Deep Pushdown Automata

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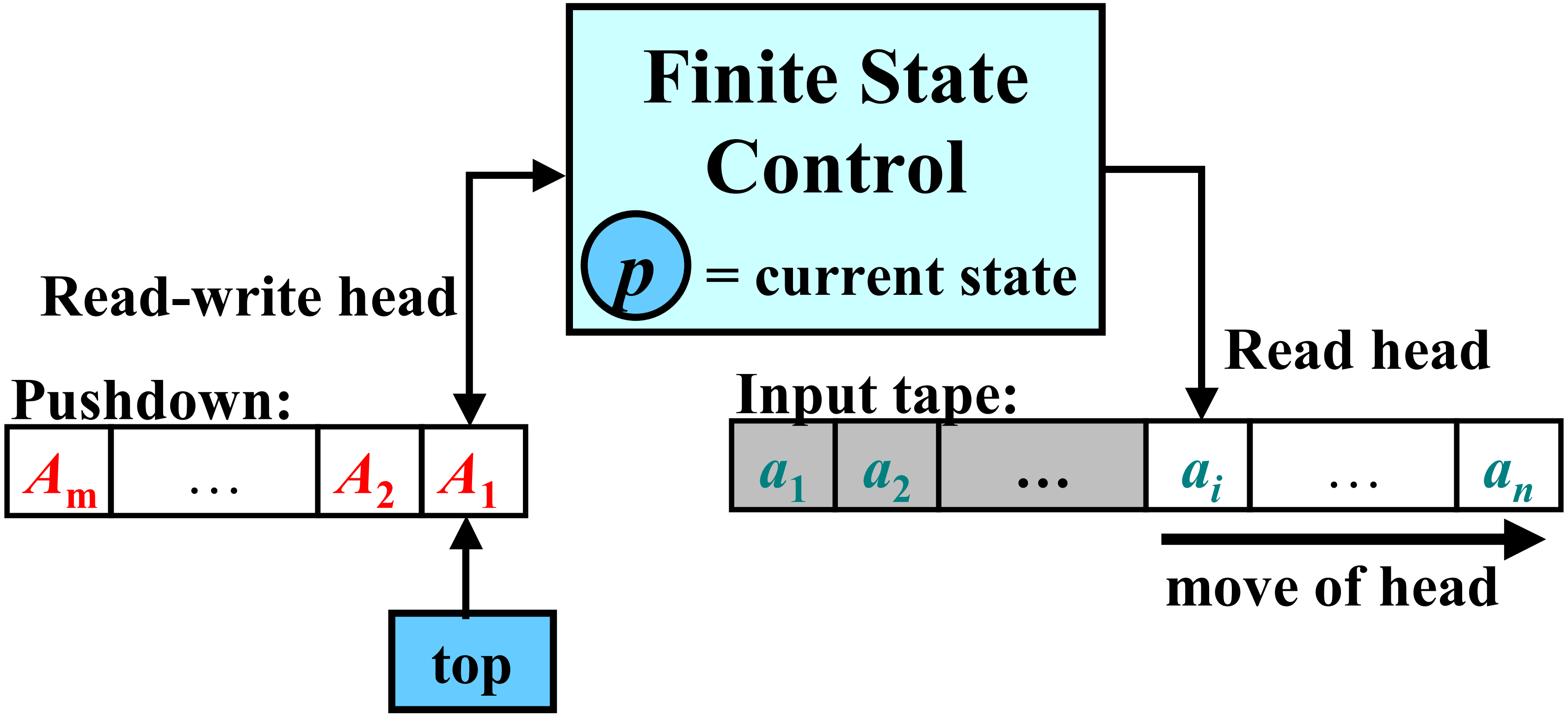
Based on

Meduna, A.: Deep Pushdown Automata,

***Acta Informatica*, 2006**

Pushdown Automaton (PDA)



Inspiration

- conversion of CFG to PDA that acts as a general top-down parser
  - if pd top = **input** symbol, **pop**
  - if pd top = **non-input** symbol, **expand**

PDA as a general Top-Down Parser

- Configuration:

$$(\mathit{state}, \mathit{input}, \mathit{pd})$$

- **Pop:**

$$(q, ax, a\alpha) \;{}_{p}\!\Rightarrow (q, x, \alpha)$$

- **Expansion:**

$$(q, x, A\alpha) \;{}_{e}\!\Rightarrow (p, x, \beta\alpha)$$

by rule $qA \rightarrow p\beta$

Final state

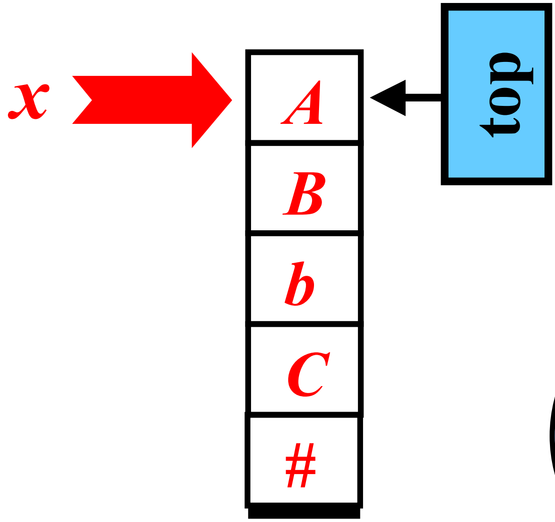
- Acceptance: $(s, x, S) \Rightarrow^{*} (f, \varepsilon, \varepsilon)$

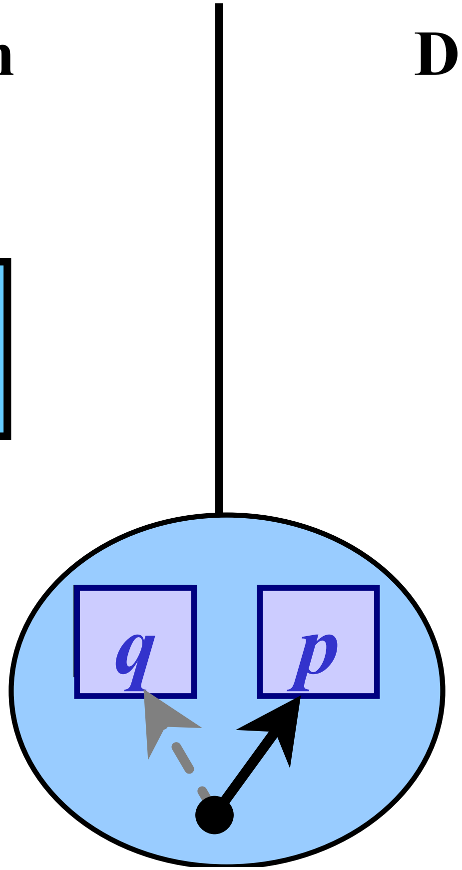
Deep Pushdown Automata: Fundamental Modification

