Context-Free Languages and Their Models

Context-Free Grammar (CFG)

Gist: A *grammar* is based on a finite set of grammatical rules, by which it generates strings of its language.

Illustration:

Start nonterminal

Grammar G:

Nonterminals: A, B, S

Terminals: a, b, c, d

Rules:
$S \rightarrow AB$,
$A \rightarrow aAb$,
$A \rightarrow ab$,
$B \rightarrow bBa$,
$B \rightarrow ba$

S

Rule: $S \rightarrow AB$

AB

Rule: $A \rightarrow ab$

abB

Rule: $B \rightarrow bBa$

$abbBa$

Rule: $B \rightarrow ba$

$abbbaa \in L(G)$

Context-Free Grammar: Definition

Definition: *A context-free grammar* (CFG) is a quadruple $G = (N, T, P, S)$, where

- N is an alphabet of *nonterminals*
- T is an alphabet of *terminals*, $N \cap T = \emptyset$
- P is a finite set of *rules* of the form $A \to x$, where $A \in N$, $x \in (N \cup T)^*$
- $S \in N$ is the *start nonterminal*

Mathematical Note on Rules:

- Strictly mathematically, P is a relation from N to $(N \cup T)^*$
- Instead of $(A, x) \in P$, we write $A \to x \in P$

- $A \to x$ means that A can be replaced with x
- $A \to \varepsilon$ is called ***ε-rule***

Convention

- $A, \ldots, F, S$: nonterminals
- S : the start nonterminal
- $a, \ldots, d$: terminals
- $U, \ldots, Z$: members of $(N \cup T)$
- $u, \ldots, z$: members of $(N \cup T)^*$
- π : sequence of productions

A subset of rules of the form:

$$A \rightarrow x_1, A \rightarrow x_2, \ldots, A \rightarrow x_n$$

can be simply written as:

$$A \rightarrow x_1 \mid x_2 \mid \ldots \mid x_n$$

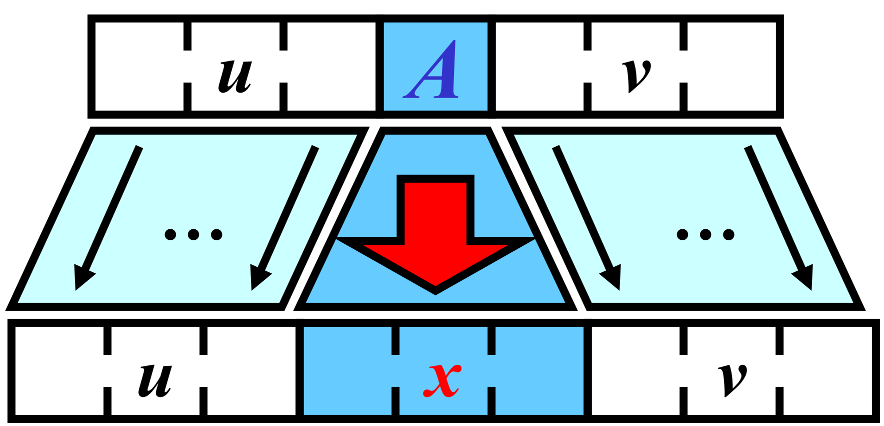
Derivation Step

Gist: A change of a string by a rule.

Definition: Let $G = (N, T, P, S)$ be a CFG. Let $\boldsymbol{u}, \boldsymbol{v} \in (N \cup T)^*$ and $p = A \rightarrow \boldsymbol{x} \in P$. Then, $\boldsymbol{u}A\boldsymbol{v}$ *directly derives* $\boldsymbol{uxv}$ *according to* p in G, written as $\boldsymbol{u}A\boldsymbol{v} \Rightarrow \boldsymbol{uxv}$ $[p]$ or, simply, $\boldsymbol{u}A\boldsymbol{v} \Rightarrow \boldsymbol{uxv}$.

Note: If $\boldsymbol{u}A\boldsymbol{v} \Rightarrow \boldsymbol{uxv}$ in G, we also say that G makes a *derivation step* from $\boldsymbol{u}A\boldsymbol{v}$ to $\boldsymbol{uxv}$.

Rule: $A \rightarrow \boldsymbol{x}$



Sequence of Derivation Steps 1/2

Gist: Several consecutive derivation steps.

Definition: Let $u \in (N \cup T)^*$. G makes a *zero-step derivation* from u to u; in symbols,

$$u \Rightarrow^0 u\ [\varepsilon] \text{ or, simply, } u \Rightarrow^0 u$$

Definition: Let $u_0,\ldots,u_n \in (N \cup T)^*$, $n \geq 1$, and $u_{i\text{-}1} \Rightarrow u_i\ [p_i]$, $p_i \in P$, for all $i = 1,\ldots, n$; that is

$$u_0 \Rightarrow u_1\ [p_1] \Rightarrow u_2\ [p_2] \ldots \Rightarrow u_n\ [p_n]$$

Then, G makes *n derivation steps* from u_0 to u_n,

$$u_0 \Rightarrow^n u_n\ [p_1 \ldots p_n] \text{ or, simply, } u_0 \Rightarrow^n u_n$$

Sequence of Derivation Steps 2/2

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 1$, then u_0 *properly derives* u_n in G, written as $u_0 \Rightarrow^+ u_n$ [π].

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 0$, then u_0 *derives* u_n in G, written as $u_0 \Rightarrow^* u_n$ [π].

Example: Consider

aAb ⇒ ***aaBbb*** [**1**: ***A*** → ***aBb***], and

aaBbb ⇒ ***aacbb*** [**2**: ***B*** → ***c***].

Then, ***aAb*** $\Rightarrow^2$ ***aacbb*** [**1 2**],

aAb $\Rightarrow^+$ ***aacbb*** [**1 2**],

aAb $\Rightarrow^*$ ***aacbb*** [**1 2**]

Generated Language

Gist: *G generates* a terminal string *w* by a sequence of derivation steps from *S* to *w*

Definition: Let $G = (N, T, P, S)$ be a CFG. The *language generated by* G, $L(G)$, is defined as

$$L(G) = \{w: w \in T^*, S \Rightarrow^* w\}$$

Illustration:

$G = (N, T, P, S)$, let $w = a_1a_2\ldots a_n$; $a_i \in T$ for $i = 1..n$

if $S \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \underbrace{a_1a_2\ldots a_n}_{w}$ **then** $w \in L(G)$;

otherwise, $w \notin L(G)$

Context-Free Language (CFL)

Gist: A language generated by a CFG.

Definition: Let L be a language. L is a *context-free language* (CFL) if there exists a context-free grammar that generates L.

