Self-Regulating Finite Automata

Based on

A. Meduna and T. Masopust: *Self-Regulating Finite Automata*,
Acta Cybernetica, in press.

Outline

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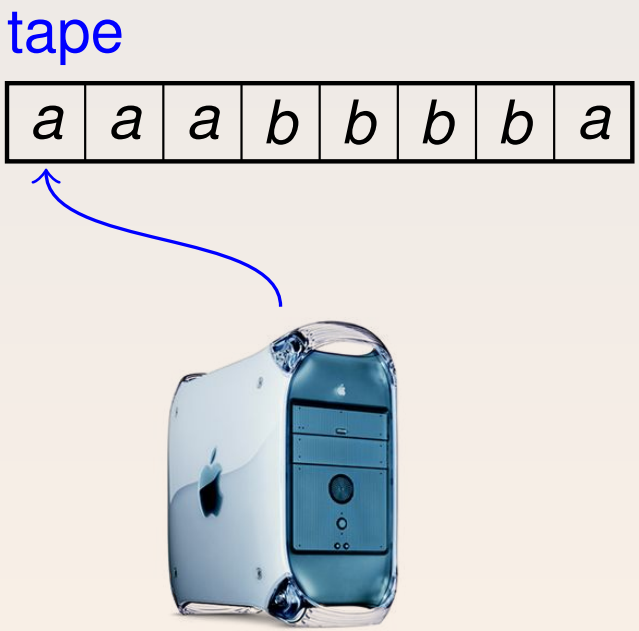
Finite Automata—Concept



Characteristics

- Finite state control
- Input cannot be modified
- Head only moves forward

Finite Automata—Definition

Definition

A finite automaton is a quintuple

$$M = (Q, \Sigma, P, q_0, F)$$

where

- Q is a finite set of states,
- Σ is an input alphabet,
- P is a finite set of rules of the form $qw \to p$, $q, p \in Q$, $w \in \Sigma^*$,
- $q_0 \in Q$ is an initial state,
- $F \subseteq Q$ is a set of final states.

Finite Automata—Language

Definition

A configuration is any member of $Q\Sigma^*$.
If

$$qwy \in Q\Sigma^* \text{ and } r.qw \to p \in P,$$

then

$$qwy \Rightarrow py\,[r].$$

The language of M is the set

$$\mathscr{L}(M) = \{w \in \Sigma^* : q_0 w \Rightarrow^* f,\ f \in F\},$$

where $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$.

Finite Automata—Example

tape

a	a	a	b	b	b	b	

Current state: q_a

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- P contains
  - $1.\ q_a a \to q_a$
  - $2.\ q_a b \to q_b$
  - $3.\ q_b b \to q_b$
- M starts in q_a
- $F = Q$
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	

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Finite Automata—Example

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Finite Automata—Example

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  - $3.\ q_b b \rightarrow q_b$
- M starts in q_a
- $F = Q$
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	

Current state: q_b

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- P contains
  - $1.q_a a \rightarrow q_a$
  - $2.q_a b \rightarrow q_b$
  - $3.q_b b \rightarrow q_b$
- M starts in q_a
- $F = Q$
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	

Current state: q_b

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- P contains
  - $1.q_a a \rightarrow q_a$
  - $2.q_a b \rightarrow q_b$
  - $3.q_b b \rightarrow q_b$
- M starts in q_a
- $F = Q$
- $\mathscr{L}(M) = \{a\}^*\{b\}^*$

Finite Automata—Example

tape

a	a	a	b	b	b	b	

Current state: q_b

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- P contains
  - $1.q_a a \rightarrow q_a$
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- M starts in q_a
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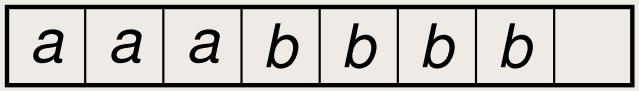
Finite Automata—Example

tape



Current state: q_b

Description

- $Q = \{q_a, q_b\}$
- $\Sigma = \{a, b\}$
- P contains
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  - $3.q_b b \rightarrow q_b$
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- $F = Q$
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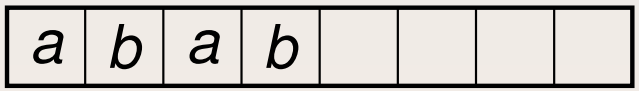
Outline

Self-Regulation—The Main Idea

tape



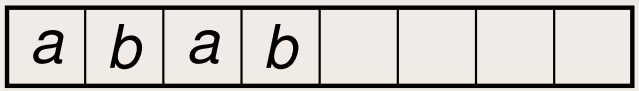
state: s, rules used: 1,2,3

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$ is a finite automaton
- P contains
  - $1.sa \rightarrow s$
  - $2.sb \rightarrow s$
  - $3.s \rightarrow t$
  - $4.ta \rightarrow t$
  - $5.tb \rightarrow t$
  - $6.t \rightarrow f$
- $R = \{(1, 4), (2, 5), (3, 6)\}$
- $\mathscr{L}(M) = \{ww : w \in \{a, b\}^*\}$