- expansion may be performed **deeper in pd**

Standard Expansion

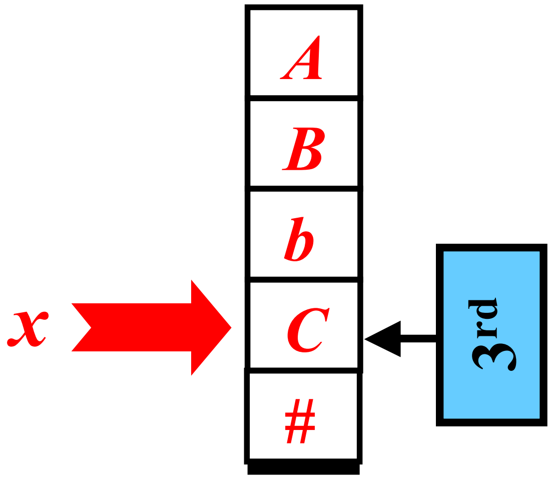
$qA \rightarrow px$





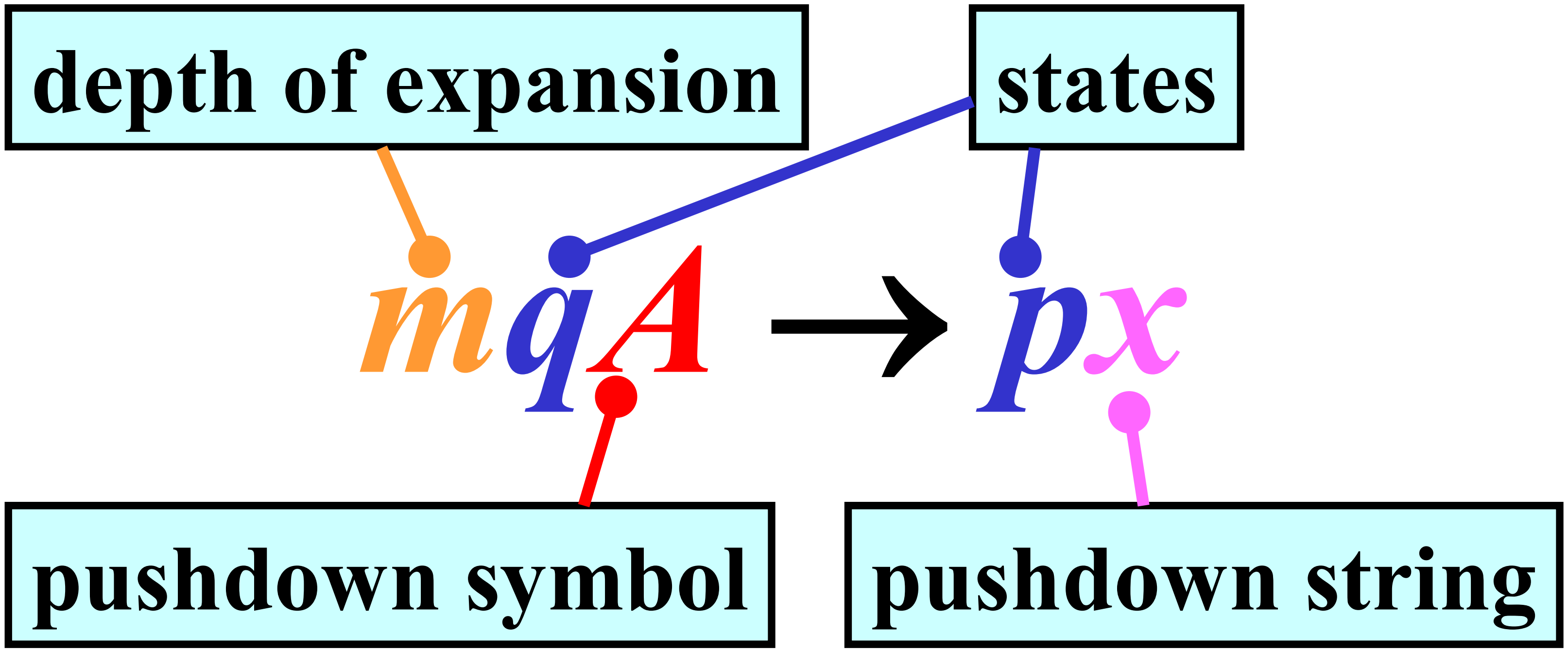
Deep Expansion

$3qC \rightarrow px$



Deep Pushdown Automata

- Same as Top-Down Parser except *deep expansions*
- *Expansion of depth m*:
  - the *m*th topmost non-input pd symbol is replaced with a string by rule



Expansion of Depth m

- ***Expansion of depth m***:

