

# #-Rewriting Systems and An Infinite Hierarchy Resulting from Them

Based upon

Křivka, Z., Meduna, A., Schönecker, R.:

Generation of Languages by Rewriting Systems that  
Resemble Automata,

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- 4. Open Problem Areas**

# #-Rewriting Systems in Formal Language Theory

- Language-defining models
- Pure rewriting systems
- Between automata and grammars:  
have states but generate languages

# Concept

**#-Rewriting System** is based on the rules of the form

$$p_{\textcolor{brown}{m}} \# \rightarrow q \textcolor{blue}{x}_0 \# \textcolor{pink}{x}_1 \dots \# \textcolor{pink}{x}_n$$

by which the system makes a computational  
step  $\Rightarrow$  as

**$m^{\text{th}}$  #**  


$$(p, \dots \# y_{m-1} \textcolor{red}{\#} y_m \# y_{m+1} \dots) \Rightarrow (q, \dots \# y_{m-1} \textcolor{blue}{x}_0 \# \textcolor{pink}{x}_1 \dots \# \textcolor{pink}{x}_n y_m \# y_{m+1} \dots)$$

# Definition 1/2

**#-Rewriting System** ( $\#RS$ ) is a quadruple

$$H = (Q, \Sigma, s, R), \text{ where}$$

- $Q$ —finite set of *states*,
- $\Sigma$ —*alphabet*,  $\# \in \Sigma$  is called a *bounder*,
- $s \in Q$ —*start state*,
- $R$ —*finite set of rules* of the form

$$\textcolor{blue}{p}_{\textcolor{orange}{m}} \# \rightarrow \textcolor{blue}{q} \textcolor{pink}{x}$$

where  $\textcolor{blue}{p}, \textcolor{blue}{q} \in Q$ ,  $\textcolor{orange}{m}$  is a positive integer,  $\textcolor{pink}{x} \in \Sigma^*$ .

## Definition 2/2

*Configuration:*  $(q, x)$ ,  $q \in Q, x \in \Sigma^*$

*Computational step:*

$(p, u\#v) \Rightarrow (q, uxv)$  [ $p_m\# \rightarrow qx \in R$ ],

where the number of  $\#$ s in  $u$  is  $m - 1$ ,

$p, q \in Q, u, x, v \in \Sigma^*$ .

*Generated language:*

$L(H) = \{w \in (\Sigma - \#)^*: (s, \#) \Rightarrow^* (q, w) \text{ in } H, q \in Q\}$ .

## Example: $\#RS$

$\#RS\ H:$

[1].  $s \ 1\# \rightarrow p \ \#\#$

[2].  $p \ 1\# \rightarrow q \ a\#b$

[3].  $q \ 2\# \rightarrow p \ \#c$

[4].  $p \ 1\# \rightarrow f \ ab$

[5].  $f \ 1\# \rightarrow f \ c$

$H$  accepts  $aabbcc$ :

## Example: $\#RS$

$\#RS\ H:$

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$H$  accepts  $aabbcc$ :

$(s, \#)$

$\Rightarrow$

## Example: $\#RS$

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$H$  accepts  $aabbcc$ :

$$\begin{array}{ccc} (s, \#) & \Rightarrow & [1] \end{array}$$

## Example: $\#RS$

$\#RS\ H:$

- [1].  $s \ 1\# \rightarrow p \ ##$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
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$H$  accepts  $aabbcc$ :

$$\begin{aligned} & (s, \#) \\ \Rightarrow & (p, ##) \quad [1] \\ \Rightarrow & \end{aligned}$$

# Example: $\#RS$

$\#RS\ H:$

- [1].  $s \ 1\# \rightarrow p \ \#\#$
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$H$  accepts  $aabbcc$ :

- $(s, \#)$
- $\Rightarrow (p, \#\#)$  [1]
- $\Rightarrow$  [2]

## Example: $\#RS$

$\#RS\ H:$

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- $\Rightarrow$

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$H$  accepts  $aabbcc$ :

- $(s, \#)$
- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\underline{\#}bc\#)$  [3]
- $\Rightarrow$  [4]

# Example: $\#RS$

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- $\Rightarrow (p, \##)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
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- $\Rightarrow (f, aabbcc\#)$  [4]
- $\Rightarrow$