Self-Regulation—The Main Idea

tape



state: s, rules used: 1

Description

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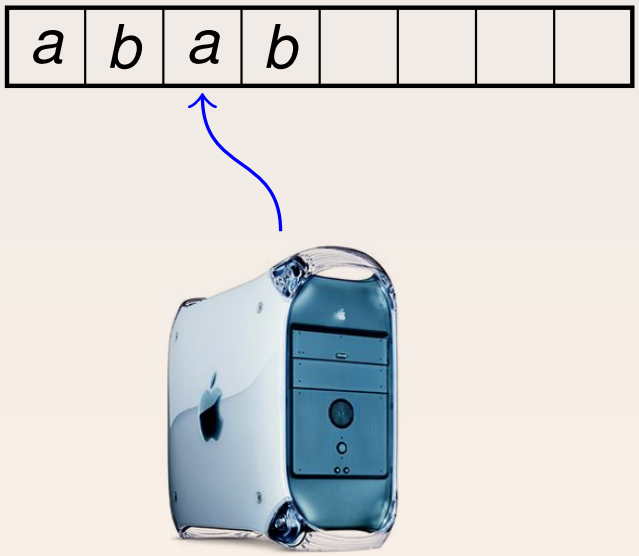

state: s, rules used: 1,2,3

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Self-Regulation—The Main Idea

tape

a	b	a	b				

state: t, rules used: 1,2,3

M makes a turn

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$ is a finite automaton
- P contains
  - $1.sa \to s$
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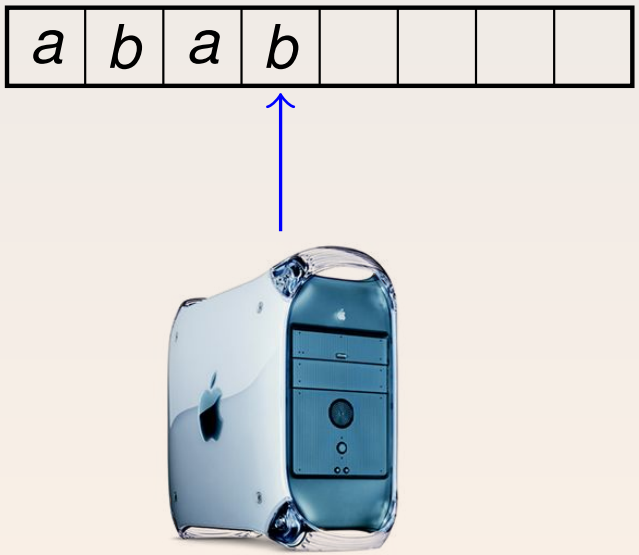
Self-Regulation—The Main Idea

tape



state: t, rules used: 4,2,3

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$ is a finite automaton
- P contains
  - $1.sa \rightarrow s$
  - $2.sb \rightarrow s$
  - $3.s \rightarrow t$
  - $4.ta \rightarrow t$
  - $5.tb \rightarrow t$
  - $6.t \rightarrow f$
- $R = \{(1, 4), (2, 5), (3, 6)\}$
- $\mathscr{L}(M) = \{ww : w \in \{a, b\}^*\}$

Self-Regulation—The Main Idea

tape

a	b	a	b				

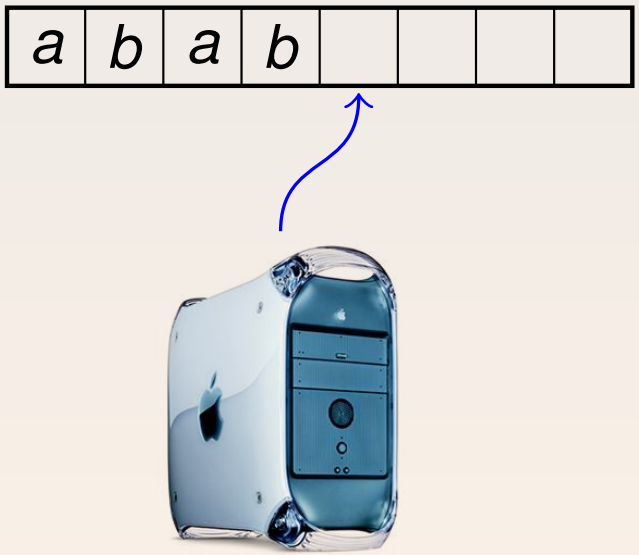
state: t, rules used: 4,5,3

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$ is a finite automaton
- P contains
  - $1.sa \rightarrow s$
  - $2.sb \rightarrow s$
  - $3.s \rightarrow t$
  - $4.ta \rightarrow t$
  - $5.tb \rightarrow t$
  - $6.t \rightarrow f$
- $R = \{(1,4), (2,5), (3,6)\}$
- $\mathscr{L}(M) = \{ww : w \in \{a, b\}^*\}$

Self-Regulation—The Main Idea

tape



state: f, rules used: 4,5,6

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$ is a finite automaton
- P contains
  - $1.sa \rightarrow s$
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  - $4.ta \rightarrow t$
  - $5.tb \rightarrow t$
  - $6.t \rightarrow f$
- $R = \{(1, 4), (2, 5), (3, 6)\}$
- $\mathscr{L}(M) = \{ww : w \in \{a, b\}^*\}$

Outline

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Self-Regulating Finite Automata

Definition

Let

$$N = (Q, \Sigma, P, q_0, F)$$

be a finite automaton.
A self-regulating finite automaton, SFA, is a triple

$$M = (N, q_t, R),$$

where

1. $q_t \in Q$ is a turn state, and
2. $R \subseteq \Psi \times \Psi$ is a finite relation on the alphabet of rule labels.