Example:

$G = (N, T, P, S)$, where $N = \{S\}$, $T = \{a, b\}$,

$P = \{1: S \to aSb,\ 2: S \to \varepsilon\}$

$S \Rightarrow \varepsilon$ [2]

$S \Rightarrow aSb$ [1] $\Rightarrow ab$ [2]

$S \Rightarrow aSb$ [1] $\Rightarrow aaSbb$ [1] $\Rightarrow aabb$ [2]

⋮

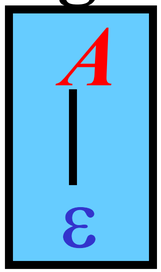
$L(G) = \{a^n b^n : n \geq 0\}$

$L = \{a^n b^n : n \geq 0\}$ is a CFL.

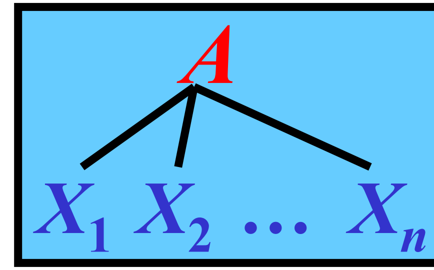
Rule Tree

- **Rule tree graphically represents a rule**

1) $A \to \varepsilon$:

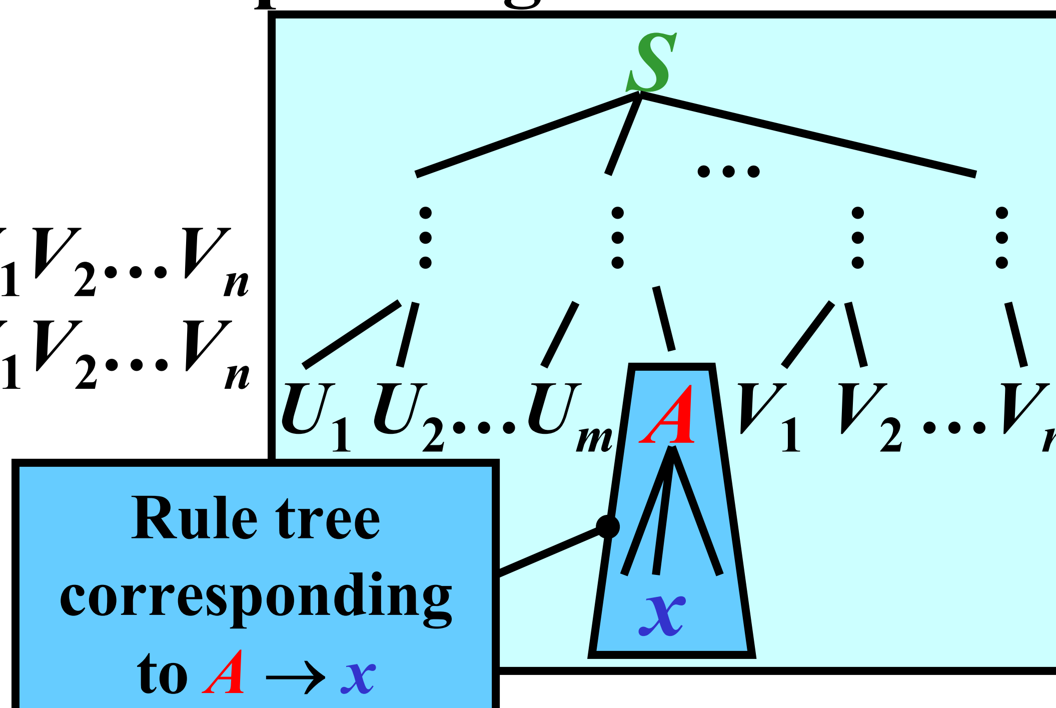


2) $A \to X_1X_2 \ldots X_n$:



- **Derivation tree corresponding to a derivation**

$$S \Rightarrow \ldots$$
$$\vdots$$
$$\Rightarrow U_1U_2 \ldots U_m A V_1V_2 \ldots V_n$$
$$\Rightarrow U_1U_2 \ldots U_m x V_1V_2 \ldots V_n$$



Derivation Tree: Example

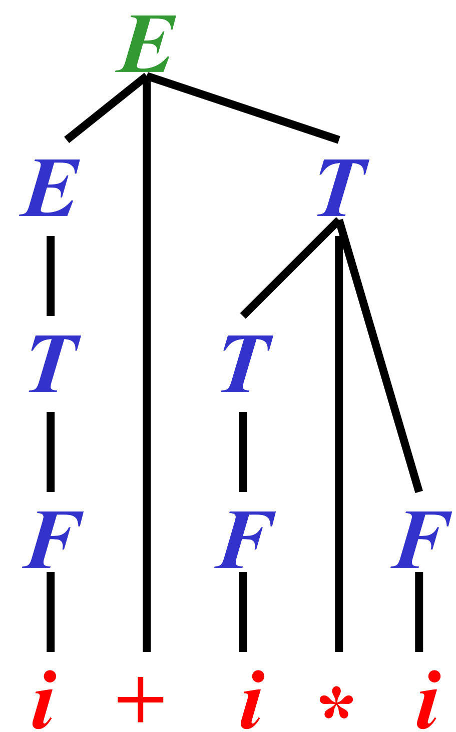
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]
$\Rightarrow E + \underline{T} * F$ [3]
$\Rightarrow E + \underline{F} * F$ [4]
$\Rightarrow \underline{E} + i * F$ [6]
$\Rightarrow T + i * \underline{F}$ [2]
$\Rightarrow \underline{T} + i * i$ [6]
$\Rightarrow \underline{F} + i * i$ [4]
$\Rightarrow i + i * i$ [6]

Derivation tree:



Leftmost Derivation

Gist: During a *leftmost derivation step*, the leftmost nonterminal is rewritten.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in T^*$, $\boldsymbol{v} \in (N \cup T)^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *leftmost way* according to $\boldsymbol{p}$ in G, written as

$$\boldsymbol{uAv} \Rightarrow_{lm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

Note: We define $\Rightarrow_{lm}^{+}$ and $\Rightarrow_{lm}^{*}$ by analogy with $\Rightarrow^{+}$ and $\Rightarrow^{*}$, respectively.