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$\#RS\ H:$

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- $\Rightarrow (f, aabbcc\#)$  [4]
- $\Rightarrow$  [5]

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$\#RS\ H:$

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$H$  accepts  $aabbcc$ :

- $(s, \#)$
- $\Rightarrow (p, \#\#)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#bc\#)$  [3]
- $\Rightarrow (f, aabb\#)$  [4]
- $\Rightarrow (f, aabbcc)$  [5]

$$L(H) = \{a^n b^n c^n : n \geq 1\}$$

# Finite index of $\#RS$

$\#$ -Rewriting systems of *index k*:

$\Rightarrow$  over configurations with  $k$  or fewer **#**s

$\#RS_k$  – the language family generated by

$\#RS$ s of index  $k$

**Example:** Index  $k = 2$ :

1.  $(p, a\#a\#b) \Rightarrow (q, aa\#aa\#b)$  [ $p_1\# \rightarrow qa\#a \in R$ ]

**OK**

2.  $(p, a\#a\#b) \not\Rightarrow (q, a\#aa##bb)$  [ $p_2\# \rightarrow qa##b \in R$ ]

**INCORRECT**

# Example: $\#RS$ of finite index

$\#RS H$ :

- [1].  $s \ 1\# \rightarrow p \ \#\#$
- [2].  $p \ 1\# \rightarrow q \ a\#b$
- [3].  $q \ 2\# \rightarrow p \ #c$
- [4].  $p \ 1\# \rightarrow f \ ab$
- [5].  $f \ 1\# \rightarrow f \ c$

$H$  accepts  $aabbcc$ :

- $(s, \#)$
- $\Rightarrow (p, \#\#)$  [1]
- $\Rightarrow (q, a\#b\#)$  [2]
- $\Rightarrow (p, a\#bc\#)$  [3]
- $\Rightarrow (f, aabb\#)$  [4]
- $\Rightarrow (f, aabbcc)$  [5]

$H$  is of index 2.

$$L(H) = \{a^n b^n c^n : n \geq 1\} \in \#RS_2$$

# Main Result: An Infinite Hierarchy

**Theorem:**  $\#RS_k \subset \#RS_{k+1}$ , for all  $k \geq 1$ .

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**Proof:**

makes use of programmed grammars (*PG*) of index  $k$

# Proof: Programmed Grammars

**Programmed Grammar (PG)** is a modification of context-free grammar based on the rules of the form:

$$\textcolor{orange}{r}: A \rightarrow \textcolor{magenta}{x}, W_{\textcolor{orange}{r}}$$

- $\textcolor{orange}{r}: A \rightarrow \textcolor{magenta}{x}$  is a context-free rule labeled by  $\textcolor{orange}{r}$ ,
- $W_{\textcolor{orange}{r}}$ —finite set of rule labels

**Derivation step ( $\Rightarrow$ ):**

after the application of rule  $\textcolor{orange}{r}$ ,  
a rule from  $W_{\textcolor{orange}{r}}$  has to be applied

## Proof: Finite index of $PG$

Programmed grammars of *index k*:

- $\Rightarrow$  over sentential forms with  $k$  or fewer occurrences of nonterminals.

$P_k$  – the language family defined by programmed grammars of index  $k$

## Example: $PG$

$PG\ G:$

- 1:  $S \rightarrow ABC, \{2, 5\}$
- 2:  $A \rightarrow aA, \{3\}$
- 3:  $B \rightarrow bB, \{4\}$
- 4:  $C \rightarrow cC, \{2, 5\}$
- 5:  $A \rightarrow a, \{6\}$
- 6:  $B \rightarrow b, \{7\}$
- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

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3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$S$

$\Rightarrow$

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

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5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

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$G$  generates  $aabbcc$ :

$S$   
 $\Rightarrow$   
[1]

## Example: $PG$

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$G$  generates  $aabbcc$ :

$$\begin{aligned} & S \\ & \Rightarrow ABC \quad [1] \\ & \Rightarrow \end{aligned}$$

## Example: $PG$

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7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{array}{rcl} S & & \\ \Rightarrow ABC & [1] & \\ \Rightarrow & [2] & \\ \Rightarrow & & \end{array}$$