All-Move Self-Regulating Finite Automata

Definition

For $n \geq 0$, an SFA M is an n-turn all-move SFA, n-all-SFA, if M accepts w as follows. There is $q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, such that

$$\mu = \underbrace{{}_0r_1\,{}_0r_2 \cdots {}_0r_k}_{k \text{ rules}}\,\underbrace{{}_1r_1\,{}_1r_2 \cdots {}_1r_k}_{k \text{ rules}} \cdots \underbrace{{}_nr_1\,{}_nr_2 \cdots {}_nr_k}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, ${}_0r_k$ is the first rule of the form $qx \rightarrow q_t$, for some $q \in Q$, $x \in \Sigma^*$, and

$$({}_jr_i, {}_{j+1}r_i) \in R$$

for all $1 \leq i \leq k$, $0 \leq j < n$.
The family of languages accepted by n-all-SFAs is denoted ALL_n.

First-Move Self-Regulating Finite Automata

Definition

For $n \geq 0$, an SFA M is an n-turn first-move SFA, n-first-SFA, if M accepts w as follows. There is $q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, such that

$$\mu = \underbrace{{}_0r_1\,{}_0r_2 \cdots {}_0r_k}_{k \text{ rules}}\,\underbrace{{}_1r_1\,{}_1r_2 \cdots {}_1r_k}_{k \text{ rules}} \cdots \underbrace{{}_nr_1\,{}_nr_2 \cdots {}_nr_k}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, ${}_0r_k$ is the first rule of the form $qx \rightarrow q_t$, for some $q \in Q$, $x \in \Sigma^*$, and

$$({}_jr_1, {}_{j+1}r_1) \in R$$

for $0 \leq j < n$.
The family of languages accepted by n-first-SFAs is denoted $FIRST_n$.

First-Move SFA—Example

tape

a	a	b	b				

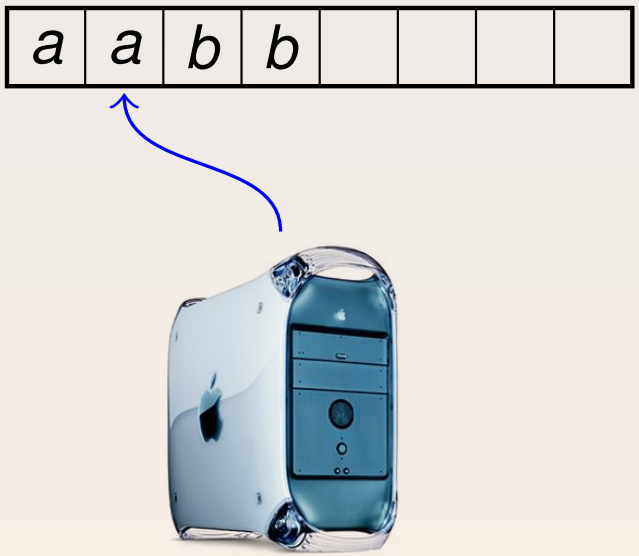
state: s, rules used: 1,2

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$
- P contains
  - $1.sa \rightarrow s$
  - $2.sa \rightarrow t$
  - $3.tb \rightarrow f$
  - $4.fb \rightarrow f$
- $R = \{(1, 3)\}$
- $\mathscr{L}(M) = \{a^n b^n : n \geq 1\}$

First-Move SFA—Example

tape



state: s, rules used: 1, 2

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$
- P contains
  - $1.sa \to s$
  - $2.sa \to t$
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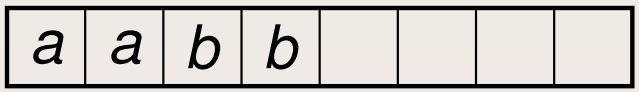
First-Move SFA—Example

tape



state: t, rules used: 1,2

M makes a turn

Description

- $M = (N, t, R)$, where
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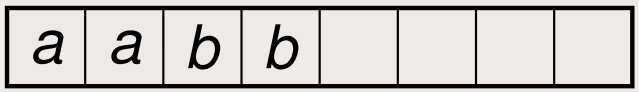
First-Move SFA—Example

tape



state: f, rules used: 3,2

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$
- P contains
  - $1.sa \rightarrow s$
  - $2.sa \rightarrow t$
  - $3.tb \rightarrow f$
  - $4.fb \rightarrow f$
- $R = \{(1, 3)\}$
- $\mathscr{L}(M) = \{a^n b^n : n \geq 1\}$

First-Move SFA—Example

tape

a	a	b	b				

state: f, rules used: 3,4

Description

- $M = (N, t, R)$, where
- $N = (\{s, t, f\}, \{a, b\}, P, s, \{f\})$
- P contains
  - $1.sa \rightarrow s$
  - $2.sa \rightarrow t$
  - $3.tb \rightarrow f$
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- $R = \{(1, 3)\}$
- $\mathscr{L}(M) = \{a^n b^n : n \geq 1\}$

Outline

Parallel Right-Linear Grammars (PRLG)

These grammars are needed in the following proof.