$$(q, w, uAz) \ {}_e\!\Rightarrow (p, w, uvz)$$

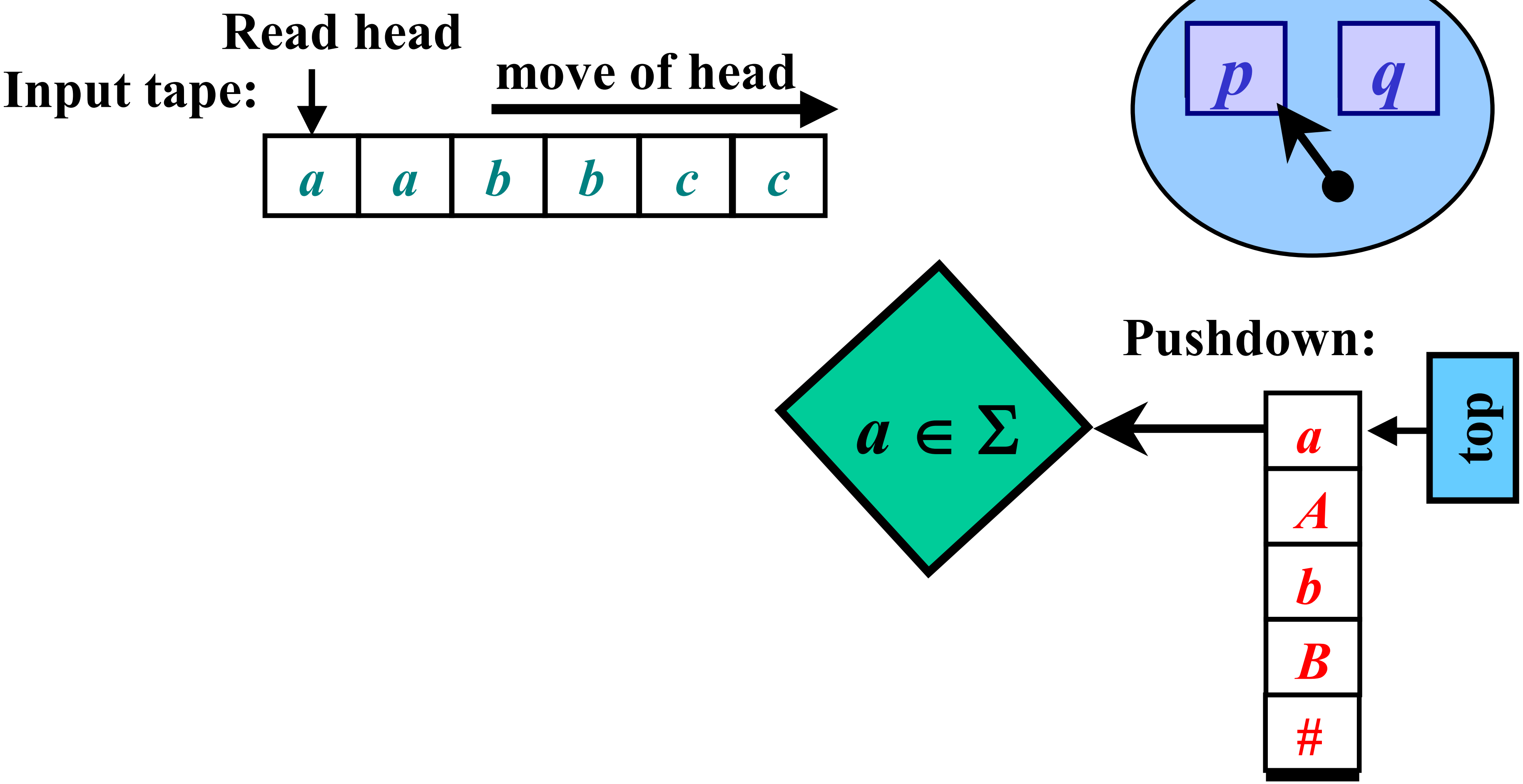
by ***rule of depth m***

$$[mqA \rightarrow pv],$$

where u contains $m - \mathbf{1}$ non-input symbols

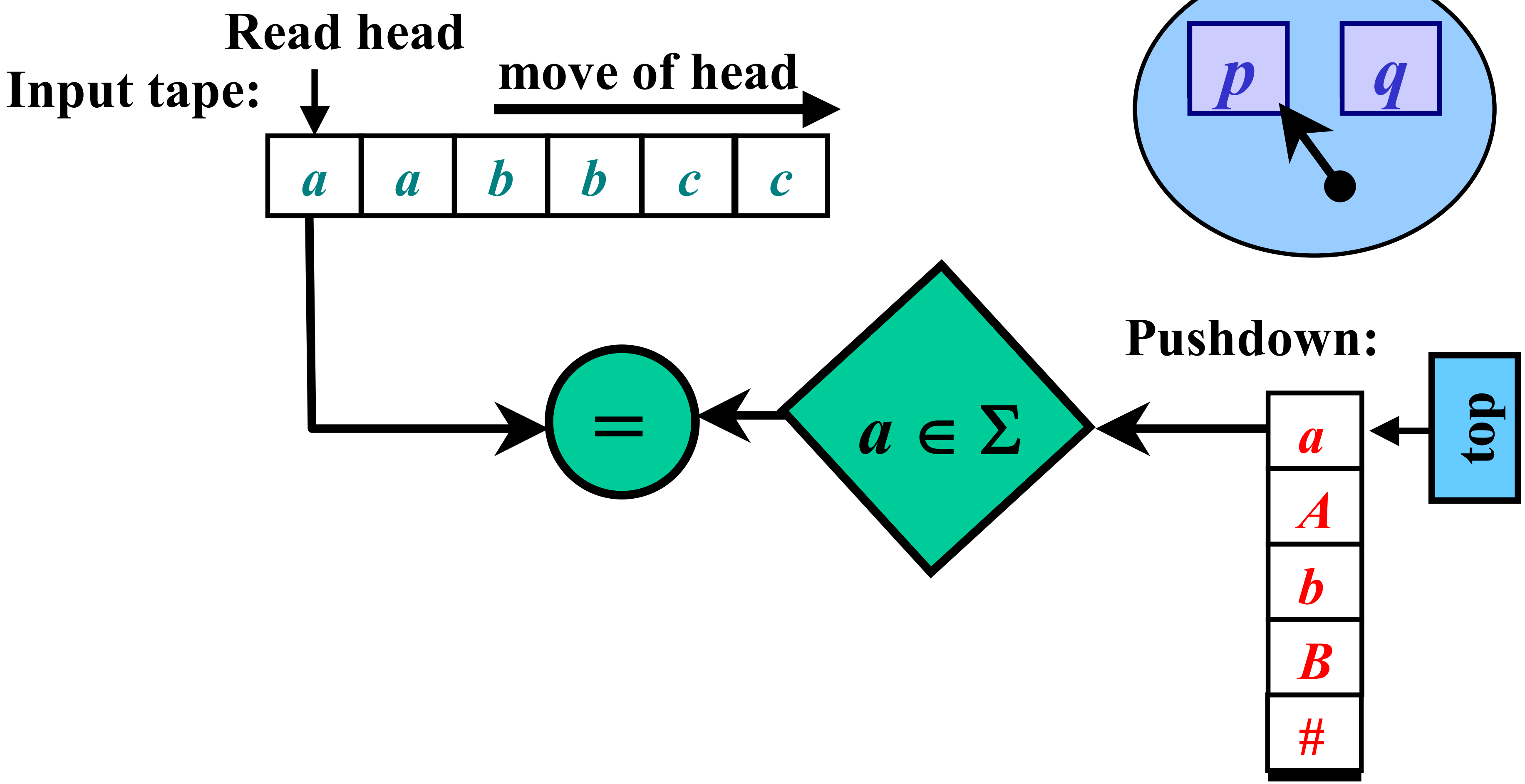
Pop: Illustration

Move: (p, $aabbcc$, $aAbB\#$) $_p\Rightarrow$ (p, $abbcc$, $AbB\#$)



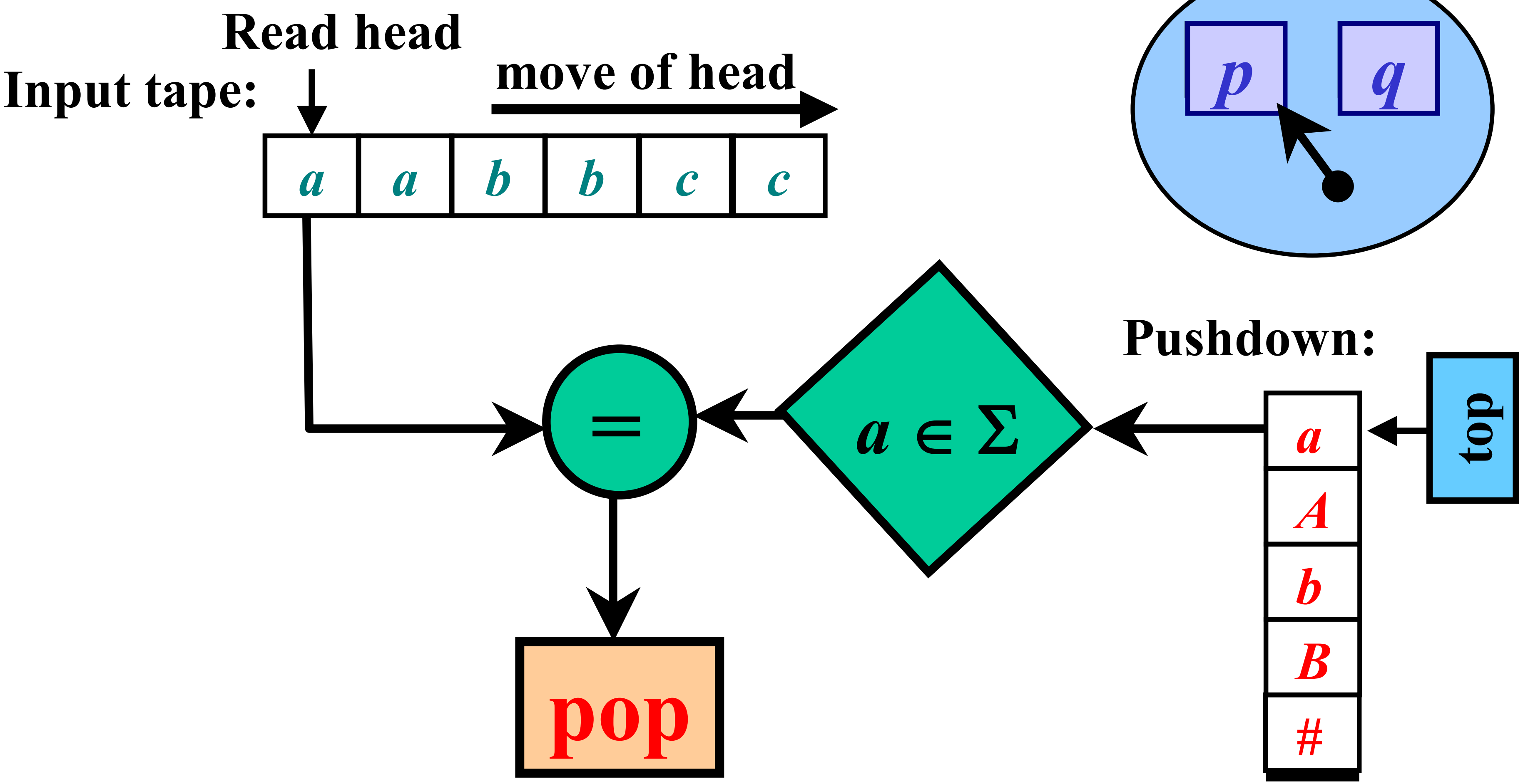
Pop: Illustration

Move: (p, *aabbcc*, *aAbB#*) $_p\Rightarrow$ (p, *abbcc*, *AbB#*)



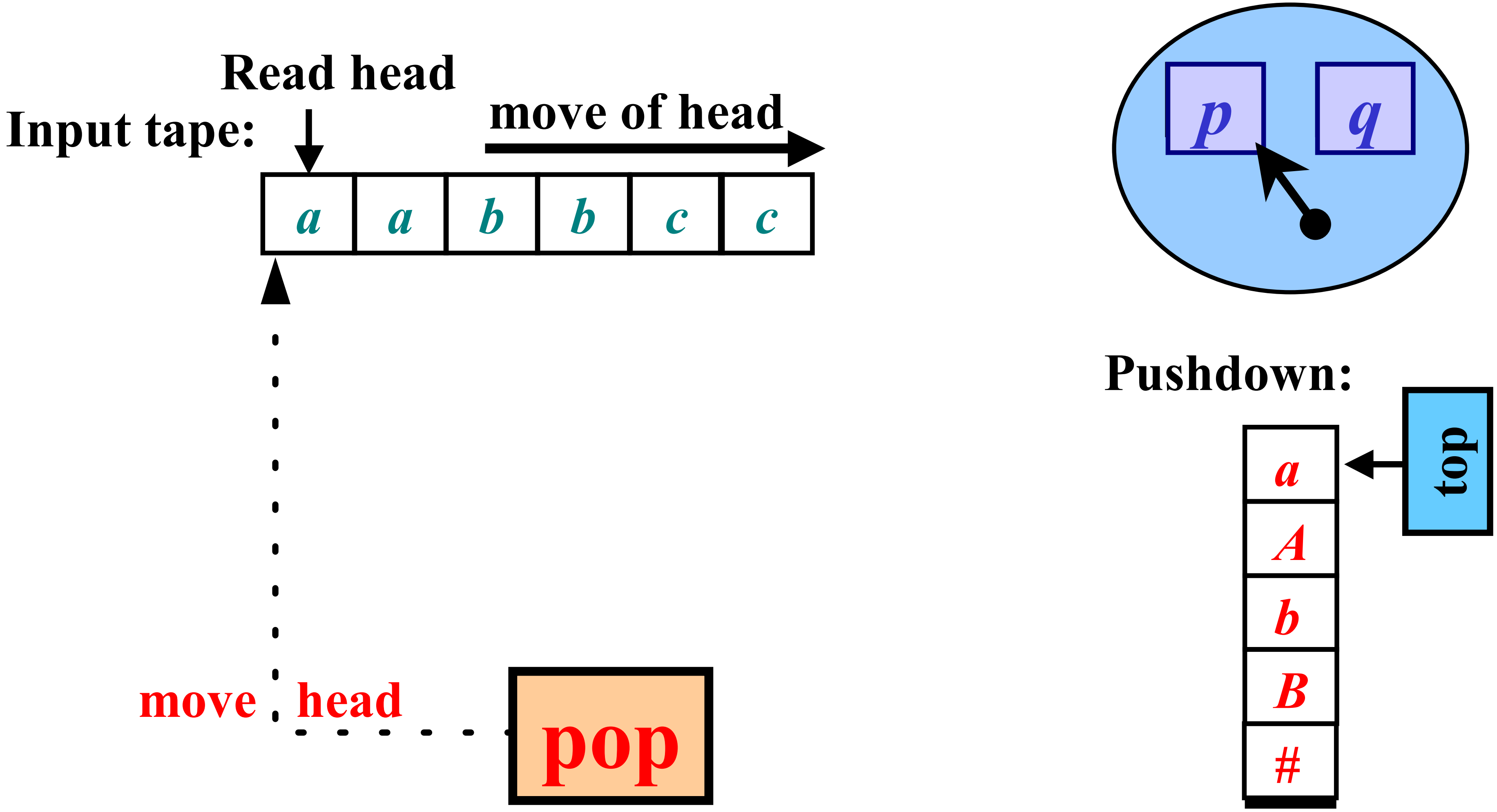
Pop: Illustration

Move: (p, $aabbcc$, $aAbB\#$) $_p\Rightarrow$ (p, $abbcc$, $AbB\#$)