## Example: $PG$

$PG\ G:$

1:  $S \rightarrow ABC, \{2, 5\}$

2:  $A \rightarrow aA, \{3\}$

3:  $B \rightarrow bB, \{4\}$

4:  $C \rightarrow cC, \{2, 5\}$

5:  $A \rightarrow a, \{6\}$

6:  $B \rightarrow b, \{7\}$

7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & S \\ & \Rightarrow ABC & [1] \\ & \Rightarrow aA\textcolor{orange}{BC} & [2] \\ & \Rightarrow \end{aligned}$$

## Example: $PG$

$PG\ G:$

$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

$$2: \textcolor{red}{A} \rightarrow aA, \{\textcolor{blue}{3}\}$$

$$3: \textcolor{red}{B} \rightarrow bB, \{\textcolor{orange}{4}\}$$

$$4: \textcolor{red}{C} \rightarrow cC, \{\textcolor{blue}{2}, 5\}$$

$$5: \textcolor{red}{A} \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{blue}{7}\}$$

$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{red}{BC} & [2] \\ & \Rightarrow & [3] \end{aligned}$$

## Example: $PG$

$PG\ G:$

$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

$$2: \textcolor{red}{A} \rightarrow aA, \{\textcolor{blue}{3}\}$$

$$3: \textcolor{red}{B} \rightarrow bB, \{\textcolor{blue}{4}\}$$

$$4: \textcolor{red}{C} \rightarrow cC, \{\textcolor{blue}{2}, 5\}$$

$$5: \textcolor{red}{A} \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{blue}{7}\}$$

$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{red}{B}C & [2] \\ & \Rightarrow aAbB\textcolor{orange}{C} & [3] \\ & \Rightarrow \end{aligned}$$

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$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

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$$3: \textcolor{red}{B} \rightarrow bB, \{\textcolor{blue}{4}\}$$

$$4: \textcolor{red}{C} \rightarrow cC, \{\textcolor{brown}{2}, \textcolor{blue}{5}\}$$

$$5: A \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{blue}{7}\}$$

$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{red}{B}C & [2] \\ & \Rightarrow aAbB\textcolor{red}{C} & [3] \\ & \Rightarrow & [4] \end{aligned}$$

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$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\textcolor{red}{B}C & [2] \\
 \Rightarrow & aAb\textcolor{red}{B}\textcolor{red}{C} & [3] \\
 \Rightarrow & a\textcolor{brown}{A}bBcC & [4] \\
 \Rightarrow &
 \end{aligned}$$

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## Example: $PG$

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$$1: \textcolor{red}{S} \rightarrow ABC, \{\textcolor{blue}{2}, 5\}$$

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$$4: \textcolor{red}{C} \rightarrow cC, \{\textcolor{blue}{2}, 5\}$$

$$5: \textcolor{red}{A} \rightarrow a, \{\textcolor{blue}{6}\}$$

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$$7: \textcolor{red}{C} \rightarrow c, \emptyset$$

$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{red}{B}C & [2] \end{aligned}$$

$$\Rightarrow aAb\textcolor{red}{B}C & [3]$$

$$\Rightarrow aAbBcC & [4]$$

$$\Rightarrow aab\textcolor{brown}{B}cC & [5]$$

$$\Rightarrow$$

## Example: $PG$

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$$5: \textcolor{red}{A} \rightarrow a, \{\textcolor{blue}{6}\}$$

$$6: \textcolor{red}{B} \rightarrow b, \{\textcolor{orange}{7}\}$$

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$G$  generates  $aabbcc$ :

$$\begin{aligned} & \textcolor{red}{S} \\ & \Rightarrow \textcolor{red}{ABC} & [1] \\ & \Rightarrow aA\textcolor{red}{B}C & [2] \\ & \Rightarrow aAb\textcolor{red}{B}C & [3] \\ & \Rightarrow aAbBcC & [4] \\ & \Rightarrow aab\textcolor{red}{B}cC & [5] \\ & \Rightarrow & [6] \end{aligned}$$

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$G$  generates  $aabbcc$ :

$$\begin{aligned}
 & S \\
 \Rightarrow & ABC & [1] \\
 \Rightarrow & aA\textcolor{red}{B}C & [2] \\
 \Rightarrow & aAb\textcolor{black}{B}\textcolor{red}{C} & [3] \\
 \Rightarrow & aAbBcC & [4] \\
 \Rightarrow & aab\textcolor{red}{B}cC & [5] \\
 \Rightarrow & aabb\textcolor{orange}{c}\textcolor{black}{C} & [6] \\
 \Rightarrow &
 \end{aligned}$$

