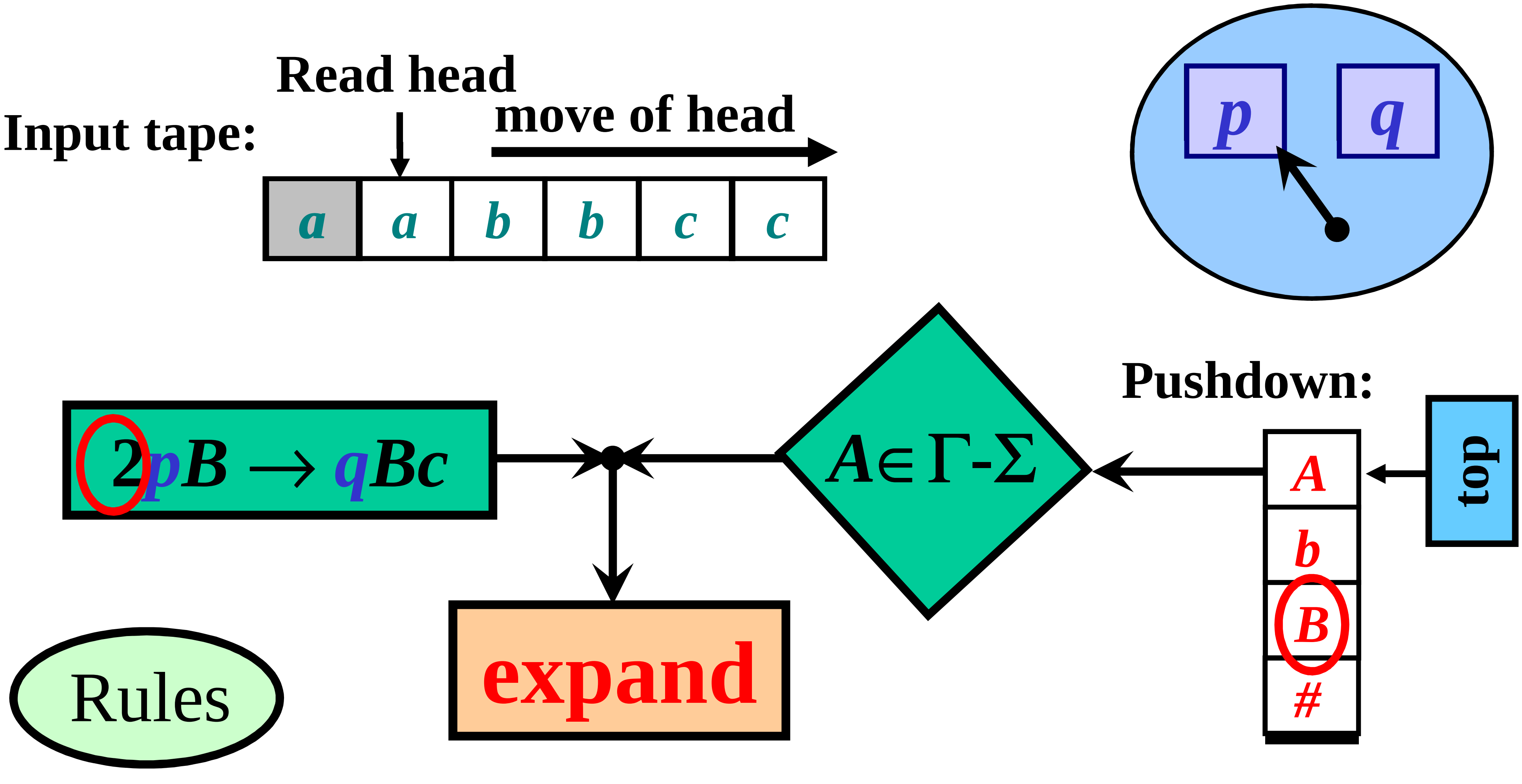


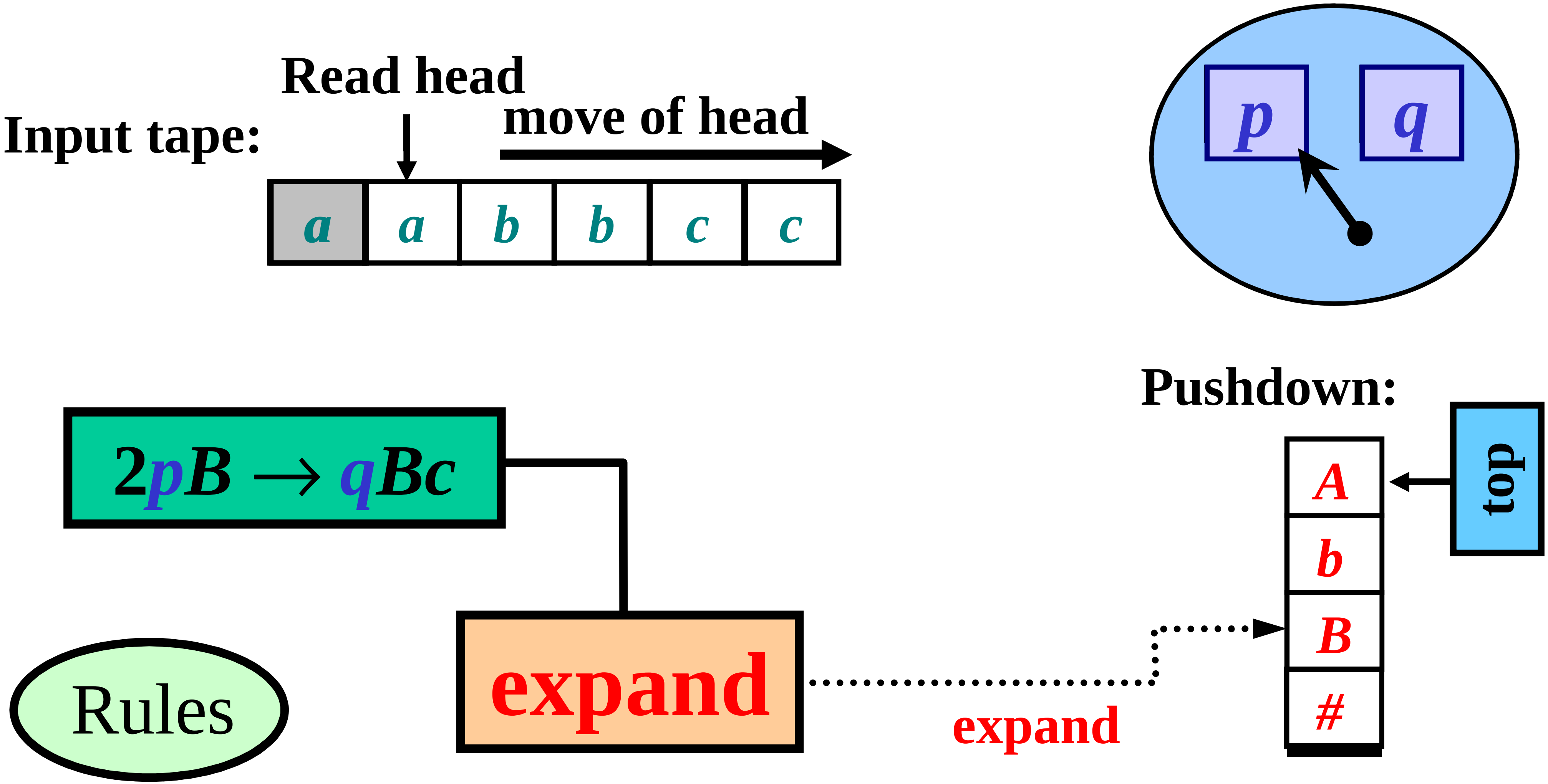
Deep Expansion: Illustration

Move: (*p*, *abbcc*, *AbB#*) $_e\Rightarrow$ (*q*, *abbcc*, *AbBc#*) [2*pB* → *qBc*]



Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#) \;_e\!\Rightarrow (q, abbcc, AbBc\#)$ $[2pB \rightarrow qBc]$