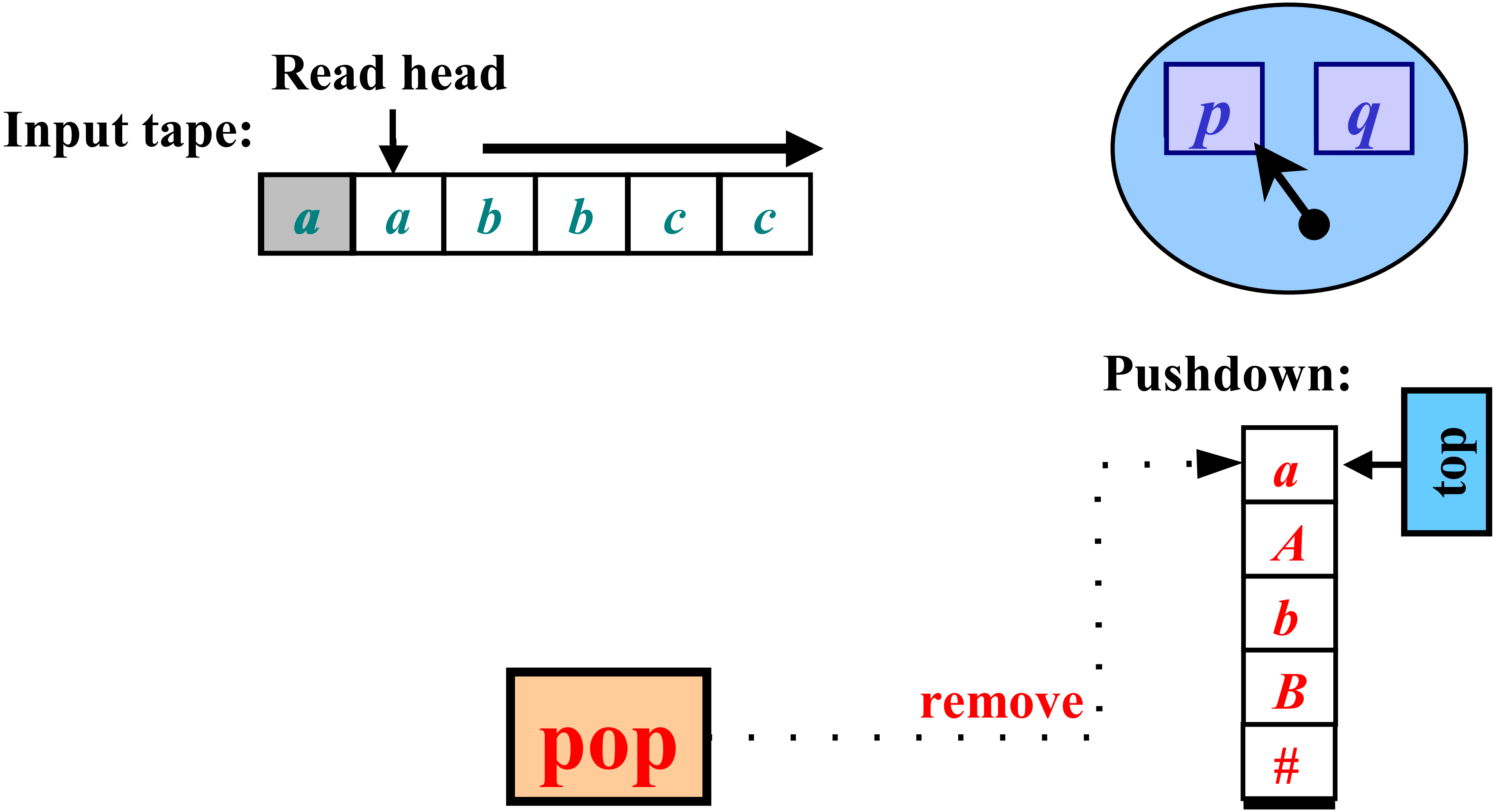
Pop: Illustration

Move: (p, $aabbcc$, $aAbB\#$) $_p\Rightarrow$ (p, $abbcc$, $AbB\#$)