Definition (Rosebrugh and Wood, 1975)

For $n > 0$, an n-PRLG is an $(n+3)$-tuple

$$G = (N_1, \ldots, N_n, T, S, P),$$

where

- N_i are pairwise disjoint nonterminal alphabets, $1 \leq i \leq n$,
- T is a terminal alphabet,
- $S \notin N_1 \cup \cdots \cup N_n$,
- P contains three kinds of rules:
  1. $S \to X_1 \ldots X_n$ $\quad X_i \in N_i$, $1 \leq i \leq n$,
  2. $X \to wY$ $\quad X, Y \in N_i$ for some i, $1 \leq i \leq n$, $w \in T^*$, and
  3. $X \to w$ $\quad X \in N$, $w \in T^*$.

PRLG—Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $S \to y \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\downarrow \quad \downarrow \dots \quad \downarrow$
   $y = y_1 x_1 y_2 x_2 \dots y_n x_n$
   $y_i \in T^*$, $x_i \in T^* N \cup T^*$, $X_i \in N_i$, $X_i \to x_i \in P$, $1 \leq i \leq n$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

$PRL_n = \{\mathscr{L}(G) : G \text{ is an } n\text{-PRLG}\}$.

PRLG—Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $S \rightarrow y \in P$,
2. $$\begin{array}{l} x = y_1 X_1 y_2 X_2 \dots y_n X_n \\ \qquad \downarrow \quad \downarrow \dots \quad \downarrow \\ y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n \end{array}$$
   $y_i \in T^*$, $x_i \in T^* N \cup T^*$, $X_i \in N_i$, $X_i \rightarrow x_i \in P$, $1 \leq i \leq n$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

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PRLG—Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $S \to y \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\quad\quad\downarrow \quad\quad \downarrow \dots \quad\quad \downarrow$
   $y = y_1\, x_1\, y_2\, x_2 \dots y_n\, x_n$
   $y_i \in T^*$, $x_i \in T^*N \cup T^*$, $X_i \in N_i$, $X_i \to x_i \in P$, $1 \leq i \leq n$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ defined as usual.

$PRL_n = \{\mathscr{L}(G) : G \text{ is an } n\text{-PRLG}\}$.

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $S \to AB$
- $A \to aA$
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Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aaAbbB$$

PRLG—Example

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Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aaAbbB \Rightarrow a^3b^3$$

PRLG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $S \to AB$
- $A \to aA$
- $A \to a$
- $B \to bB$
- $B \to b$

Consider a derivation in G:

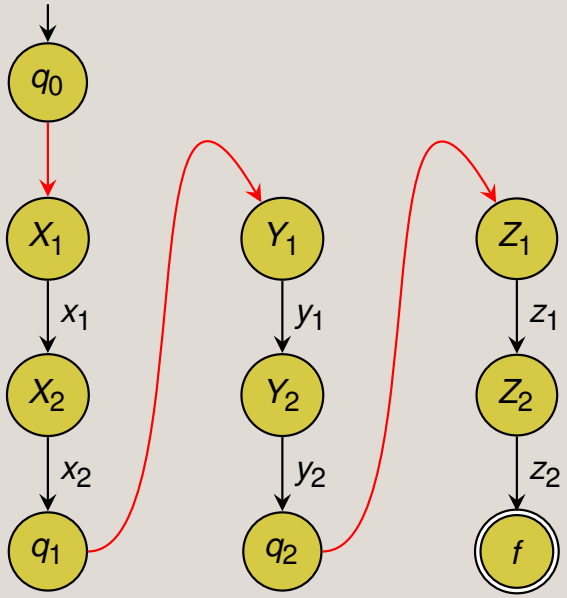
$$\mathscr{L}(G) = \{a^n b^n : n \geq 1\}$$

Lemma

Let G be an n-PRLG. There is an $(n-1)$-first-SFA M: $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea for $n = 3$.

$$S$$
$$\Downarrow$$
$$X_1 Y_1 Z_1$$
$$\Downarrow$$
$$x_1 X_2 y_1 Y_2 z_1 Z_2$$
$$\Downarrow$$
$$x_1 x_2 y_1 y_2 z_1 z_2$$



Lemma

Let M be an n-first-SFA. There is an $(n+1)$-PRLG G: $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea for $n = 2$.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
$(q_ty_0 \to r_1), (r_1y_1 \to r_2), (r_2y_2 \to q_i),$
$(q_iz_0 \to s_1), (s_1z_1 \to s_2), (s_2z_2 \to q_f)$

be an acceptance of $x_0x_1x_2y_0y_1y_2z_0z_1z_2z_2$ in M. Then,

$$
\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
&\Rightarrow x_0[q_1, 0, q_t]y_0[r_1, 1, q_i]z_0[s_1, 2, q_f] \\
&\Rightarrow x_0x_1[q_2, 0, q_t]y_0y_1[r_2, 1, q_i]z_0z_1[s_2, 2, q_{i_2}] \\
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Lemma

Let M be an n-first-SFA. There is an $(n+1)$-PRLG G: $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea for $n = 2$.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
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\end{aligned}
$$

□

PRLG vs. first-SFA

Theorem

For all $n \geq 0$, $FIRST_n = PRL_{n+1}$.