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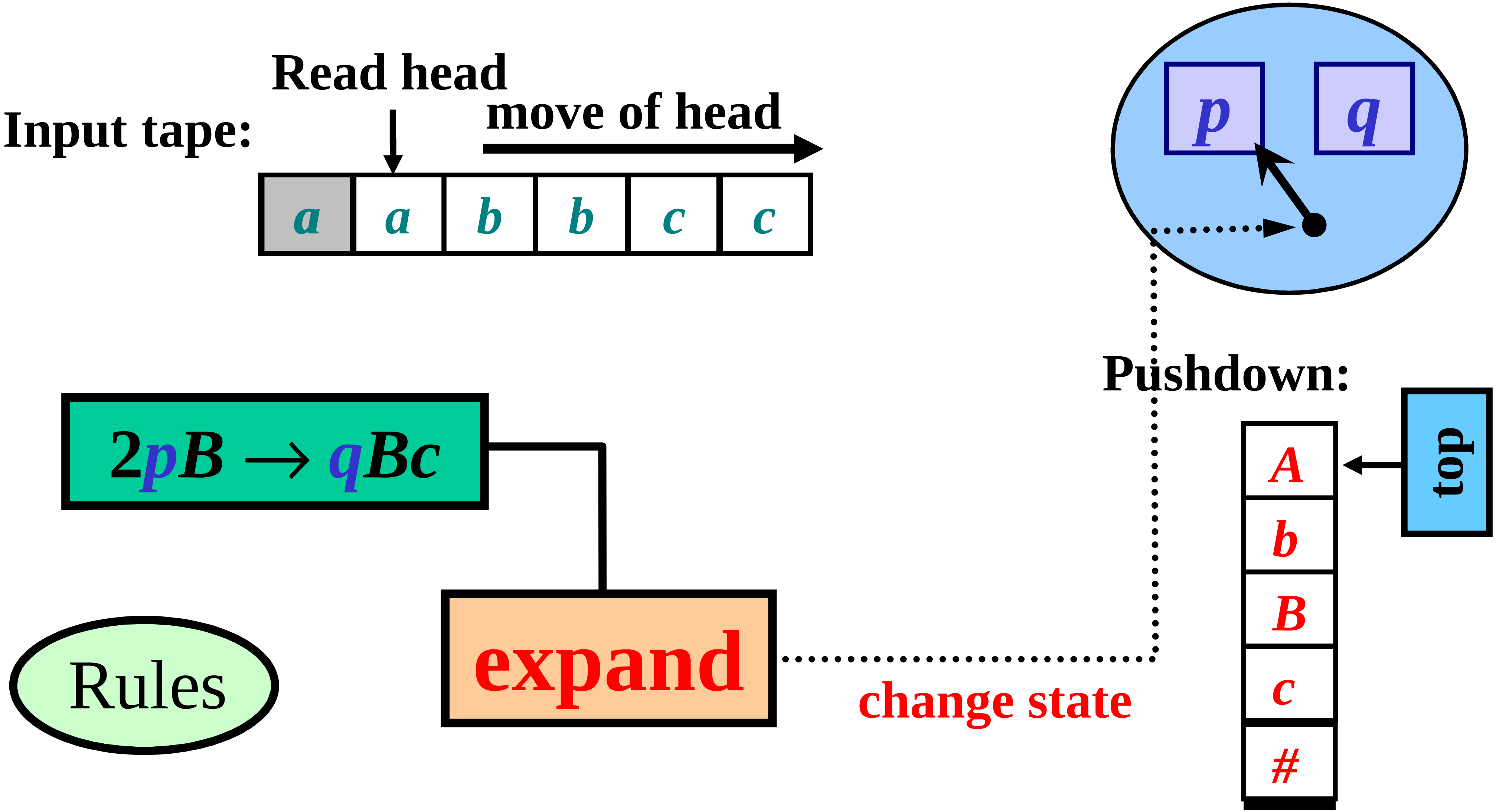