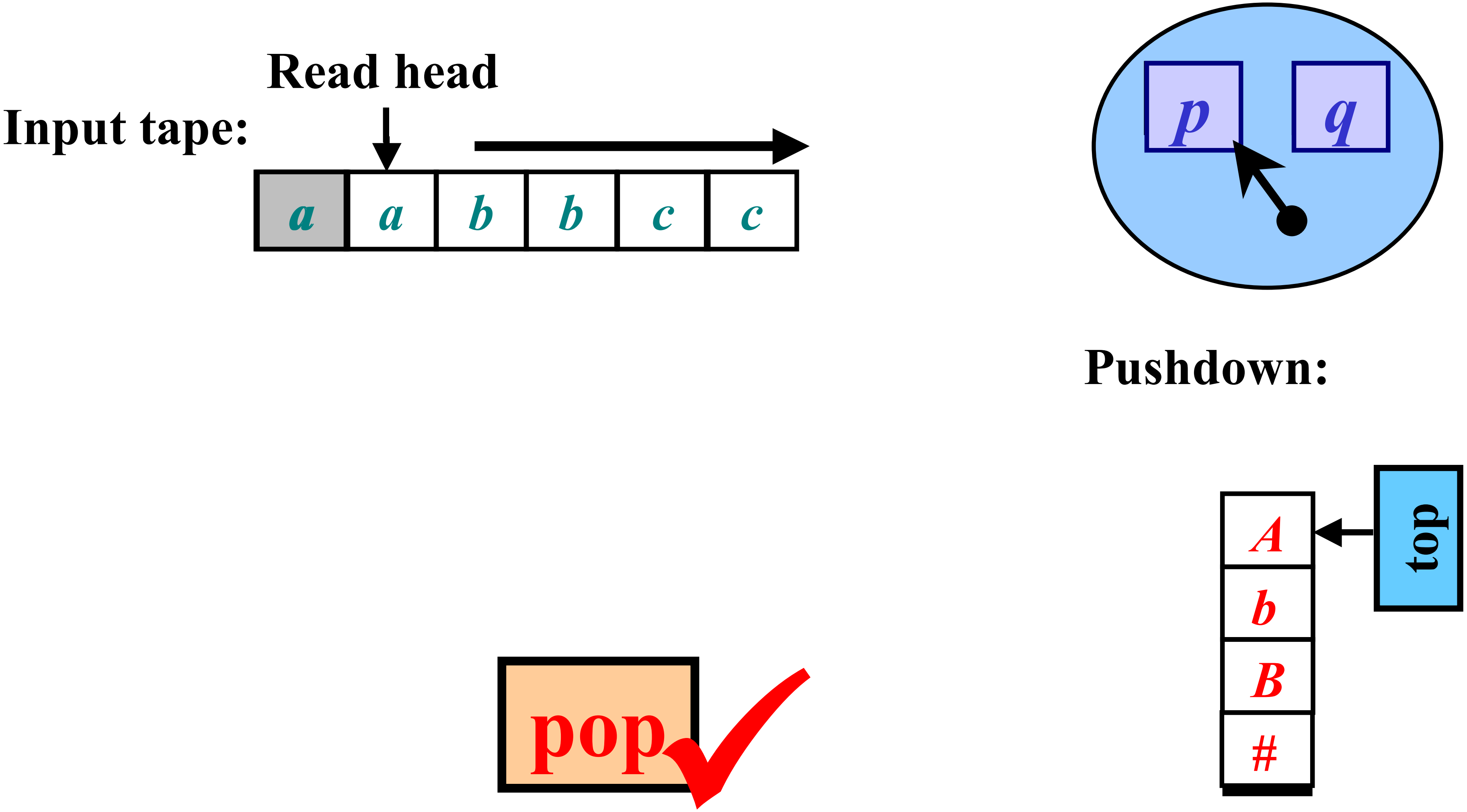
Pop: Illustration

Move: $(p, aabbcc, aAbB\#) \,_p\!\Rightarrow (p, abbcc, AbB\#)$



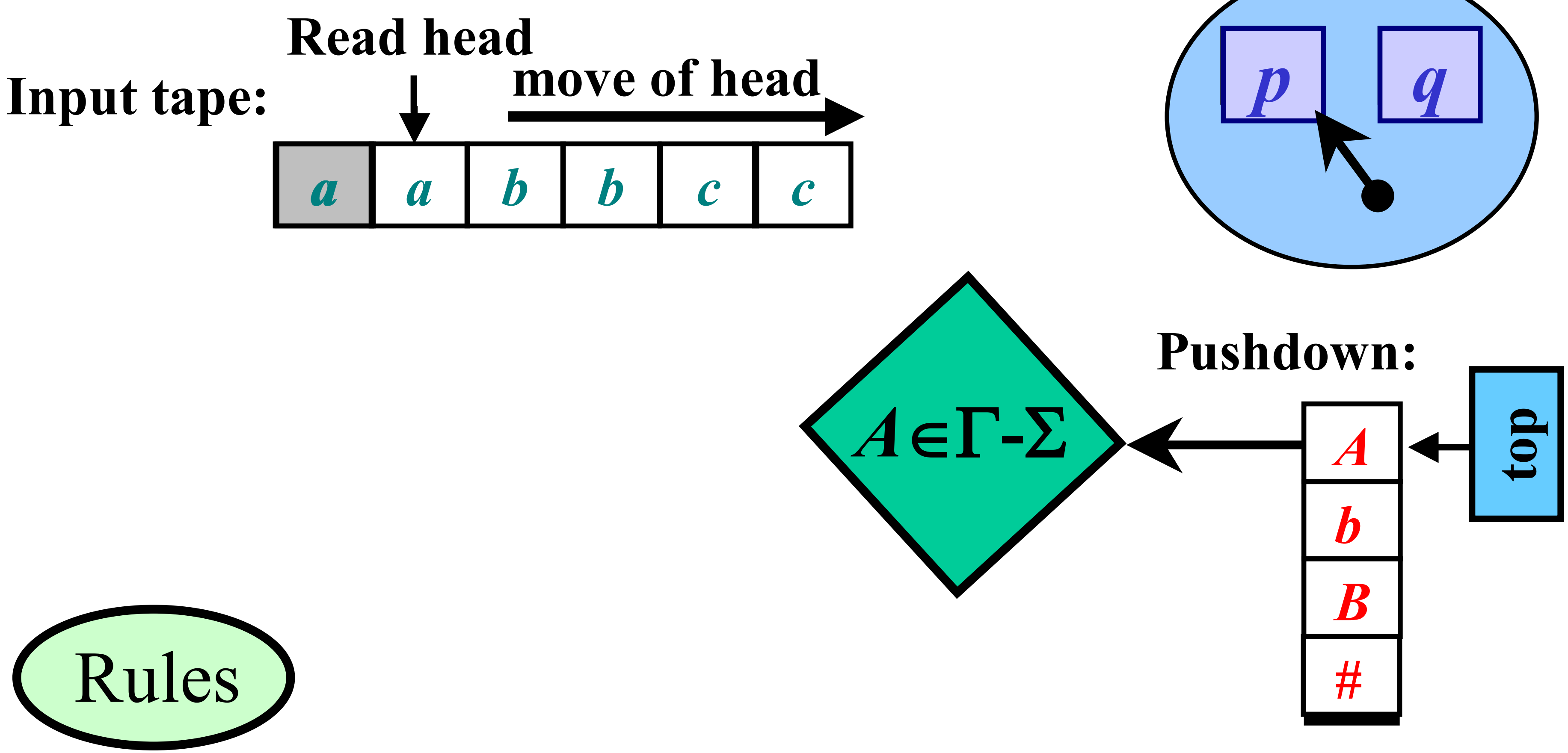
Pop: Illustration

Move: (p, *aabbcc*, *aAbB#*) $_p\Rightarrow$ (p, *abbcc*, *AbB#*)



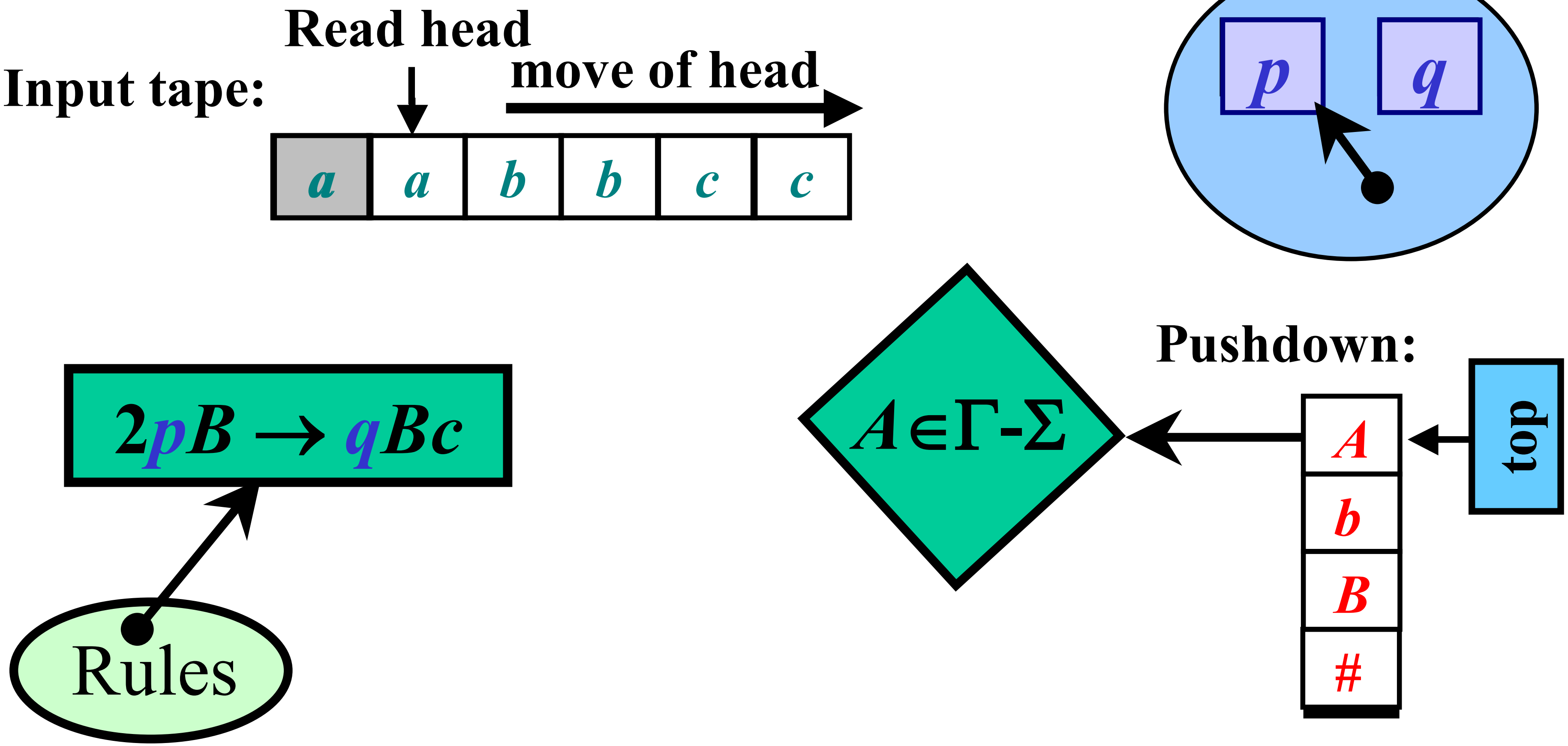
Deep Expansion: Illustration

Move: (p, $abbcc$, $AbB\#$) ${}_{e}\Rightarrow$ (q, $abbcc$, $AbBc\#$) [2$pB \rightarrow qBc$]