Leftmost Derivation: Example

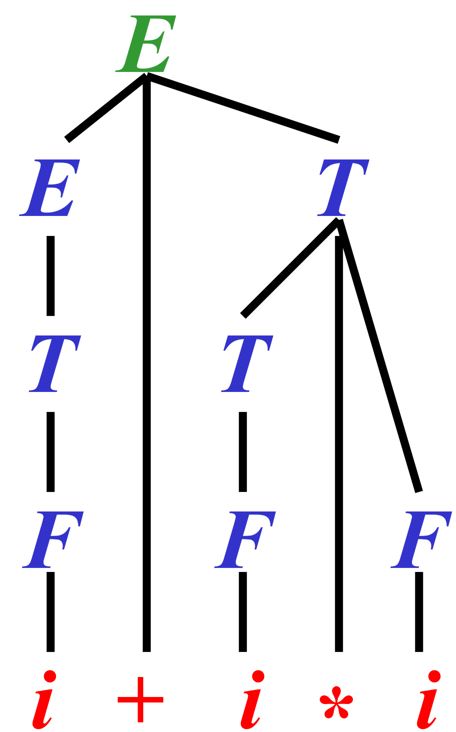
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$, 4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{lm} \underline{E} + T & [1] \\
&\Rightarrow_{lm} \underline{T} + T & [2] \\
&\Rightarrow_{lm} \underline{F} + T & [4] \\
&\Rightarrow_{lm} i + \underline{T} & [6] \\
&\Rightarrow_{lm} i + \underline{T} * F & [3] \\
&\Rightarrow_{lm} i + \underline{F} * F & [4] \\
&\Rightarrow_{lm} i + i * \underline{F} & [6] \\
&\Rightarrow_{lm} i + i * i & [6]
\end{aligned}
$$

Derivation tree:



Rightmost Derivation

Gist: During a *rightmost derivation step*, the rightmost nonterminal is rewritten.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in (N \cup T)^*$, $\boldsymbol{v} \in T^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *rightmost way* according to $\boldsymbol{p}$ in G, written as

$$\boldsymbol{uAv} \Rightarrow_{rm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

Note: We define $\Rightarrow_{rm}^{+}$ and $\Rightarrow_{rm}^{*}$ by analogy with $\Rightarrow^{+}$ and $\Rightarrow^{*}$, respectively.

Rightmost Derivation: Example

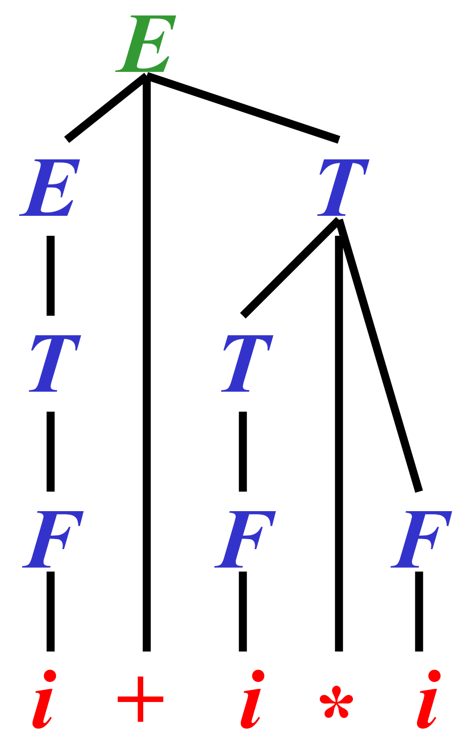
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$$\begin{aligned}
\underline{E} &\Rightarrow_{rm} E + \underline{T} & [1] \\
&\Rightarrow_{rm} E + T * \underline{F} & [3] \\
&\Rightarrow_{rm} E + \underline{T} * i & [6] \\
&\Rightarrow_{rm} E + \underline{F} * i & [4] \\
&\Rightarrow_{rm} \underline{E} + i * i & [6] \\
&\Rightarrow_{rm} \underline{T} + i * i & [2] \\
&\Rightarrow_{rm} \underline{F} + i * i & [4] \\
&\Rightarrow_{rm} i + i * i & [6]
\end{aligned}$$

Derivation tree:



Derivations: Summary

- Let $A \rightarrow x \in P$ be a rule.

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1) Derivation:

Let $u, v \in (N \cup T)^*$: $uAv \Rightarrow uxv$

Note: Any nonterminal is rewritten

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2) Leftmost derivation:

Let $u \in T^*, v \in (N \cup T)^*$: $uAv \Rightarrow_{lm} uxv$

Note: Leftmost nonterminal is rewritten

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3) Rightmost derivation:

Let $u \in (N \cup T)^*, v \in T^*$: $uAv \Rightarrow_{rm} uxv$

Note: Rightmost nonterminal is rewritten

Reduction of the Number of Derivations

Gist: Without any loss of generality, we can consider only leftmost or rightmost derivations.

Theorem: Let $G = (N, T, P, S)$ be a CFG.
The next three languages coincide

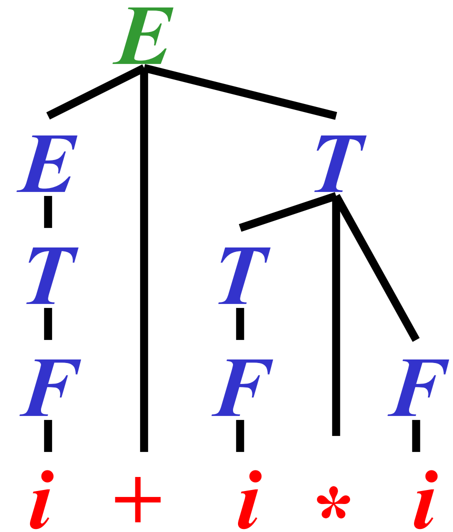
(1) $\{w: w \in T^*, S \Rightarrow_{lm}^* w\}$

(2) $\{w: w \in T^*, S \Rightarrow_{rm}^* w\}$

(3) $\{w: w \in T^*, S \Rightarrow^* w\} = L(G)$

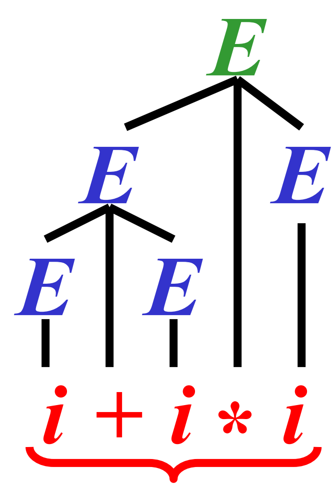
Introduction to Ambiguity

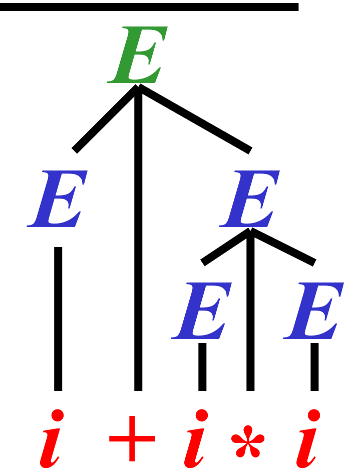
$G_{expr1} = (N, T, P, E)$, where
$N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,
$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$,
3: $T \rightarrow T*F$, 4: $T \rightarrow F$,
5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$



Theory: ☹ × Practice: ☺

$G_{expr2} = (N, T, P, E)$, where
$N = \{E\}$, $T = \{i, +, *, (,)\}$,
$P = \{$ 1: $E \rightarrow E+E$, 2: $E \rightarrow E*E$,
3: $E \rightarrow (E)$, 4: $E \rightarrow i$ $\}$





Improper during compilation

Theory: ☺ × Practice: ☹

Note: $L(G_{expr1}) = L(G_{expr2})$

Grammatical Ambiguity

Definition: Let $G = (N, T, P, S)$ be a CFG. If there exists $x \in L(G)$ with more than one derivation tree, then G is *ambiguous*; otherwise, G is *unambiguous*.