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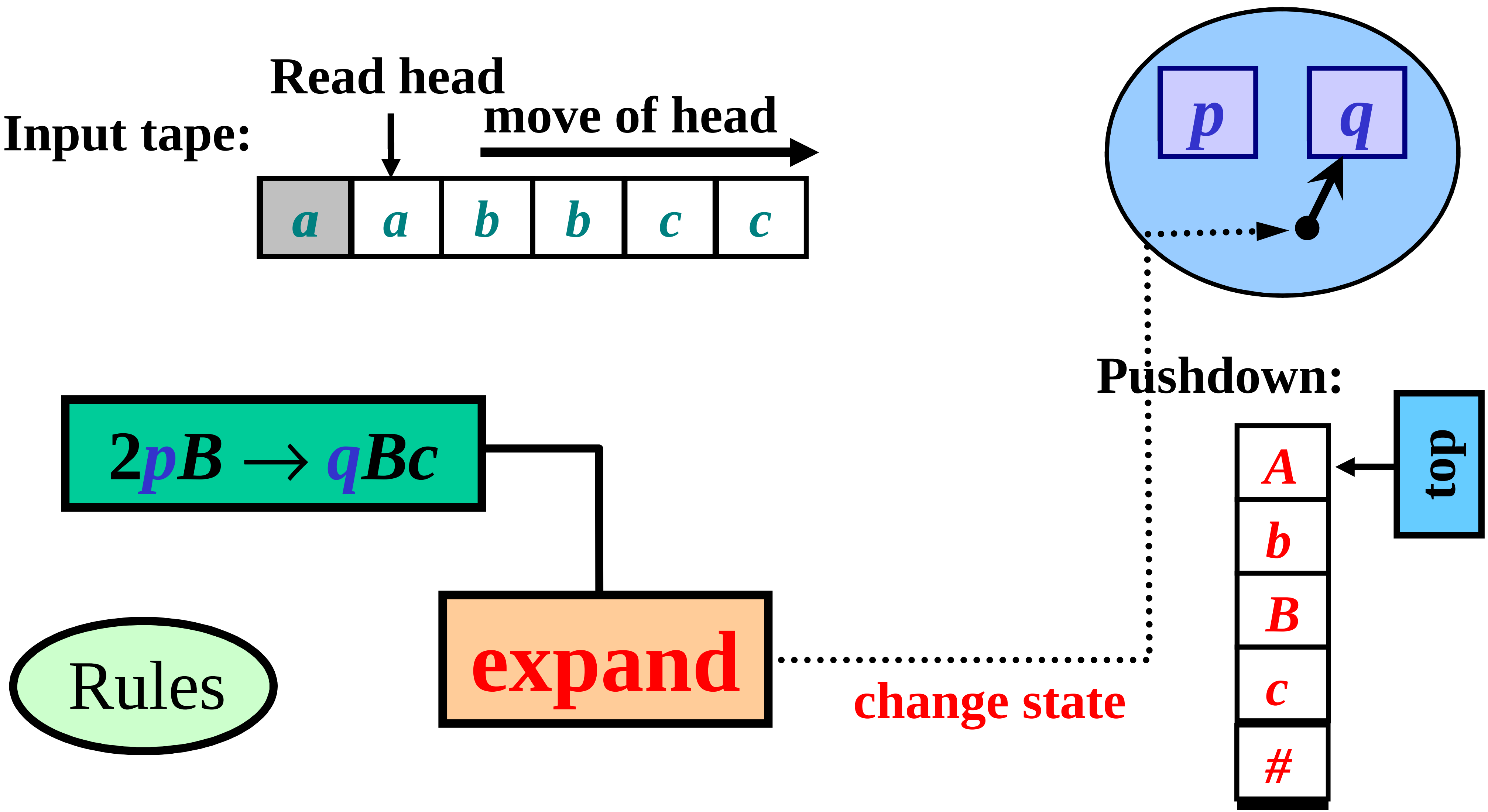