This theorem follows from the previous two lemmas:

Lemma

Let G be an n-PRLG. There is an $(n-1)$-first-SFA M such that

$$\mathscr{L}(G) = \mathscr{L}(M).$$

Lemma

Let M be an n-first-SFA. There is an $(n+1)$-PRLG G such that

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Language Families of First-Move SFAs

Corollary (Rosebrugh and Wood, 1975)

1. $REG = FIRST_0 \subset FIRST_1 \subset FIRST_2 \subset \cdots \subset CS.$
2. $FIRST_1 \subset CF.$
3. $FIRST_2 \not\subseteq CF.$
4. $CF \not\subseteq FIRST_n$ *for any* $n \geq 0.$

Right-Linear Simple Matrix Grammar (RLSMG)

These grammars are needed in the following proof.

Definition (Wood, 1975)

For $n > 0$, an n-RLSMG is an $(n+3)$-tuple

$$G = (N_1, \ldots, N_n, T, S, P),$$

where

- N_i are pairwise disjoint nonterminal alphabets, $1 \leq i \leq n$,
- T is a terminal alphabet,
- $S \notin N_1 \cup \cdots \cup N_n$,
- P contains three kinds of matrix rules:
  1. $[S \to X_1 \ldots X_n]$ $\quad X_i \in N_i,\ 1 \leq i \leq n,$
  2. $[X_1 \to w_1 Y_1, \ldots, X_n \to w_n Y_n]$ $\quad w_i \in T^*,\ X_i, Y_i \in N_i,\ 1 \leq i \leq n,$
  3. $[X_1 \to w_1, \ldots, X_n \to w_n]$ $\quad X_i \in N_i,\ w_i \in T^*,\ 1 \leq i \leq n.$

RLSMG–Derivation Step

For $x, y \in (N \cup T \cup \{S\})^*$,

$$x \Rightarrow y$$

if and only if

1. either $x = S$ and $[S \to y] \in P$,
2. $x = y_1 X_1 y_2 X_2 \dots y_n X_n$
   $\downarrow \quad \downarrow \dots \quad \downarrow$
   $y = y_1 x_1 y_2 x_2 \dots y_n x_n$
   $y_i \in T^*$, $x_i \in T^* N \cup T^*$, $X_i \in N_i$, $1 \le i \le n$,
   $[X_1 \to x_1, \dots, X_n \to x_n] \in P$.

$\mathscr{L}(G) = \{w \in T^* : S \Rightarrow^+ w\}$, $\Rightarrow^+$ is defined as usual.

$RLSM_n = \{\mathscr{L}(G) : G \text{ is an } n\text{-RLSMG}\}$.

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RLSMG—Example

Example

Let $G = (\{A\}, \{B\}, \{a, b\}, S, P)$ be a 2-PRLG, where P contains rules

- $[S \to AB]$
- $[A \to aA, B \to aB]$
- $[A \to bA, B \to bB]$
- $[A \to \varepsilon, B \to \varepsilon]$

Consider a derivation in G:

$$S$$

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Consider a derivation in G:

$$S \Rightarrow AB$$

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Consider a derivation in G:

$$S \Rightarrow AB \Rightarrow aAaB$$

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- $[A \to \varepsilon, B \to \varepsilon]$

Consider a derivation in G:

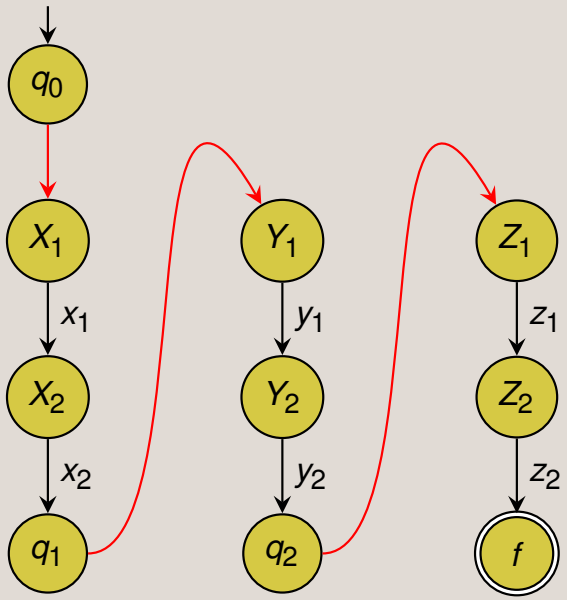
$$\mathscr{L}(G) = \{ww : w \in \{a, b\}^*\}$$

Lemma

Let G be an n-RLSMG. There is an $(n-1)$-all-SFA M: $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea for $n = 3$.

$$S$$
$$\Downarrow$$
$$X_1 Y_1 Z_1$$
$$\Downarrow$$
$$x_1 X_2 y_1 Y_2 z_1 Z_2$$
$$\Downarrow$$
$$x_1 x_2 y_1 y_2 z_1 z_2$$



Lemma

Let M be an n-all-SFA. There is an $(n+1)$-RLSMG G: $\mathscr{L}(G) = \mathscr{L}(M)$.