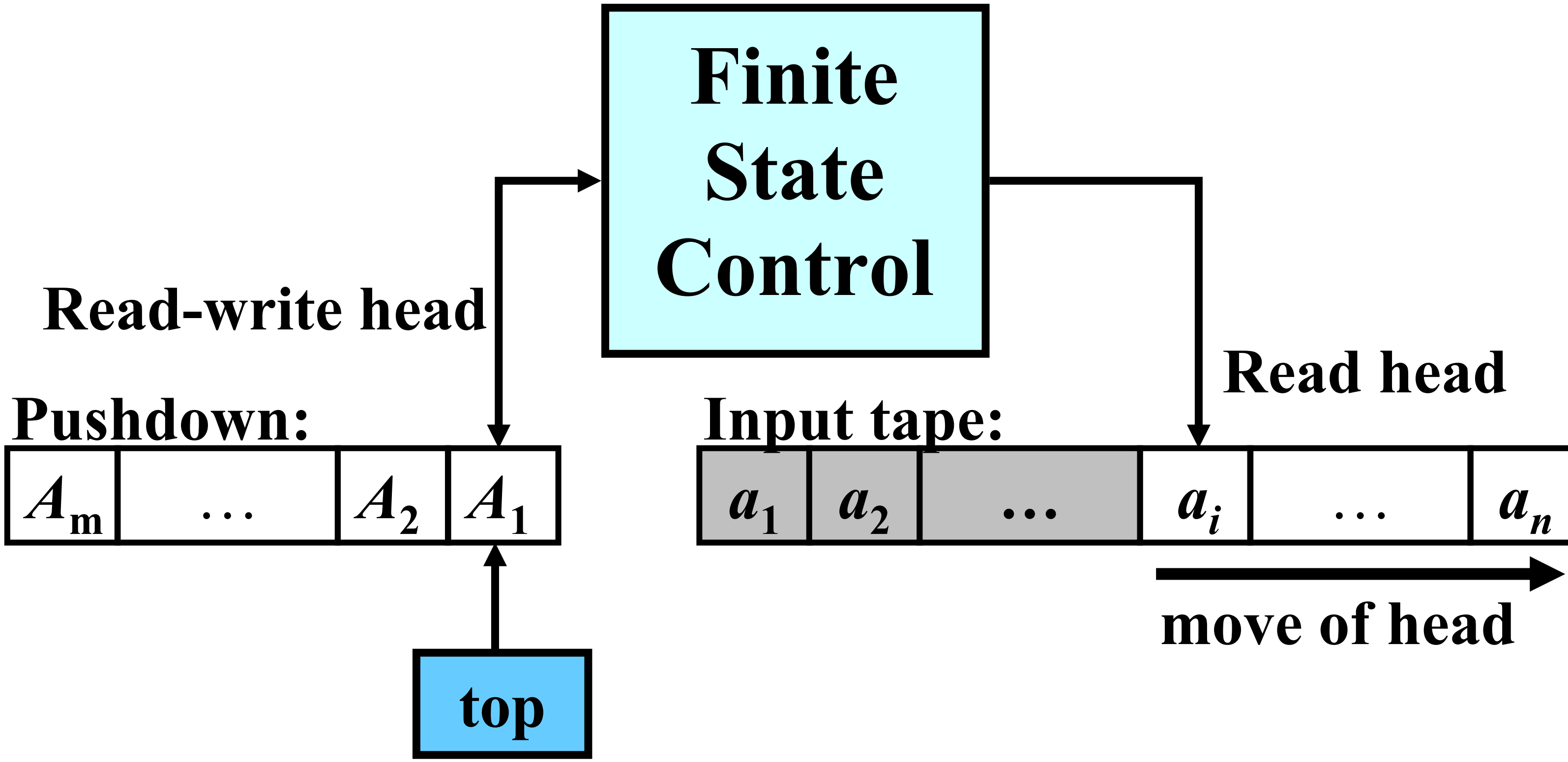
Definition: A CFL, L, is *inherently ambiguous* if L is generated by no unambiguous grammar.

Example:

- G_{expr1} is **unambiguous**, because for every $x \in L(G_{expr1})$ there exists **only one derivation tree**
- G_{expr2} is **ambiguous**, because for $i+i*i \in L(G_{expr2})$ there exist **two derivation trees**
- $L_{expr} = L(G_{expr1}) = L(G_{expr2})$ is **not inherently ambiguous** because G_{expr1} **is unambiguous**

Pushdown Automata (PDA)

Gist: An FA extended by a pushdown store.



Pushdown Automata: Definition

Definition: *A pushdown automaton* (PDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where

- Q is a *finite set of states*
- Σ is an *input alphabet*
- Γ is a *pushdown alphabet*
- R is a *finite set of rules* of the form: $Apa \rightarrow wq$ where $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $w \in \Gamma^*$
- $s \in Q$ is the *start state*
- $S \in \Gamma$ is the *start pushdown symbol*
- $F \subseteq Q$ is a set of *final states*

Notes on PDA Rules

Mathematical note on rules:

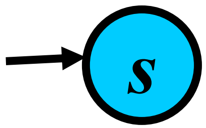
- Strictly mathematically, R is a relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
- Instead of $(Apa, wq) \in R$, however, we write $Apa \rightarrow wq \in R$

- **Interpretation of** $Apa \rightarrow wq$: if the current state is p, current input symbol is a, and the topmost symbol on the pushdown is A, then M can read a, replace A with w and change state p to q.

