#-Rewriting Systems
and An Infinite Hierarchy Resulting from Them

Based upon

Křivka, Z., Meduna, A., Schönecker, R.:
Generation of Languages by Rewriting Systems that Resemble Automata,
In: *IJFCS* Vol. 17, No. 5, 2006

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Contents

#-Rewriting Systems in Formal Language Theory

- Language-defining models
- Pure rewriting systems
- Between automata and grammars:
  have states but generate languages

Concept

#-Rewriting System is based on the rules of the form

$$p_m\# \to q\ x_0\#x_1 \ldots \#x_n$$

by which the system makes a computational step ⇒ as

m**th** #

↓

$$(p, \ldots\#y_{m\text{-}1}\#y_m\#y_{m+1}\cdots) \Rightarrow$$

$$(q, \ldots\#\,y_{m\text{-}1}x_0\#x_1\ldots\#x_ny_m\#\,y_{m+1}\cdots)$$

Definition 1/2

#-Rewriting System (#***RS***) is a quadruple

$$H = (Q, \Sigma, s, R), \text{ where}$$

- Q—finite set of *states*,
- Σ—*alphabet*, # $\in \Sigma$ is called a *bounder*,
- $s \in Q$—*start state*,
- R—*finite set of rules* of the form

$$\boldsymbol{p}\,{}_{\boldsymbol{m}}\# \rightarrow \boldsymbol{q}\boldsymbol{x}$$

where $\boldsymbol{p}$, $\boldsymbol{q} \in Q$, $\boldsymbol{m}$ is a positive integer, $\boldsymbol{x} \in \Sigma^*$.

Definition 2/2

Configuration: (q, x), $q \in Q$, $x \in \Sigma^*$

Computational step:

$$(p, u\#v) \Rightarrow (q, uxv)\ [p_m\# \rightarrow qx \in R],$$

where the number of #s in u is $m - 1$,

$$p, q \in \mathrm{Q}, u, x, v \in \Sigma^*.$$

Generated language:

$$L(H) = \{w \in (\Sigma - \#)^*: (s, \#) \Rightarrow^* (q, w) \text{ in } H, q \in Q\}.$$

Example: #RS

#RS H:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

H accepts $aabbcc$:

Example: #*RS*

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$

$\Rightarrow$

Example: #*RS*

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \underline{\#})$

$\Rightarrow$ [1]

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \underline{\#}\#)$	[1]
$\Rightarrow$	[2]

Example: #*RS*

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow (q, a\#b\#)$ [2]

$\Rightarrow$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\underline{\#})$	[2]
$\Rightarrow$	[3]

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow (q, a\#b\#)$ [2]

$\Rightarrow (p, a\#bc\#)$ [3]

$\Rightarrow$

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow (q, a\#b\#)$ [2]

$\Rightarrow (p, a\underline{\#}bc\#)$ [3]

$\Rightarrow$ [4]

Example: #*RS*

#*RS H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\#)$	[2]
$\Rightarrow (p, a\#bc\#)$	[3]
$\Rightarrow (f, aabbc\#)$	[4]
$\Rightarrow$	

Example: #*RS*

#*RS H*:

[1]. $s\ _1\# \rightarrow p\ \#\#$

[2]. $p\ _1\# \rightarrow q\ a\#b$

[3]. $q\ _2\# \rightarrow p\ \#c$

[4]. $p\ _1\# \rightarrow f\ ab$

[5]. $f\ _1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\#)$	[2]
$\Rightarrow (p, a\#bc\#)$	[3]
$\Rightarrow (f, aabbc\underline{\#})$	[4]
$\Rightarrow$	[5]

Example: #*RS*

#*RS* *H*:

[1]. $s\ _1\# \rightarrow p\ \#\#$

[2]. $p\ _1\# \rightarrow q\ a\#b$

[3]. $q\ _2\# \rightarrow p\ \#c$

[4]. $p\ _1\# \rightarrow f\ ab$

[5]. $f\ _1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$

$\Rightarrow (p, \#\#)$ [1]

$\Rightarrow (q, a\#b\#)$ [2]

$\Rightarrow (p, a\#bc\#)$ [3]

$\Rightarrow (f, aabbc\#)$ [4]

$\Rightarrow (f, aabbcc)$ [5]

Example: #*RS*

#*RS* *H*:

[1]. $s\ {}_1\# \rightarrow p\ \#\#$

[2]. $p\ {}_1\# \rightarrow q\ a\#b$

[3]. $q\ {}_2\# \rightarrow p\ \#c$

[4]. $p\ {}_1\# \rightarrow f\ ab$

[5]. $f\ {}_1\# \rightarrow f\ c$

***H* accepts *aabbcc*:**

$(s, \#)$	
$\Rightarrow (p, \#\#)$	[1]
$\Rightarrow (q, a\#b\#)$	[2]
$\Rightarrow (p, a\#bc\#)$	[3]
$\Rightarrow (f, aabbc\#)$	[4]
$\Rightarrow (f, aabbcc)$	[5]

$$L(H) = \{a^n b^n c^n : n \geq 1\}$$

Finite index of #*RS*

#-Rewriting systems of ***index k***:

$\Rightarrow$ over configurations with ***k*** or fewer #s

$\#\boldsymbol{RS}_k$ – the language family generated by #*RS*s of index *k*

Example: Index $k = 2$:

1. $(p, a\#a\#b) \Rightarrow (q, aa\#aa\#b)$ $[p_1\# \rightarrow qa\#a \in R]$

OK

2. $(p, a\#a\#b) \not\Rightarrow (q, a\#aa\#\#bb)$ $[p_2\# \rightarrow qa\#\#b \in R]$

INCORRECT

Example: #*RS* of finite index

#*RS H*:

[1]. $s\ {}_1\# \to p\ \#\#$

[2]. $p\ {}_1\# \to q\ a\#b$

[3]. $q\ {}_2\# \to p\ \#c$

[4]. $p\ {}_1\# \to f\ ab$

[5]. $f\ {}_1\# \to f\ c$

***H* accepts *aabbcc*:**

	$(s, \#)$	
$\Rightarrow$	$(p, \#\#)$	[1]
$\Rightarrow$	$(q, a\#b\#)$	[2]
$\Rightarrow$	$(p, a\#bc\#)$	[3]
$\Rightarrow$	$(f, aabbc\#)$	[4]
$\Rightarrow$	$(f, aabbcc)$	[5]

***H* is of index 2.**

$$L(H) = \{a^n b^n c^n : n \geq 1\} \in \#RS_2$$

Main Result: An Infinite Hierarchy

Theorem: $\#RS_k \subset \#RS_{k+1}$, for all $k \geq 1$.

Proof:

makes use of programmed grammars (*PG*) of index k

Proof: Programmed Grammars

Programmed Grammar (***PG***) is a modification of context-free grammar based on the rules of the form:

$$r\text{: } A \rightarrow x,\ W_r$$

- r: $A \rightarrow x$ is a context-free rule labeled by r,
- W_r—finite set of rule labels

Derivation step ($\Rightarrow$):

after the application of rule r,
a rule from W_r has to be applied

Proof: Finite index of *PG*

Programmed grammars of ***index k***:

- $\Rightarrow$ over sentential forms with ***k*** or fewer occurrences of nonterminals.

$\boldsymbol{P_k}$ – the language family defined by programmed grammars of index k

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow$ [1]

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow$ [2]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

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***G* generates *aabbcc*:**

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$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

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Example: *PG*

***PG G*:**

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6: $B \rightarrow b, \{7\}$

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***G* generates *aabbcc*:**

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$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow$

Example: *PG*

***PG G*:**

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5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow$ [4]

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

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5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

S	
$\Rightarrow ABC$	[1]
$\Rightarrow aABC$	[2]
$\Rightarrow aAbBC$	[3]
$\Rightarrow aAbBcC$	[4]
$\Rightarrow$	[5]

Example: *PG*

***PG G*:**

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***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow aabBcC$ [5]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

2: $A \rightarrow aA, \{3\}$

3: $B \rightarrow bB, \{4\}$

4: $C \rightarrow cC, \{2, 5\}$

5: $A \rightarrow a, \{6\}$

6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

$$
\begin{aligned}
& S \\
\Rightarrow\ & ABC && [1] \\
\Rightarrow\ & aABC && [2] \\
\Rightarrow\ & aAbBC && [3] \\
\Rightarrow\ & aAbBcC && [4] \\
\Rightarrow\ & aabBcC && [5] \\
\Rightarrow\ & && [6]
\end{aligned}
$$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