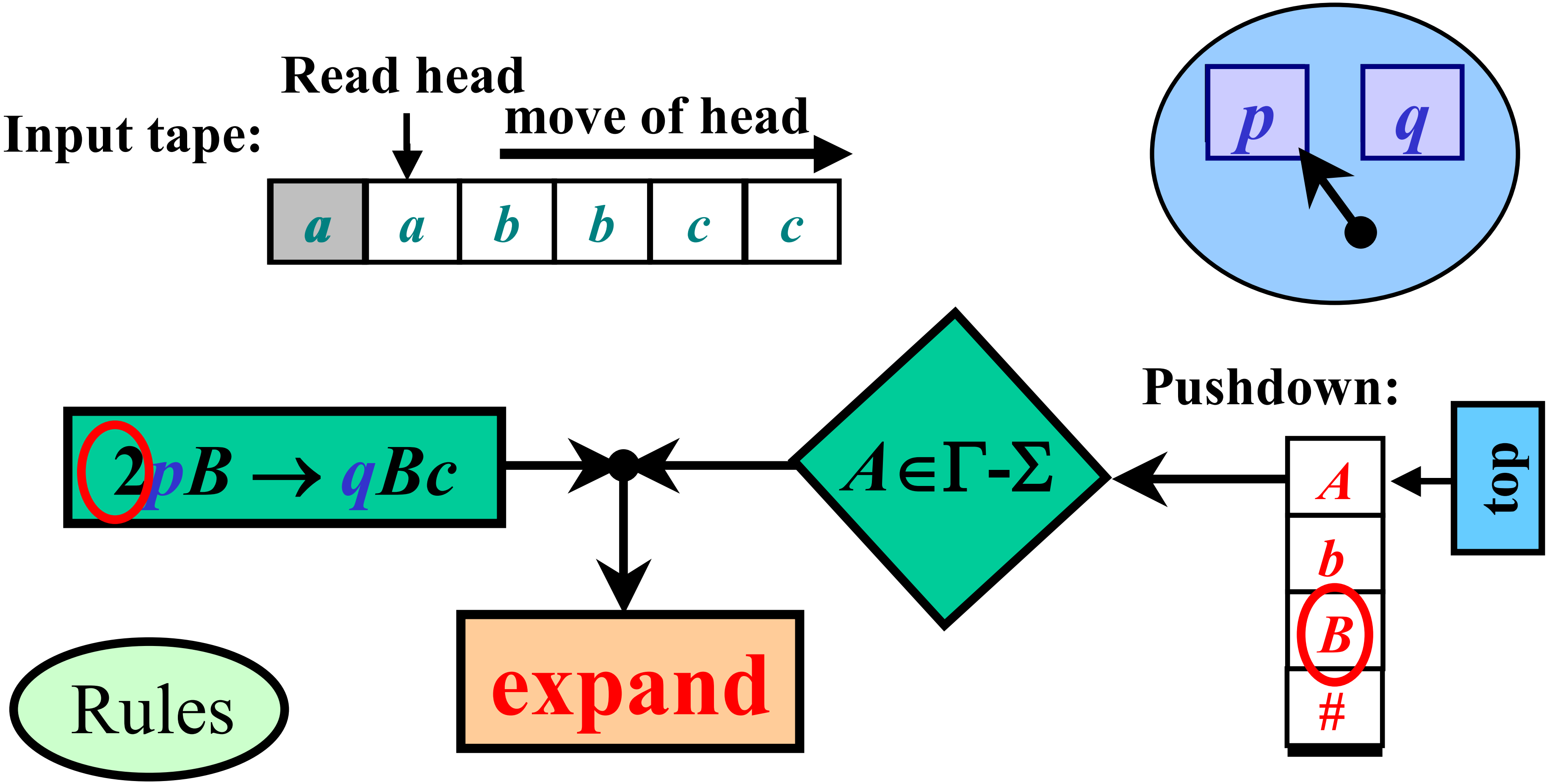
Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#)\ {}_e\!\Rightarrow (q, abbcc, AbBc\#)\ [2pB \rightarrow qBc]$



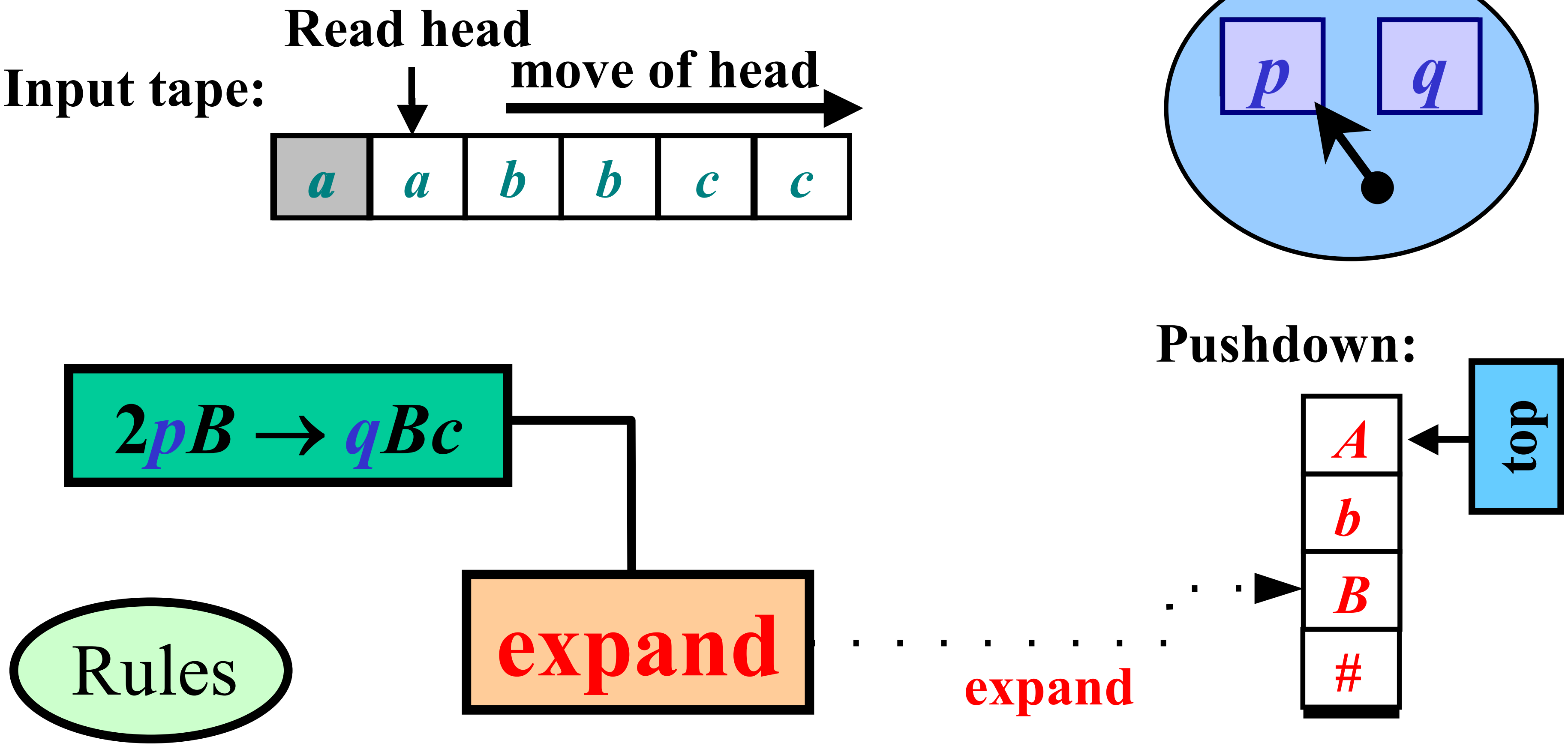
Deep Expansion: Illustration

Move: (p, *abbcc*, *AbB#*) $_e\Rightarrow$ (q, *abbcc*, *AbBc#*) [2$pB \rightarrow qBc$]