Proof idea for $n = 2$.

Let $\mu = (q_0x_0 \to q_1), (q_1x_1 \to q_2), (q_2x_2 \to q_t),$
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be an acceptance of $x_0x_1x_2y_0y_1y_2z_0z_1z_2z_2$ in M. Then,

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\begin{aligned}
S &\Rightarrow [q_0x_0, q_1, 0, q_t][q_ty_0, r_1, 1, q_i][q_iz_0, s_1, 2, q_f] \\
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\end{aligned}
$$

□

RLSMG vs. all-SFA

Theorem

For all $n \geq 0$, $ALL_n = RLSM_{n+1}$.

This theorem follows from two previous lemmas:

Lemma

Let G be an n-RLSMG. There is an $(n-1)$-all-SFA M such that

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Language Families of All-Move SFAs

Corollary (Wood, 1975)

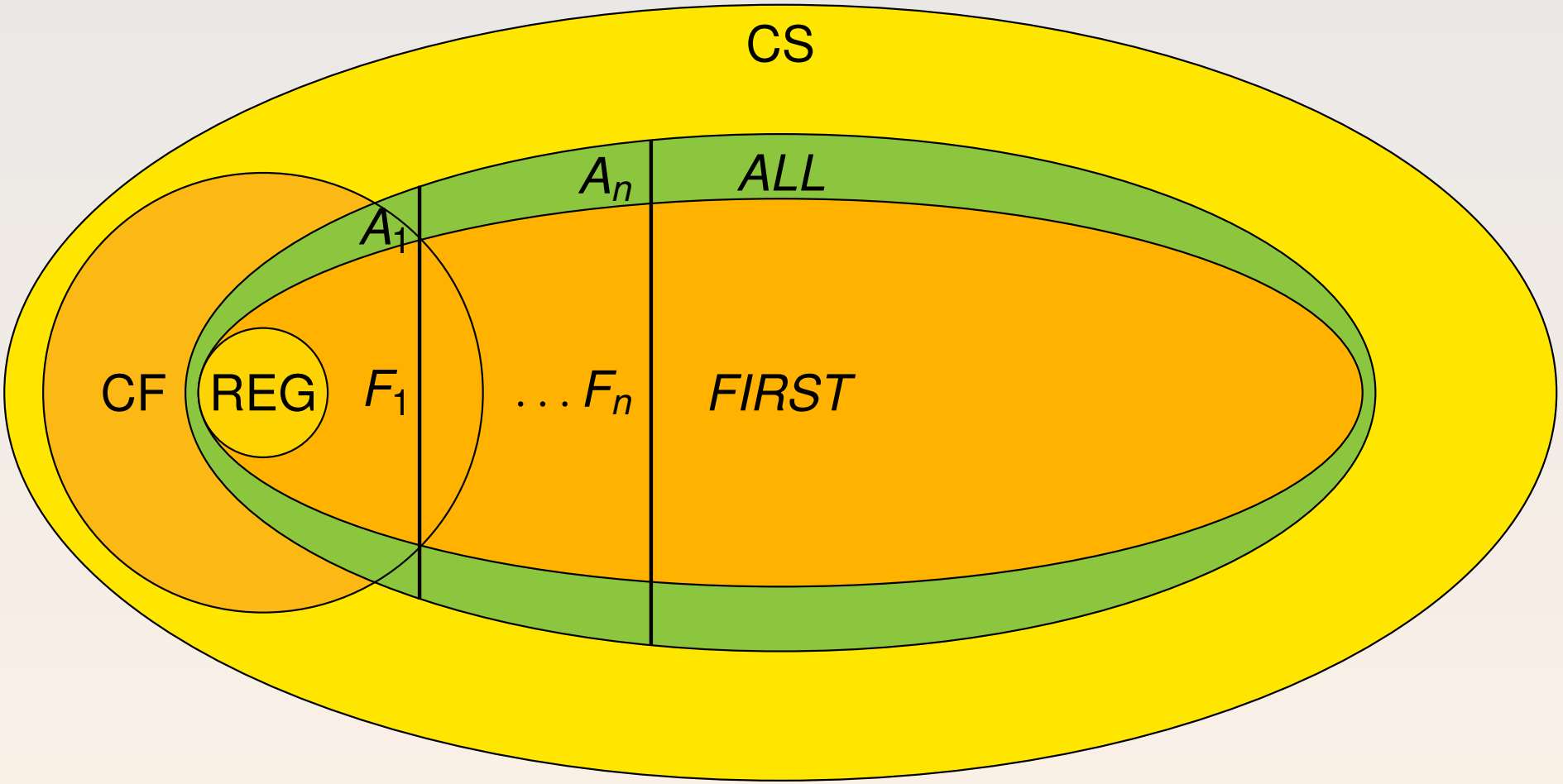
1. $REG = ALL_0 \subset ALL_1 \subset ALL_2 \subset \cdots \subset CS$.
2. $ALL_1 \not\subseteq CF$.
3. $CF \not\subseteq ALL_n$, *for* $n \geq 0$.