Deep Expansion: Illustration

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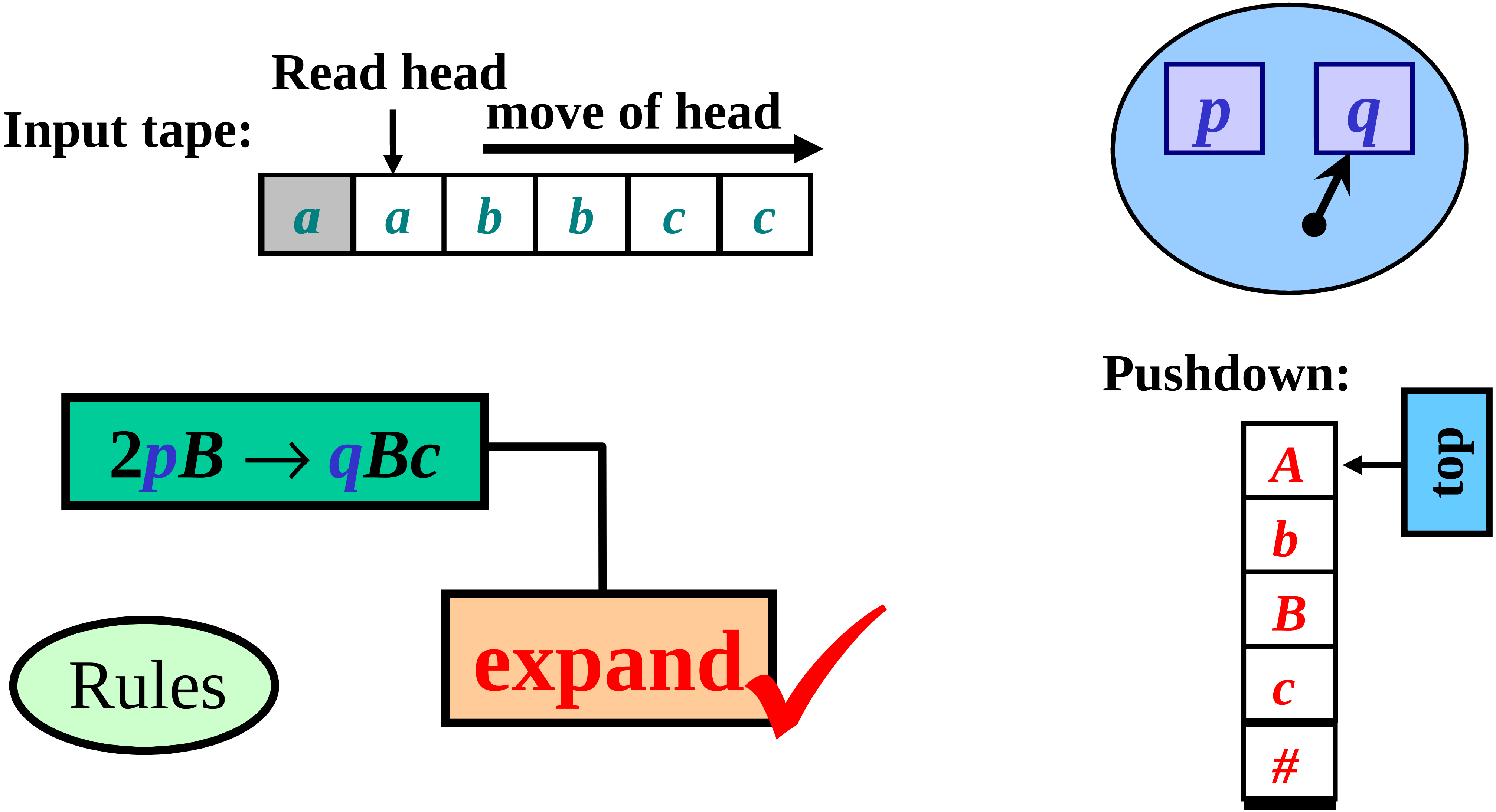

Example: Deep PDA

Deep PDA M:

[1]. $1sS \rightarrow qAB$

[2]. $1qA \rightarrow paAb$

[3]. $1qA \rightarrow fab$

[4]. $2pB \rightarrow qBc$

[5]. $1fB \rightarrow fc$

M accepts $aabbcc$:

$(s, aabbcc, S\#)$

${}_e\Rightarrow (q, aabbcc, AB\#)$ [1]

${}_e\Rightarrow (p, aabbcc, aAbB\#)$ [2]

${}_p\Rightarrow (p, abbcc, AbB\#)$

${}_e\Rightarrow (q, abbcc, AbBc\#)$ [4]

${}_e\Rightarrow (f, abbcc, abbBc\#)$ [3]

${}_p\Rightarrow (f, bbcc, bbBc\#)$

${}_p\Rightarrow^2 (f, cc, Bc\#)$

${}_e\Rightarrow (f, cc, cc\#)$ [5]

${}_p\Rightarrow (f, c, c\#)$

${}_p\Rightarrow (f, \varepsilon, \#)$

$$L(M) = \{a^n b^n c^n : n \geq 1\} \in PD_2$$

Definition 1/3

A deep pushdown automaton is a 7-tuple

$$M = (Q, \Sigma, \Gamma, R, s, S, F), \text{ where}$$

- Q – *states,*
- $\Sigma \subseteq \Gamma$ – *input alphabet,*
- Γ – *pushdown alphabet, bottom symbol* $\# \in \Gamma - \Sigma$
- R – *finite set of rules* of the form

$$mqA \rightarrow pw \quad \text{or} \quad mq\# \rightarrow pv\#$$

- $s \in Q$ – *start state*
- $S \in \Gamma$ – *start pushdown symbol*
- $F \subseteq Q$ – *final states*

Definition 2/3

- if an input symbol is on pd top, ***M*** **pops** the pd as

$$(q, au, az)\ {}_p\!\Rightarrow (q, u, z), \quad a \in \Sigma$$

- no explicit rule needed in *R*

- if a non-input symbol is on pd top, ***M*** **expands** the pd as

$$(q, w, uAz)\ {}_e\!\Rightarrow (p, w, uvz)\ [mqA \rightarrow pv],$$

where u contains $m - 1$ non-input symbols

Definition 3/3

- ***M*** is ***of depth n***, denotated by $_{\boldsymbol{n}}\boldsymbol{M}$, if n is the minimal positive integer such that each of M's rules is of depth n or less.