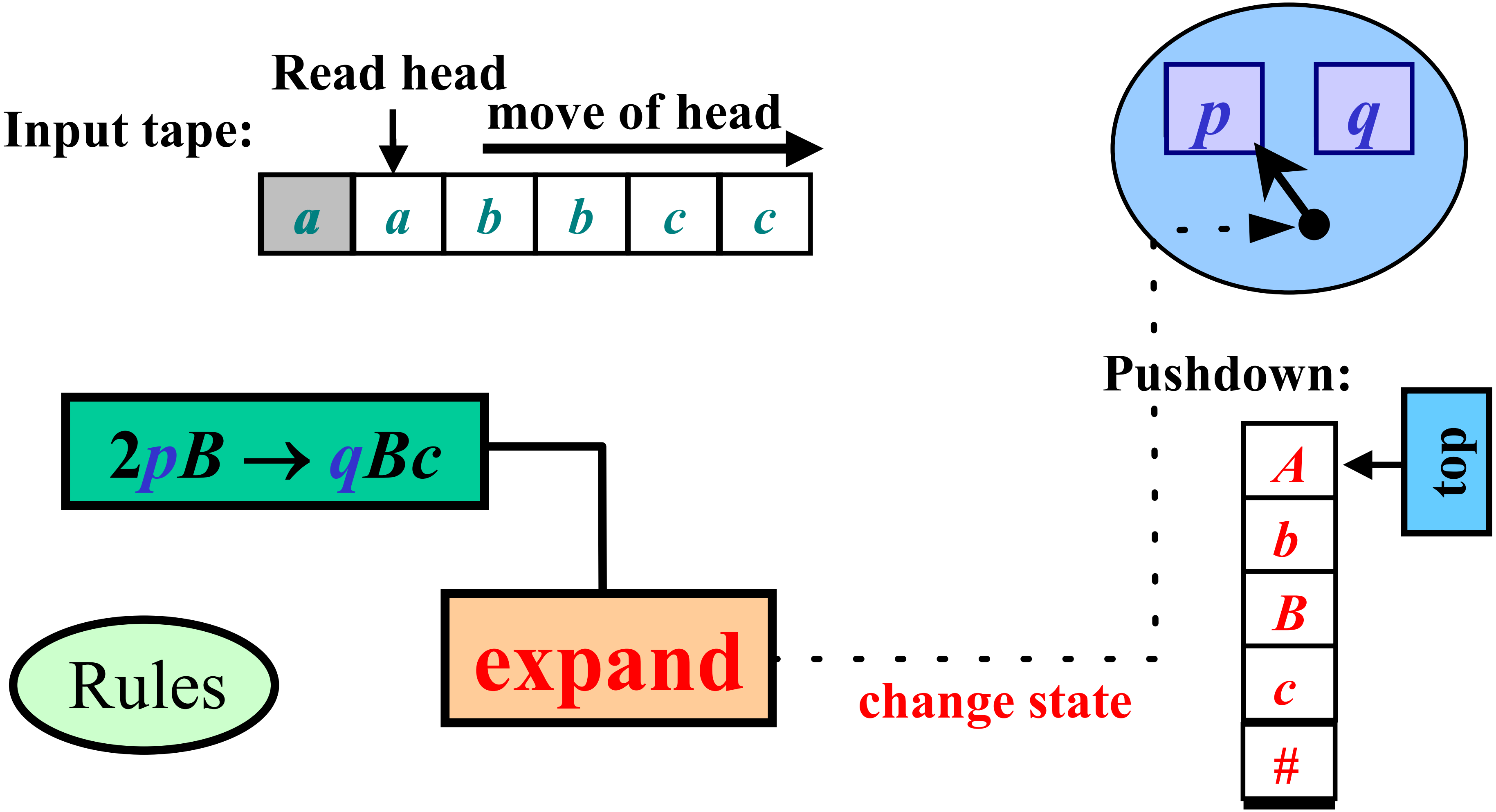
Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#)\ {}_e\!\Rightarrow (q, abbcc, AbBc\#)\ [2pB \rightarrow qBc]$



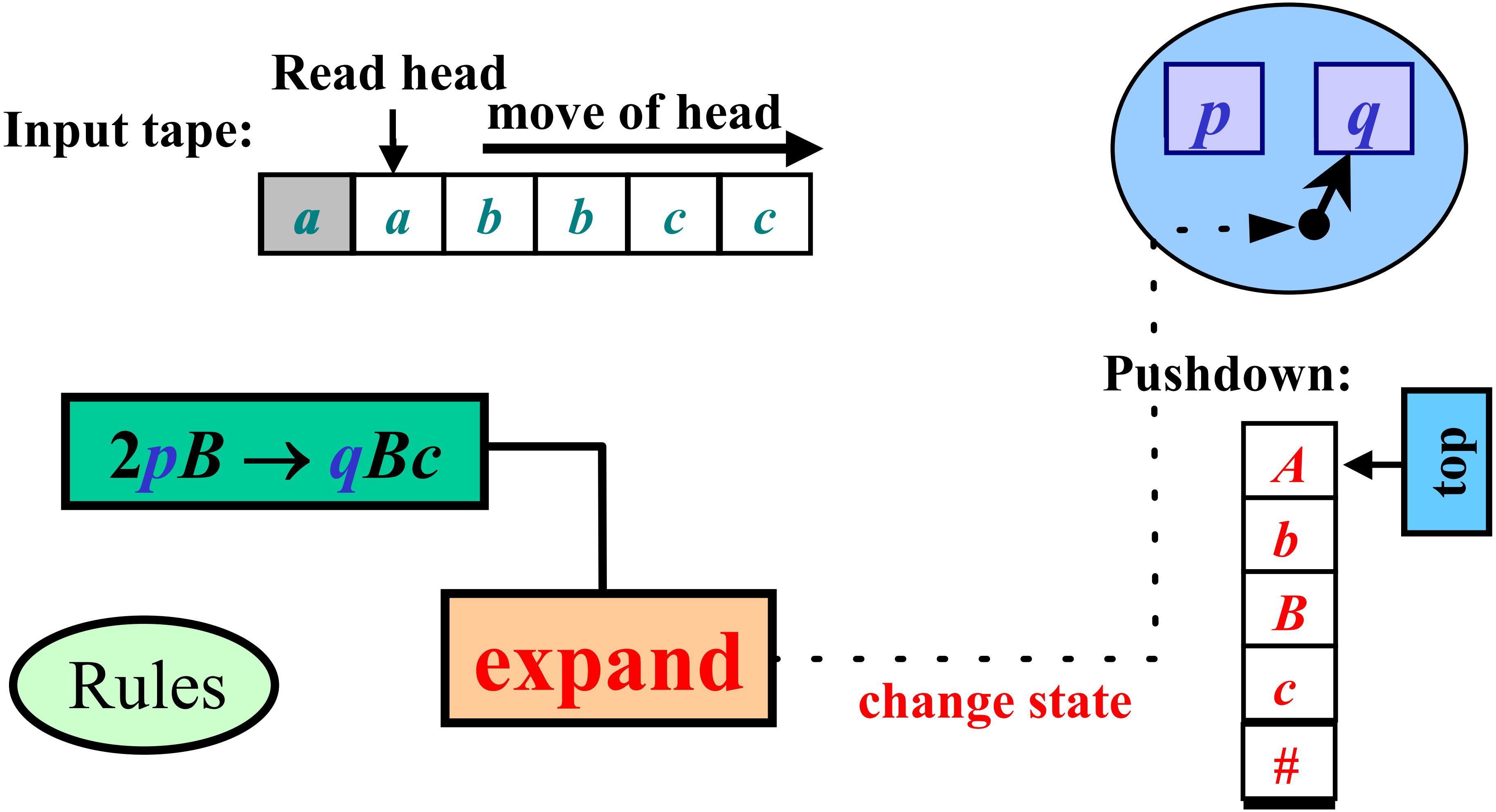
Deep Expansion: Illustration

Move: (p, $abbcc$, $AbB\#$) $_e\Rightarrow$ (q, $abbcc$, $AbBc\#$) [$2pB \rightarrow qBc$]