## Example: $PG$

$PG\ G:$

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- 2:  $A \rightarrow aA, \{3\}$
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$G$  generates  $aabbcc$ :

- $S$
- $\Rightarrow ABC [1]$
- $\Rightarrow aA$
- $\Rightarrow aAbBC [3]$
- $\Rightarrow aAbBcC [4]$
- $\Rightarrow aabBcC [5]$
- $\Rightarrow aabbC [6]$
- $\Rightarrow aabbcc [7]$

# Example: $PG$

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- 7:  $C \rightarrow c, \emptyset$

$G$  generates  $aabbcc$ :

- $S$
- $\Rightarrow ABC [1]$
- $\Rightarrow aA$
- $\Rightarrow aAbBC [3]$
- $\Rightarrow aAbBcC [4]$
- $\Rightarrow aabBcC [5]$
- $\Rightarrow aabbC [6]$
- $\Rightarrow aabbcc [7]$

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in P_3$$

Proof:  $P_k = \#RS_k$ ,  $k \geq 1$

$P_k \subseteq \#RS_k$ :

Let  $G$  be a  $PG$  of index  $k$ . Construct a  $\#RS H$  of index  $k$ , so  $H$  simulates derivation step

$$a\underline{A}bBc \Rightarrow_G a\textcolor{magenta}{dXY}bBc \quad [p: A \rightarrow \textcolor{magenta}{dXY}, \{q, o\}] \Rightarrow_G \dots [\textcolor{orange}{q}]$$

as

$$\begin{aligned} (\langle \underline{A}B, \textcolor{blue}{p} \rangle, a\#b\#c) &\Rightarrow_H (\langle XYB, \textcolor{blue}{q} \rangle, a\textcolor{magenta}{d}\#\#b\#c) \\ [\langle \underline{A}B, p \rangle \textcolor{red}{1}^\# \rightarrow \langle XYB, q \rangle \textcolor{magenta}{d}\#\#] \end{aligned}$$

# Proof: $\#RS_k = P_k$ , $k \geq 1$

$\#RS_k \subseteq P_k$ :

Let  $H$  be a  $\#RS$  of index  $k$ . Construct a  $PG$   $G$  of index  $k$ , so  $G$  simulates a computational step

$$(\mathbf{p}, a\underline{\#} b \# c) \Rightarrow_H (\mathbf{q}, a\underline{a} \# \underline{b} b \# c) [\mathbf{p}_1 \# \rightarrow \mathbf{q} \ a \# b]$$

as

$$a\langle \mathbf{p}, 1, \underline{2} \rangle b\langle \mathbf{p}, 2, \underline{2} \rangle c$$

- 1) Renumbering:  $\Rightarrow_G a\langle \mathbf{q}', 1, \underline{2} \rangle b\langle \mathbf{p}, 2, \underline{2} \rangle c$   
 $\Rightarrow_G a\langle \mathbf{q}', 1, \underline{2} \rangle b\langle \mathbf{q}', 2, \underline{2} \rangle c$
- 2) Rewriting:  $\Rightarrow_G a\underline{a}\langle \mathbf{q}', 1, \underline{2} \rangle \underline{b} b\langle \mathbf{q}', 2, \underline{2} \rangle c$
- 3) Finalization:  $\Rightarrow_G aa\langle \mathbf{q}, 1, \underline{2} \rangle bb\langle \mathbf{q}', 2, \underline{2} \rangle c$   
 $\Rightarrow_G aa\langle \mathbf{q}, 1, \underline{2} \rangle bb\langle \mathbf{q}, 2, \underline{2} \rangle c$

Proof:  $\#RS_k \subset \#RS_{k+1}$ ,  $k \geq 1$

Recall that:

- $P_k \subset P_{k+1}$ , for all  $k \geq 1$
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As  $P_k = \#RS_k$ , for all  $k \geq 1$ , we have

Theorem:  $\#RS_k \subset \#RS_{k+1}$ , for all  $k \geq 1$ .

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# Future Investigation

- Determinism
- Unlimited index
- Other variants:
  - Right-linear
  - Context-sensitive
  - Parallel

# Discussion