- **Note**: if $a = \varepsilon$, no symbol is read

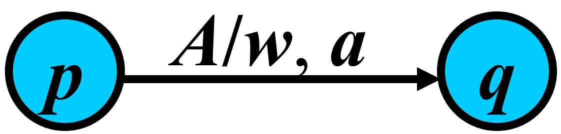
Graphical Representation

represents $q \in Q$



represents the initial state $s \in Q$

represents a final state $f \in F$



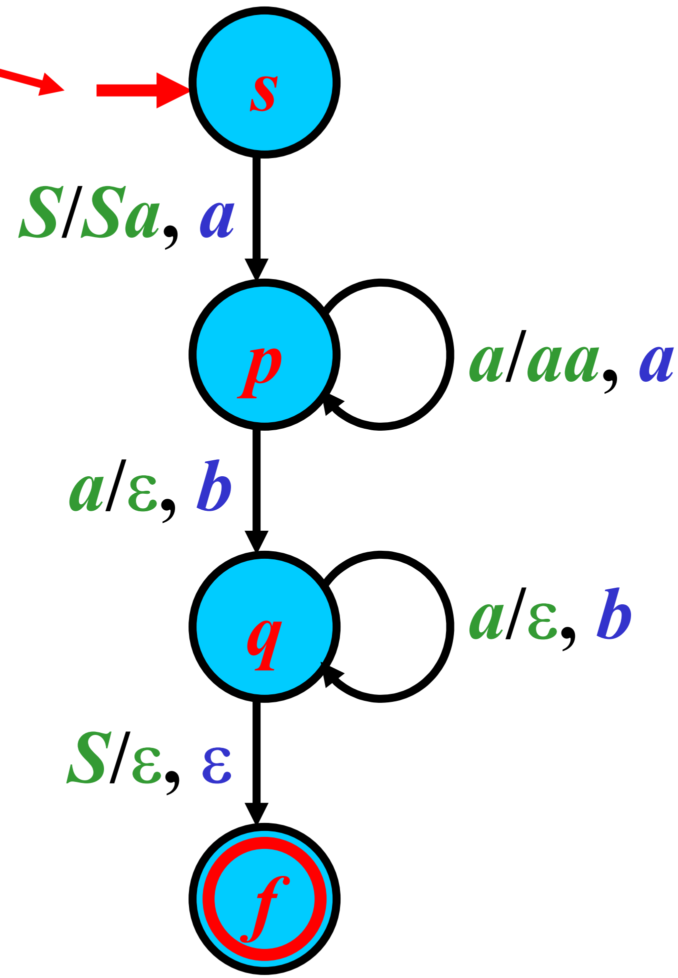
denotes $Apa \to wq \in R$

Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

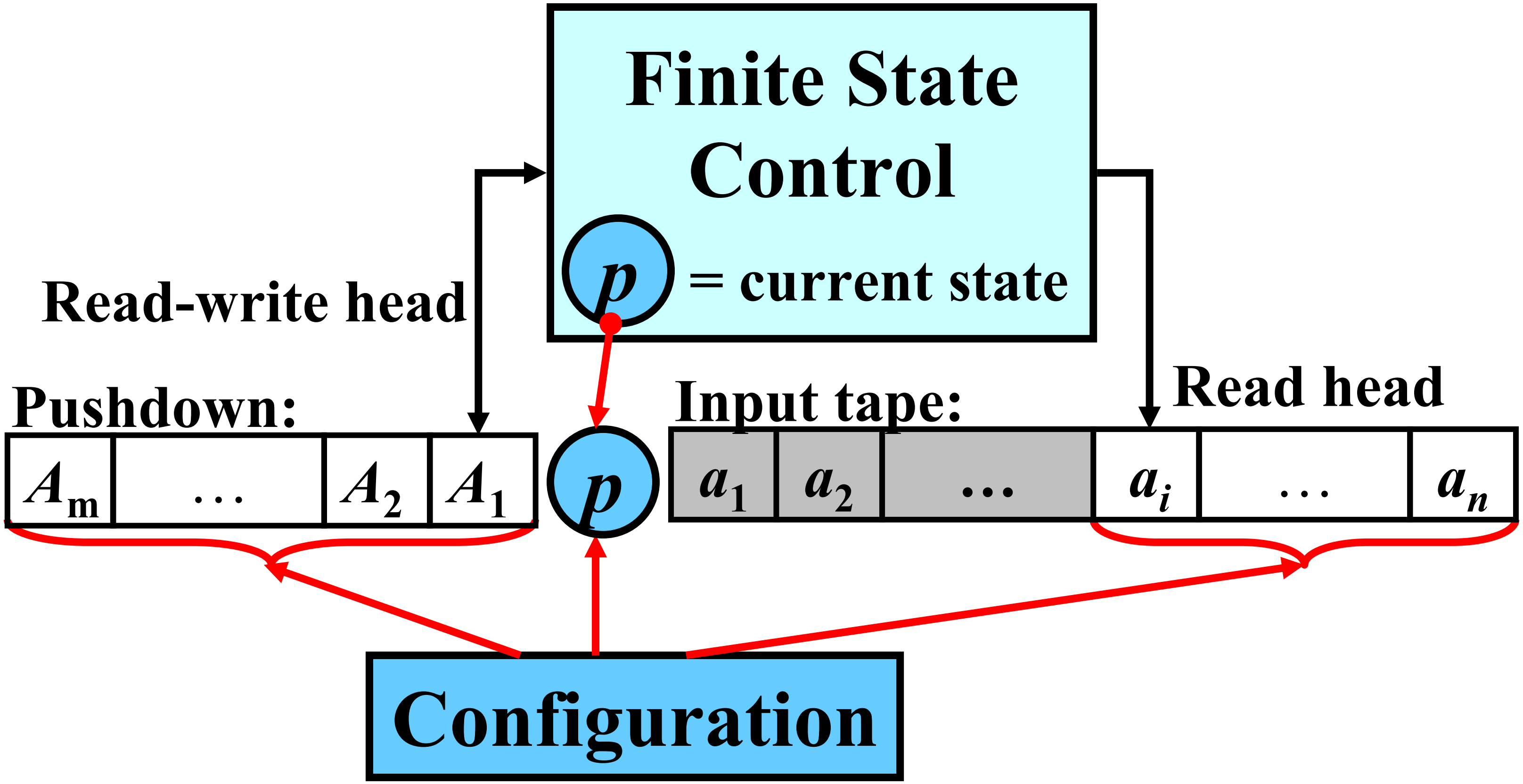
- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$
  $aqb \rightarrow q,$
  $Sq \rightarrow f\}$
- $F = \{f\}$



PDA Configuration

Gist: Instantaneous description of PDA

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.
A configuration of M is a string $\chi \in \Gamma^* Q \Sigma^*$



Move

Gist: A computational step made by a PDA

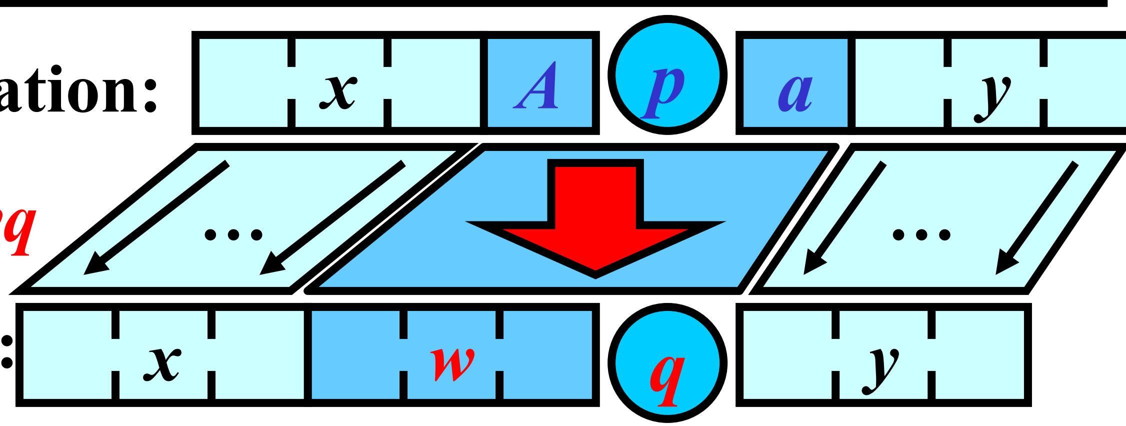
Definition: Let $xApay$ and $xwqy$ be two configurations of a PDA, M, where
$x, w \in \Gamma^*$, $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$.
Let $r = Apa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xApay$ to $xwqy$ according to r, written as $xApay \vdash xwqy$ [r] or, simply, $xApay \vdash xwqy$.