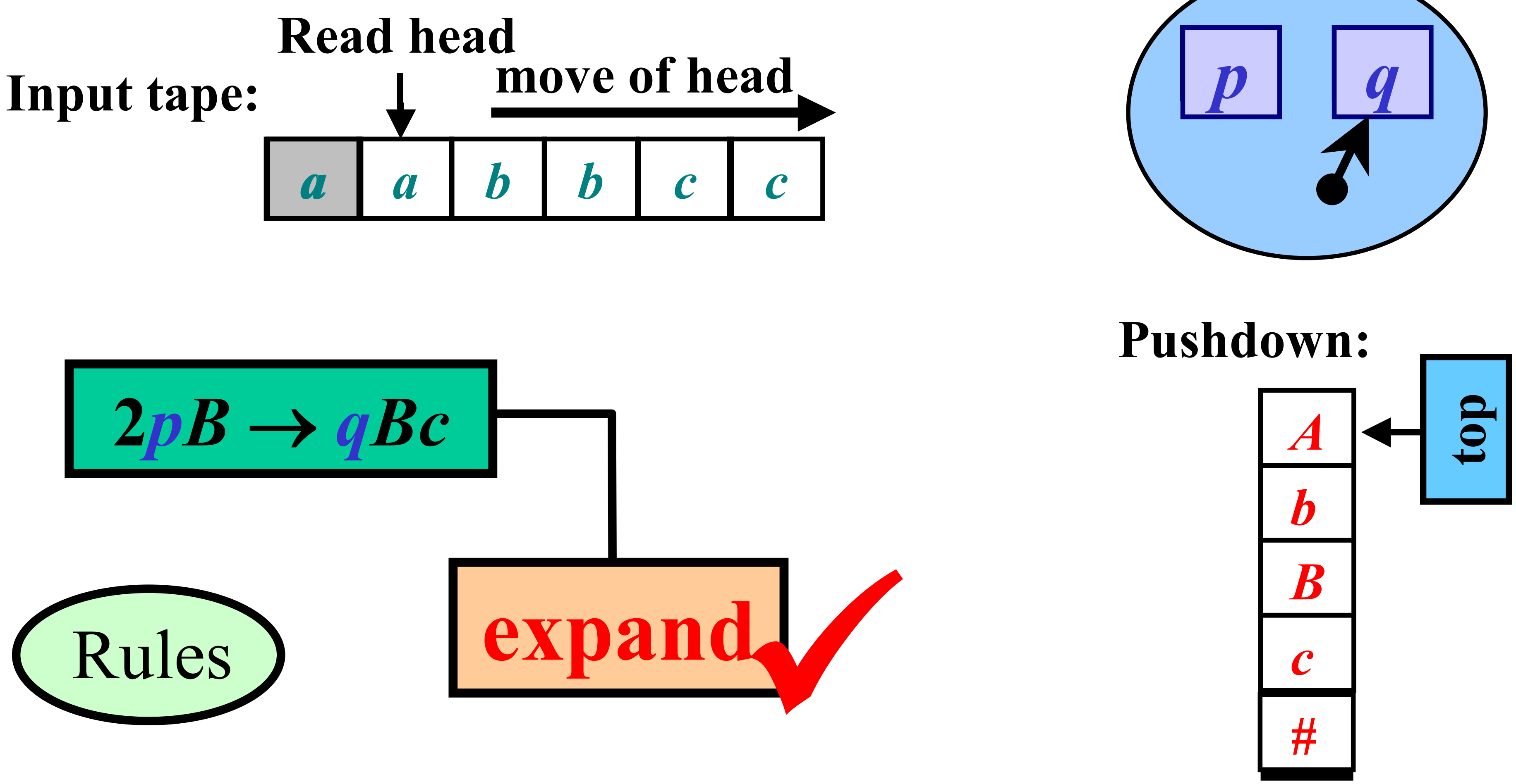
Deep Expansion: Illustration

Move: (p, $abbcc$, $AbB\#$) ${}_e\Rightarrow$ (q, $abbcc$, $AbBc\#$) [$2pB \rightarrow qBc$]



Deep Expansion: Illustration

Move: (p, *abbcc*, *AbB#*) $_e\Rightarrow$ (q, *abbcc*, *AbBc#*) [**2**$pB \rightarrow qBc$]



Example: Deep PDA

Deep PDA *M*:

[1]. $1sS \rightarrow qAB$

[2]. $1qA \rightarrow paAb$

[3]. $1qA \rightarrow fab$

[4]. $2pB \rightarrow qBc$

[5]. $1fB \rightarrow fc$

***M* accepts *aabbcc*:**

$(s, aabbcc, S\#)$

${}_e\Rightarrow (q, aabbcc, AB\#)$ [1]

${}_e\Rightarrow (p, aabbcc, aAbB\#)$ [2]

${}_p\Rightarrow (p, abbcc, AbB\#)$

${}_e\Rightarrow (q, abbcc, AbBc\#)$ [4]

${}_e\Rightarrow (f, abbcc, abbBc\#)$ [3]

${}_p\Rightarrow (f, bbcc, bbBc\#)$

${}_p\Rightarrow^2 (f, cc, Bc\#)$

${}_e\Rightarrow (f, cc, cc\#)$ [5]

${}_p\Rightarrow (f, c, c\#)$

${}_p\Rightarrow (f, \varepsilon, \#)$

$$L(M) = \{a^n b^n c^n : n \geq 1\} \in PD_2$$

Definition 1/3

A deep pushdown automaton is a 7-tuple

$$M = (Q, \Sigma, \Gamma, R, s, S, F), \text{ where}$$

- Q – *states*,
- $\Sigma \subseteq \Gamma$ – *input alphabet*,
- Γ – *pushdown alphabet*, *bottom symbol* $\# \in \Gamma - \Sigma$
- R – *finite set of rules* of the form

$$\boldsymbol{mqA \rightarrow pw} \text{ or } \boldsymbol{mq\# \rightarrow pv\#}$$

- $s \in Q$ – *start state*
- $S \in \Gamma$ – *start pushdown symbol*
- $F \subseteq Q$ – *final states*

Definition 2/3

- if an input symbol is on pd top, ***M* pops** the pd as

$$(q, au, az) \;_{p}\!\Rightarrow (q, u, z), \quad a \in \Sigma$$

- no explicit rule needed in *R*

- if a non-input symbol is on pd top, ***M* expands** the pd as

$$(q, w, uAz) \;_{e}\!\Rightarrow (p, w, uvz) \; [mqA \to pv],$$

where u contains $m - 1$ non-input symbols

Definition 3/3

- $\boldsymbol{M}$ is ***of depth n***, denotated by $_{\boldsymbol{n}}\boldsymbol{M}$, if n is the minimal positive integer such that each of M's rules is of depth n or less.

- Language accepted by $_nM$, $L(_nM)$, is defined as

$$L(_nM) = \{\boldsymbol{w} \in \Sigma^*: (\boldsymbol{s}, \boldsymbol{w}, \boldsymbol{S}\#) \Rightarrow^* (\boldsymbol{f}, \boldsymbol{\varepsilon}, \#) \text{ in } {}_nM \text{ with } \boldsymbol{f} \in F\}.$$

Main Result and its Proof

- ***PD**$_n$* – the language family defined by DeepPDAs of depth *n*.

Theorem: $PD_n \subset PD_{n+1}$, for all $n \geq 1$.

Proof (Sketch):

- State grammars (Kasai, 1970) are needed in the proof
- State grammar is a modification of CFG based on rules of the form

$$(q, A) \rightarrow (p, v)$$

Proof 1/6: State Grammar

- ***State grammar G*** = ($\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{T}$, $\boldsymbol{P}$, $\boldsymbol{S}$)
  - $\boldsymbol{V}$ – total alphabet, $\boldsymbol{W}$ – states, $\boldsymbol{T} \subseteq \boldsymbol{V}$ – terminals,
  - $\boldsymbol{P}$ – set of rules of the form $(q, A) \rightarrow (p, v)$
  - $\boldsymbol{S} \in (\boldsymbol{V} - \boldsymbol{T})$ – start symbol,

- ***Configuration*** – (q, x)
- ***Derivation step***:

$$(q, uAz) \Rightarrow (p, uvz) \; [(q, A) \rightarrow (p, v)]$$

and for every nonterminal $\boldsymbol{B}$ in u, $\boldsymbol{P}$ contains no rule with $(q, \boldsymbol{B})$ on the left-hand side

Proof 2/6: *n*-limited Step

- ***n-limited derivation step***:
  each derivation step within the first ***n*** non-terminals

$$(q, uAz)\ {}_n\Rightarrow (p, uvz) \text{ and}$$

uA has ***n*** or fewer non-terminals

- ***n-limited state language:***

$$L(G, n) = \{w \in T^*: (q, S)\ {}_n\Rightarrow^* (p, w)\}$$

- ST_n – the family of *n*-limited state languages

Proof 3/6: Example

State Grammar G:

[1]. $(1, S) \rightarrow (2, AC)$

[2]. $(2, A) \rightarrow (3, aAb)$

[3]. $(2, A) \rightarrow (4, ab)$

[4]. $(3, C) \rightarrow (2, Cc)$

[5]. $(4, C) \rightarrow (4, c)$

$W = \{1, 2, 3, 4\}$

G generates $aabbcc$:

$(S, 1) \Rightarrow (AC, 2)$ [1]

$\Rightarrow (aAbC, 3)$ [2]

$\Rightarrow (aAbCc, 2)$ [4]

$\Rightarrow (aabbCc, 4)$ [3]

$\Rightarrow (aabbcc, 4)$ [5]

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in ST_2$$

Proof 4/6: $\boldsymbol{PD_n} \subseteq \boldsymbol{ST_n}$, $n \geq 1$

- G simulates the application of $\boldsymbol{ipA} \rightarrow \boldsymbol{qy} \in R$:
  - make a left-to-right scan of the pd until the $\boldsymbol{i}$th occurence of a non-terminal
  - if $X_{\boldsymbol{i}} = \boldsymbol{A}$, then replace $\boldsymbol{A}$ with $\boldsymbol{y}$ and return to the beginning of the sentential form
  - rightmost symbol is always a special $\boldsymbol{a'}$, and G completes the simulation by changing $\boldsymbol{a'}$ to $\boldsymbol{a}$

Proof 5/6: $\boldsymbol{ST_n} \subseteq \boldsymbol{PD_n}$, $n \geq 1$

- ${}_nM$ simulates G's n-limited derivations in pd:
  - always records the first $\boldsymbol{n}$ non-terminals from current G's sentential form in its state
  - fewer than $\boldsymbol{n}$ non-terminals are extended by #s
  - reads the string, empties pd, enters $\$ \in F$

Proof 6/6: $PD_n \subset PD_{n+1}$, $n \geq 1$

1) As $PD_n \subseteq ST_n$ and $ST_n \subseteq PD_n$ for all $n \geq 1$, $ST_n = PD_n$.

2) Kasai (1970): $ST_n \subset ST_{n+1}$, for all $n \geq 1$.

For all $n \geq 1$, $PD_n = ST_n \subset ST_{n+1} = PD_{n+1}$

Q. E. D.

Note: $\boldsymbol{PD_n} \subset \boldsymbol{CS}$, $n \geq 1$

For every $n \geq 1$, there exists a context-sensitive language L not included in $\boldsymbol{PD_n}$.

Open Problem Areas

- Determinism
- Rules of form $mqA \rightarrow p\varepsilon$

Discussion