Comparison

Theorem

1. $FIRST_0 = ALL_0 = REG$.
2. For all $n > 0$, $FIRST_n \subset ALL_n$.
3. $FIRST_n \not\subseteq ALL_{n-1}$, $n \geq 1$.
4. $ALL_n - FIRST \neq \emptyset$, $n \geq 1$, where $FIRST = \bigcup_{m=1}^{\infty} FIRST_m$.

Comparison

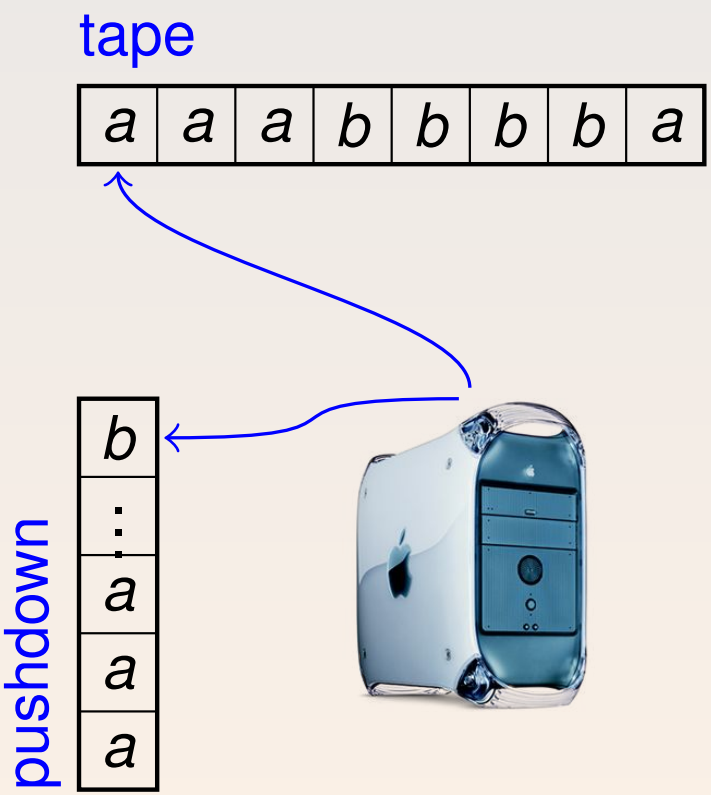


$F_n = FIRST_n$, the family of languages accepted by n-first-SFAs
$A_n = ALL_n$, the family of languages accepted by n-all-SFAs

Outline

Outline

Pushdown Automata—Concept



Characteristics

- Finite state control
- Input cannot be modified
- Head only moves forward
- Potentially infinite pushdown store
- Pushdown top can be modified
- Pushdown-head reads the top symbol

Pushdown Automata—Definition

Definition

A pushdown automaton is a quintuple

$$M = (Q, \Sigma, \Gamma, P, q_0, Z_0, F),$$

where

- Q is a finite set of states,
- Σ and Γ are input and pushdown alphabets, respectively,
- P is a finite set of rules of the form $Aqw \to zp$, $A \in \Gamma$, $q, p \in Q$, $w \in \Sigma^*$, $z \in \Gamma^*$,
- $q_0 \in Q$ is an initial state,
- Z_0 is an initial pushdown symbol,
- $F \subseteq Q$ is a set of final states.

Pushdown Automata—Language

Definition

A configuration is a member of $\Gamma^* Q \Sigma^*$.
If

$$xAqwy \in \Gamma^* Q \Sigma^* \text{ and } r.Aqw \to zp \in P,$$

then

$$xAqwy \Rightarrow xzpy\,[r].$$

The language of M is the set

$$\mathscr{L}(M) = \{w \in \Sigma^* : Z_0 q_0 w \Rightarrow^* f,\ f \in F\}.$$

Self-Regulating Pushdown Automata

Definition

Let

$$N = (Q, \Sigma, \Gamma, P, q_0, Z_0, F)$$

be a pushdown automaton.
A self-regulating pushdown automaton, SPDA, is a triple

$$M = (N, q_t, R),$$

where

1. $q_t \in Q$ is a turn state, and
2. $R \subseteq \Psi \times \Psi$ is a finite relation on the alphabet of rule labels.

All-Move Self-Regulating Pushdown Automata

Definition

For $n \geq 0$, an SPDA M is n-turn all-move SPDA, n-all-SPDA, if M accepts w as follows. There is $Z_0 q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, such that

$$\mu = \underbrace{{}_0r_1\,{}_0r_2 \cdots {}_0r_k}_{k \text{ rules}}\,\underbrace{{}_1r_1\,{}_1r_2 \cdots {}_1r_k}_{k \text{ rules}} \cdots \underbrace{{}_nr_1\,{}_nr_2 \cdots {}_nr_k}_{k \text{ rules}},$$

where $k \in \mathbb{N}$, ${}_0r_k$ is the first rule of the form $Zqx \rightarrow zq_t$, for some $Z \in \Gamma$, $q \in Q$, $x \in \Sigma^*$, $z \in \Gamma^*$, and

$$({}_jr_i, {}_{j+1}r_i) \in R$$

for all $1 \leq i \leq k$, $0 \leq j < n$.
The family of languages accepted by n-all-SPDAs is denoted $ALL\text{-}SPDA_n$.