Note: if $a = \varepsilon$, no input symbol is read

Configuration:

Rule: $Apa \rightarrow wq$

New configuration:



Sequence of Moves 1/2

Gist: Several consecutive computational steps

Definition: Let χ be a configuration. *M* makes *zero moves* from χ to χ; in symbols,

$$\chi \vdash^0 \chi\ [\varepsilon] \text{ or, simply, } \chi \vdash^0 \chi$$

Definition: Let $\chi_0, \chi_1, \ldots, \chi_n$ be a sequence of configurations, $n \geq 1$, and $\chi_{i-1} \vdash \chi_i\ [r_i]$, $r_i \in R$, for all $i = 1, \ldots, n$; that is,

$$\chi_0 \vdash \chi_1\ [r_1] \vdash \chi_2\ [r_2] \ldots \vdash \chi_n\ [r_n]$$

Then *M* makes *n moves* from χ_0 to χ_n,

$$\chi_0 \vdash^n \chi_n\ [r_1 \ldots r_n] \text{ or, simply, } \chi_0 \vdash^n \chi_n$$

Sequence of Moves 2/2

If $\chi_0 \vdash^n \chi_n$ [ρ] for some $n \geq 1$, then
$\chi_0 \vdash^+ \chi_n$ [ρ] or, simply, $\chi_0 \vdash^+ \chi_n$

If $\chi_0 \vdash^n \chi_n$ [ρ] for some $n \geq 0$, then
$\chi_0 \vdash^* \chi_n$ [ρ] or, simply, $\chi_0 \vdash^* \chi_n$

Example: Consider

AApabc $\vdash$ ***ABqbc*** [**1**: ***Apa*** → ***Bq***], and
ABqbc $\vdash$ ***ABCrc*** [**2**: ***Bqb*** → ***BCr***].

Then,
AApabc $\vdash^2$ ***ABCrc*** [**1 2**],
AApabc $\vdash^+$ ***ABCrc*** [**1 2**],
AApabc $\vdash^*$ ***ABCrc*** [**1 2**]

Accepted Language: Three Types

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.

1) The *language that M accepts* ***by final state***, denoted by $\boldsymbol{L(M)_f}$, is defined as

$$L(M)_f = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z \in \Gamma^*, f \in F\}$$

2) The *language that M accepts* ***by empty pushdown***, denoted by $\boldsymbol{L(M)_\varepsilon}$, is defined as

$$L(M)_\varepsilon = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in Q\}$$

3) The *language that M accepts* ***by final state and empty pushdown***, denoted by $\boldsymbol{L(M)_{f\varepsilon}}$, is defined as

$$L(M)_{f\varepsilon} = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in F\}$$

PDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$
  $aqb \rightarrow q,$
  $Sq \rightarrow f\}$
- $F = \{f\}$

Question: $aabb \in L(M)_{f\varepsilon}$?

| S | s | a | a | b | b |

Rule: $Ssa \rightarrow Sap$

| S | a | p | a | b | b |

Rule: $apa \rightarrow aap$

| S | a | a | p | b | b |

Rule: $apb \rightarrow q$

| S | a | q | b |

Rule: $aqb \rightarrow q$

| S | q |

Rule: $Sq \rightarrow f$

| f |

Empty pushdown

Final state

Answer: YES

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

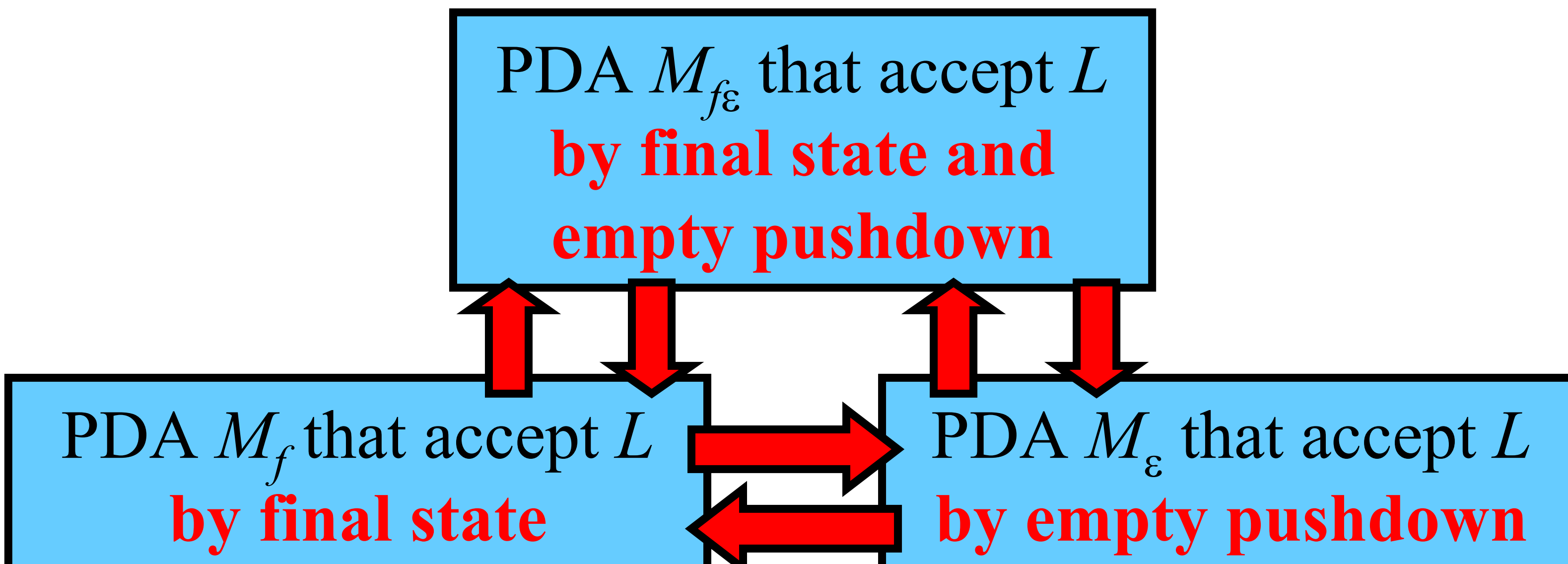
Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{a^n b^n : n \geq 1\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for a PDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for a PDA M_ε

Note: There exist these conversions:



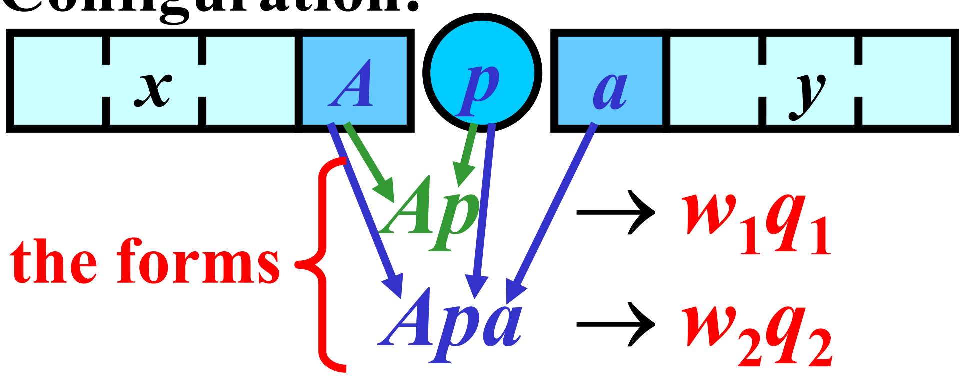
Deterministic PDA (DPDA)

Gist: Deterministic PDA makes no more than one move from any configuration.

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. M is a *deterministic PDA* if for each rule $Apa \rightarrow wq \in R$, it holds that $R - \{Apa \rightarrow wq\}$ contains no rule with the left-hand side equal to Apa or Ap.

Illustration:

Configuration:



No more that one rule of the forms

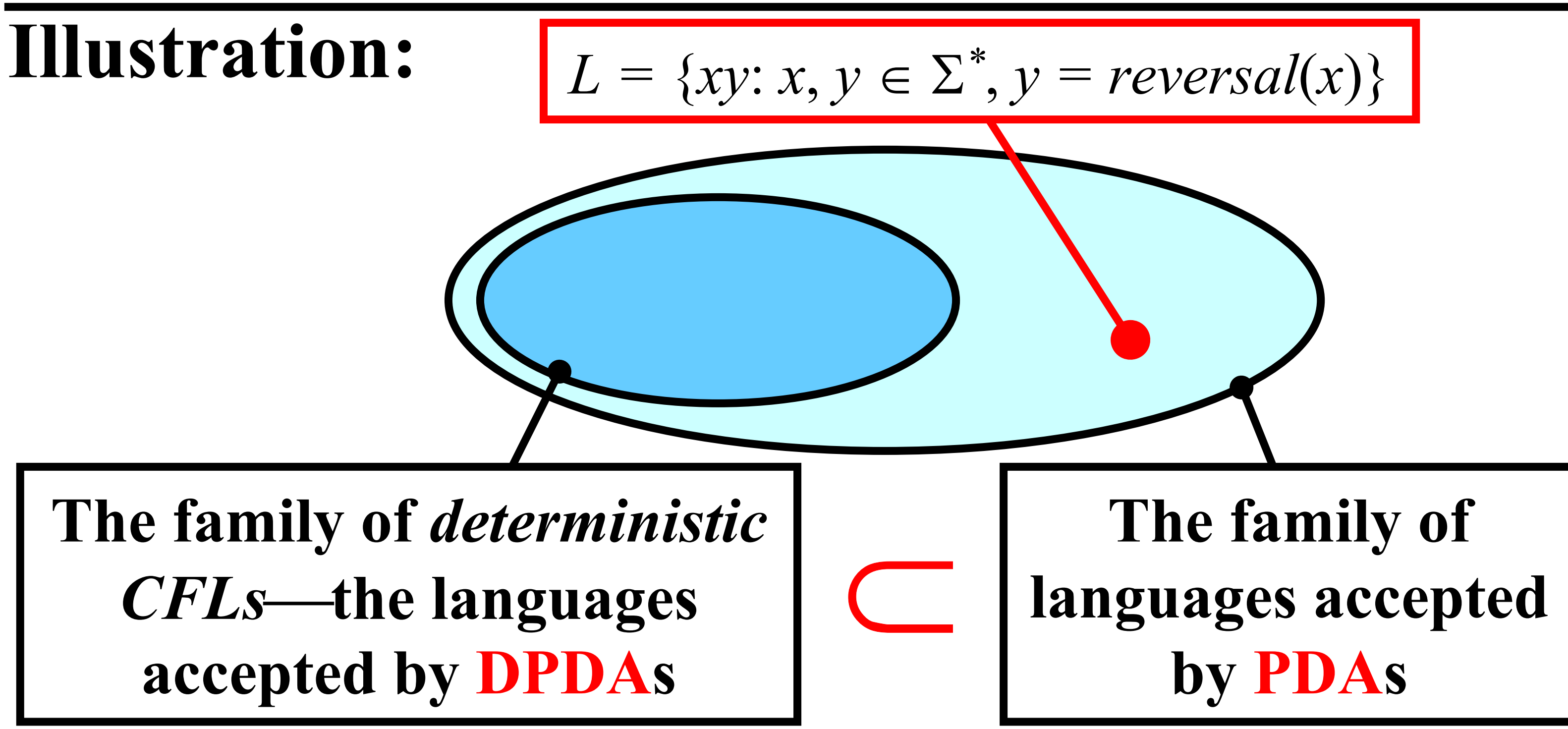
PDAs are Stronger than DPDAs

Theorem: There exists no DPDA $M_{f\varepsilon}$ that accepts

$$L = \{xy: x, y \in \Sigma^*, y = reversal(x)\}$$

Proof: See page 431 in [Meduna: Automata and Languages]

Illustration:



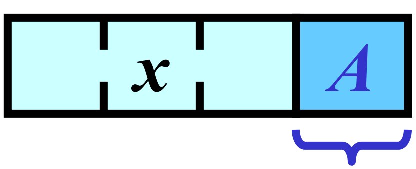
Extended PDA (EPDA)

Gist: The pushdown top of an EPDA represents a string rather than a single symbol.