4: $C \rightarrow cC$, {2, 5}

5: $A \rightarrow a$, {6}

6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\varnothing$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow aabBcC$ [5]

$\Rightarrow aabbcC$ [6]

$\Rightarrow$

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC, \{2, 5\}$

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3: $B \rightarrow bB, \{4\}$

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6: $B \rightarrow b, \{7\}$

7: $C \rightarrow c, \emptyset$

***G* generates *aabbcc*:**

S

$\Rightarrow ABC$ [1]

$\Rightarrow aABC$ [2]

$\Rightarrow aAbBC$ [3]

$\Rightarrow aAbBcC$ [4]

$\Rightarrow aabBcC$ [5]

$\Rightarrow aabbcC$ [6]

$\Rightarrow aabbcc$ [7]

Example: *PG*

***PG G*:**

1: $S \rightarrow ABC$, {2, 5}

2: $A \rightarrow aA$, {3}

3: $B \rightarrow bB$, {4}

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6: $B \rightarrow b$, {7}

7: $C \rightarrow c$, $\emptyset$

***G* generates *aabbcc*:**

S	
$\Rightarrow ABC$	[1]
$\Rightarrow aABC$	[2]
$\Rightarrow aAbBC$	[3]
$\Rightarrow aAbBcC$	[4]
$\Rightarrow aabBcC$	[5]
$\Rightarrow aabbcC$	[6]
$\Rightarrow aabbcc$	[7]

$$L(G) = \{a^n b^n c^n: n \geq 1\} \in P_3$$

Proof: $\boldsymbol{P_k = \#RS_k}$, $k \geq 1$

$\boldsymbol{P_k \subseteq \#RS_k}$:

Let G be a PG of index k. Construct a $\#RS$ H of index k, so H simulates derivation step

$$\boldsymbol{a\underline{A}bBc} \Rightarrow_G \boldsymbol{adXYbBc}\ [p\text{: } \boldsymbol{A} \rightarrow \boldsymbol{dXY}, \{\boldsymbol{q}, \boldsymbol{o}\}] \Rightarrow_G \ldots [\boldsymbol{q}]$$

as

$$(\langle \boldsymbol{\underline{A}B}, \boldsymbol{p}\rangle, \boldsymbol{a\underline{\#}b\#c}) \Rightarrow_H (\langle \boldsymbol{XYB}, \boldsymbol{q}\rangle, \boldsymbol{ad\#\#b\#c})$$

$$[\langle \boldsymbol{\underline{A}B}, \boldsymbol{p}\rangle\ {}_1\# \rightarrow \langle \boldsymbol{XYB}, \boldsymbol{q}\rangle\ \boldsymbol{d\#\#}]$$

Proof: $\#RS_k = P_k$, $k \geq 1$

$\#RS_k \subseteq P_k$:

Let H be a $\#RS$ of index k. Construct a PG G of index k, so G simulates a computational step

$$(p, a\#b\#c) \Rightarrow_H (q, aa\#bb\#c)\ [p_1\# \rightarrow q\ a\#b]$$

as

$$a\langle p, 1, 2\rangle b\langle p, 2, 2\rangle c$$

1) Renumbering: $\Rightarrow_G a\langle q'', 1, 2\rangle b\langle p, 2, 2\rangle c$

$\Rightarrow_G a\langle q'', 1, 2\rangle b\langle q', 2, 2\rangle c$

2) Rewriting: $\Rightarrow_G aa\langle q', 1, 2\rangle bb\langle q', 2, 2\rangle c$

3) Finalization: $\Rightarrow_G aa\langle q, 1, 2\rangle bb\langle q', 2, 2\rangle c$

$\Rightarrow_G aa\langle q, 1, 2\rangle bb\langle q, 2, 2\rangle c$

Proof: $\#\boldsymbol{RS}_k \subset \#\boldsymbol{RS}_{k+1}$, $k \geq 1$

Recall that:

- $\boldsymbol{P}_k \subset \boldsymbol{P}_{k+1}$, for all $k \geq 1$

As $\boldsymbol{P}_k = \#\boldsymbol{RS}_k$, for all $k \geq 1$, we have

Theorem: $\#\boldsymbol{RS}_k \subset \#\boldsymbol{RS}_{k+1}$, for all $k \geq 1$.

Future Investigation

- Determinism
- Unlimited index
- Other variants:
  - Right-linear
  - Context-sensitive
  - Parallel

Discussion