First-Move Self-Regulating Pushdown Automata

Definition

For $n \geq 0$, an SPDA M is n-turn first-move SPDA, n-first-SPDA, if M accepts w as follows. There is $Z_0 q_0 w \Rightarrow^* f\,[\mu]$, $f \in F$, such that

$$\mu = \underbrace{{}_0r_1\,{}_0r_2 \cdots {}_0r_k}_{k \text{ rules}}\,\underbrace{{}_1r_1\,{}_1r_2 \cdots {}_1r_k}_{k \text{ rules}} \cdots \underbrace{{}_nr_1\,{}_nr_2 \cdots {}_nr_k}_{k \text{ rules}},$$

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for $0 \leq j < n$.
The family of languages accepted by n-first-SPDAs is denoted $FIRST\text{-}SPDA_n$.

Outline

All-Move Self-Regulating Pushdown Automata

Observation: $ALL\text{-}SPDA_0 = FIRST\text{-}SPDA_0 = CF$.

Theorem

$ALL\text{-}SPDA_1 = RE$.

Proof Idea.

$L \in RE$, then there are CFGs G and H in GNF such that

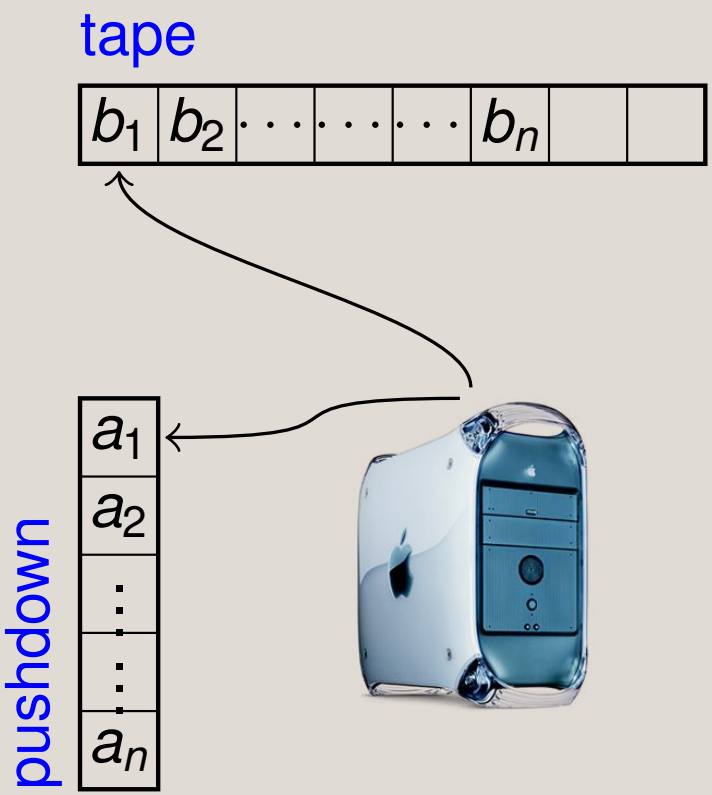
$$L = h(\mathscr{L}(G) \cap \mathscr{L}(H)).$$

On the pushdown, M simulates

1. G that generates a string x so if a is on the top, M reads $h(a)$; then,
2. H that generates x, which is verified by R (no input is read).

All-Move Self-Regulating Pushdown Automata

Proof Idea (cont.).



Characteristics

- $x = b_1 b_2 \ldots b_n$
- $b_i = h(a_i)$
- check if $a_1 a_2 \ldots a_n \in \mathscr{L}(G)$
- if so, $h(a_1 a_2 \ldots a_n) \in h(\mathscr{L}(G))$
- check if $a_1 a_2 \ldots a_n \in \mathscr{L}(H)$
- if so, $a_1 a_2 \ldots a_n \in \mathscr{L}(G) \cap \mathscr{L}(H)$

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Proof Idea—Construction.

$$M = (\{q_0, q, q_t, p, f\}, \Delta, \Sigma \cup N_G \cup N_H \cup \{Z\}, P, q_0, Z, \{f\}, R)$$

$Z \notin \Sigma \cup N_G \cup N_H$, with R and P made as

1. add $(Zq_0 \to ZS_G q, Zq_t \to ZS_H p)$ to R
2. add $(Aq \to B_n \dots B_1 aq, Cp \to D_m \dots D_1 ap)$ to R if $A \to aB_1 \dots B_n \in P_G$ and $C \to aD_1 \dots D_m \in P_H$
3. add $(aqh(a) \to q, ap \to p)$ to R
4. add $(Zq \to Zq_t, Zp \to f)$ to R

Outline

Open Problems

1. What is $\mathit{FIRST}\text{-}\mathit{SPDA}_n$, for $n \geq 1$?
2. Determinism.

References

J. Dassow and G. Paun.
Regulated Rewriting in Formal Language Theory.
Springer-Verlag, Berlin, 1989.

R. D. Rosebrugh and D. Wood.
Restricted parallelism and right linear grammars.
Utilitas mathematica, 7:151–186, 1975.

A. Meduna and T. Masopust.
Self-Regulating Finite Automata.
Acta Cybernetica, in press.

D. Wood.
m-parallel *n*-right linear simple matrix languages.
Utilitas mathematica, 8:3–28, 1975.

Discussion