- Language accepted by $_nM$, $L(_nM)$, is defined as

$$L(_nM) = \{\boldsymbol{w} \in \Sigma^*: (\boldsymbol{s}, \boldsymbol{w}, \boldsymbol{S}\#) \Rightarrow^* (\boldsymbol{f}, \varepsilon, \#) \text{ in } {}_nM \text{ with } \boldsymbol{f} \in F\}.$$

Main Result and its Proof

- PD_n – the language family defined by DeepPDAs of depth n.

Theorem: $PD_n \subset PD_{n+1}$, for all $n \geq 1$.

Proof (Sketch):

- State grammars (Kasai, 1970) are needed in the proof
- State grammar is a modification of CFG based on rules of the form

$$(q, A) \rightarrow (p, v)$$

Proof 1/6: State Grammar

- ***State grammar G*** = (***V***, ***W***, ***T***, ***P***, ***S***)
  - ***V*** – total alphabet, ***W*** – states, ***T*** ⊆ ***V*** – terminals,
  - ***P*** – set of rules of the form (***q***, ***A***) → (***p***, ***v***)
  - ***S*** ∈ **(*V* – *T*)** – start symbol,

- ***Configuration*** – (***q***, ***x***)

- ***Derivation step***:

$$(q, uAz) \Rightarrow (p, uvz)\ [(q, A) \rightarrow (p, v)]$$

and for every nonterminal ***B*** in ***u***, ***P*** contains no rule with (***q***, ***B***) on the left-hand side

Proof 2/6: *n*-limited Step

- ***n-limited derivation step***:
  each derivation step within the first ***n*** non-terminals

$$(q, uAz)\ {}_n\!\Rightarrow (p, uvz) \text{ and}$$

$$uA \text{ has } n \text{ or fewer non-terminals}$$

- ***n-limited state language:***

$$L(G, n) = \{w \in T^*: (q, S)\ {}_n\!\Rightarrow^* (p, w)\}$$

- ST_n – the family of *n*-limited state languages

Proof 3/6: Example

State Grammar G:

[1]. $(1, S) \rightarrow (2, AC)$

[2]. $(2, A) \rightarrow (3, aAb)$

[3]. $(2, A) \rightarrow (4, ab)$

[4]. $(3, C) \rightarrow (2, Cc)$

[5]. $(4, C) \rightarrow (4, c)$

$W = \{1, 2, 3, 4\}$

G generates $aabbcc$:

$(1, S) \Rightarrow (2, AC)$ [1]

$\Rightarrow (3, aAbC)$ [2]

$\Rightarrow (2, aAbCc)$ [4]

$\Rightarrow (4, aabbCc)$ [3]

$\Rightarrow (4, aabbcc)$ [5]

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in ST_2$$

Proof 4/6: $\boldsymbol{PD_n} \subseteq \boldsymbol{ST_n}$, $n \geq 1$

- G simulates the application of $\boldsymbol{ipA} \rightarrow \boldsymbol{qy} \in R$:
  - make a left-to-right scan of the pd until the ***i***th

    occurence of a non-terminal
  - if X_i = ***A***, then replace ***A*** with ***y*** and return to the beginning of the sentential form
  - rightmost symbol is always a special ***a'***, and G completes the simulation by changing ***a'*** to ***a***

Proof 5/6: $\boldsymbol{ST_n} \subseteq \boldsymbol{PD_n}$, $n \geq 1$

- ${}_nM$ simulates the n-limited derivations of G in pd:
  - always records the first $\boldsymbol{n}$ non-terminals from the current sentential form of G in its state
  - fewer than $\boldsymbol{n}$ non-terminals are extended by #s
  - reads the string, empties pd, enters $\$ \in F$

Proof 6/6: $\boldsymbol{PD_n} \subset \boldsymbol{PD_{n+1}}$, $n \geq 1$

1) As $PD_n \subseteq ST_n$ and $ST_n \subseteq PD_n$ for all $n \geq 1$, $ST_n = PD_n$.

2) Kasai (1970): $ST_n \subset ST_{n+1}$, for all $n \geq 1$.

For all $n \geq 1$, $PD_n = ST_n \subset ST_{n+1} = PD_{n+1}$

Q. E. D.

Open Problem Areas

- Determinism
- Rules of form $\boldsymbol{mqA \rightarrow p\varepsilon}$

References

- **Meduna, A.: Deep Pushdown Automata**, ***Acta Informatica*, 2006**