Definition: *An Extended Pushdown automaton* (EPDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where $Q, \Sigma, \Gamma, s, S, F$ are defined as in an PDA and R is a *finite set of rules* of the form: $vpa \rightarrow wq$, where $v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$

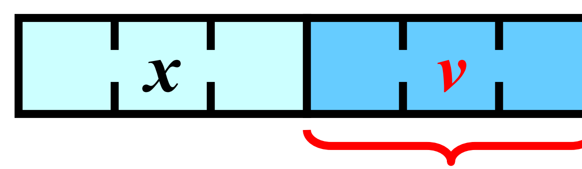
Illustration:

Pushdown of PDA:



PDA has a single symbols as the pushdown top

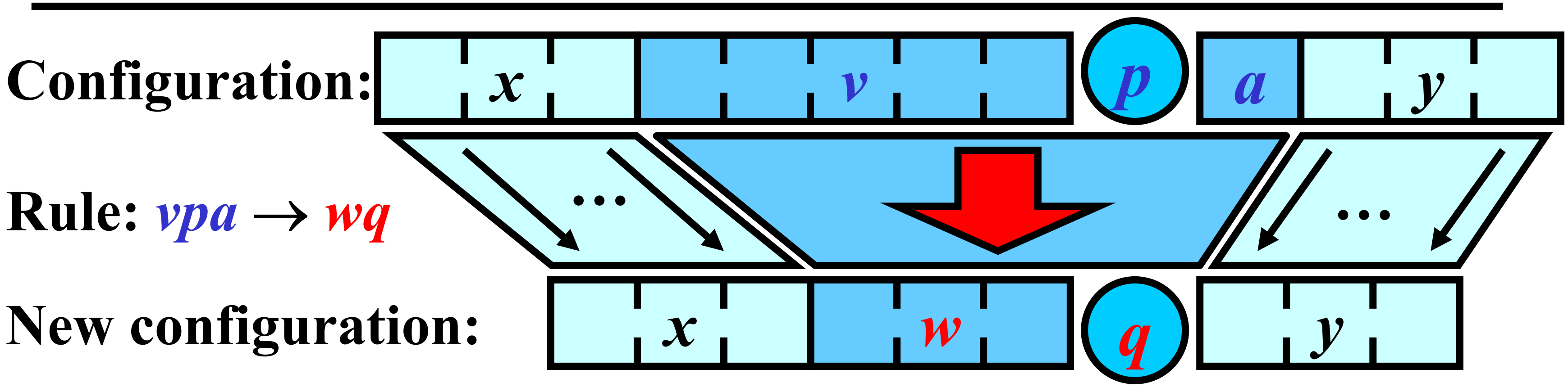
Pushdown of EPDA:



EPDA has a string as the pushdown top

Move in EPDA

Definition: Let $xvpay$ and $xwqy$ be two configurations of an EPDA, M, where $x, v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$. Let $r = vpa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xvpay$ to $xwqy$ according to r, written as $xvpay \vdash xwqy$ [r] or $xvpay \vdash xwqy$.



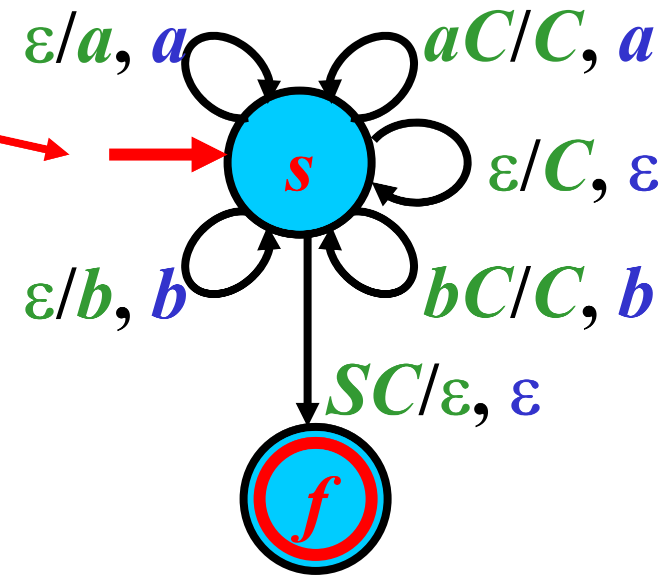
Note: $\vdash^n$, $\vdash^+$, $\vdash^*$, $L(M)_f$, $L(M)_\varepsilon$, and $L(M)_{f\varepsilon}$ are defined analogically to the corresponding definitions for PDA.

EPDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, b, S, C\}$;
- $R = \{\ sa \rightarrow as,\ sb \rightarrow bs,\ s \rightarrow Cs,\ aCsa \rightarrow Cs,\ bCsb \rightarrow Cs,\ SCs \rightarrow f\ \}$
- $F = \{f\}$



Question: $abba \in L_{f\varepsilon}(M)$?

$Ssabba \vdash Sasbba \vdash Sabsba$
$\vdash SabCsba \vdash SaCsa$
$\vdash SCs \vdash f$

Answer: YES

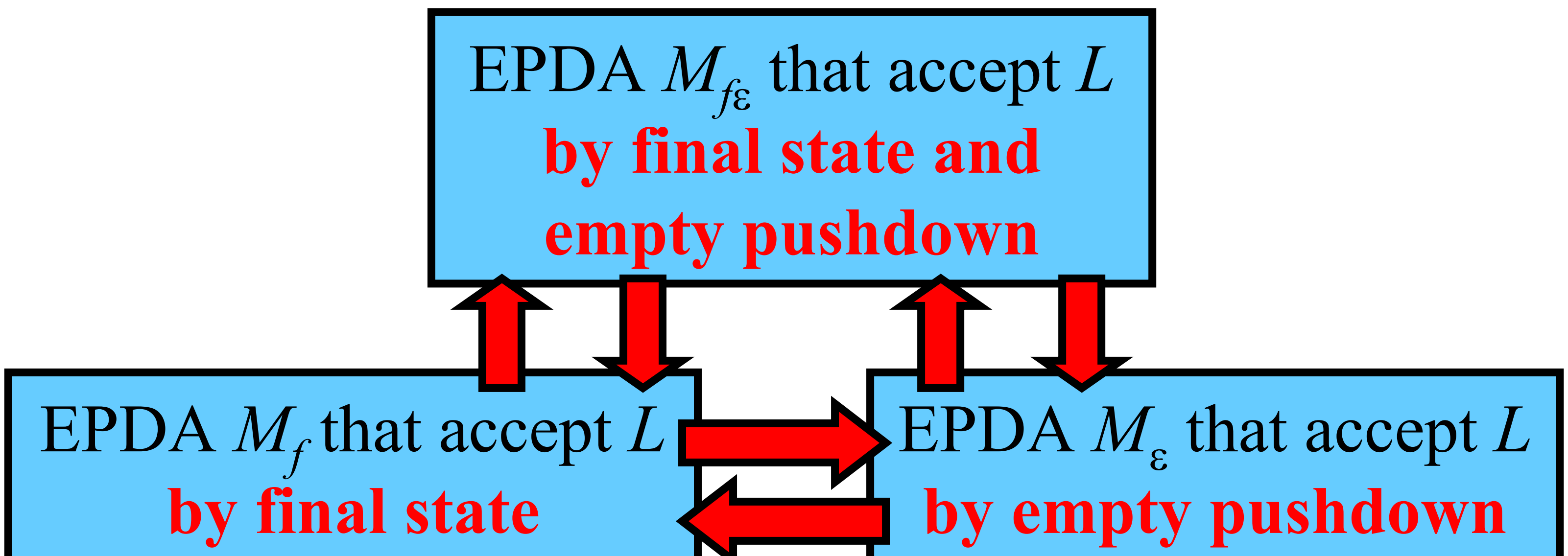
Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{xy: x, y \in \Sigma^*, y = \text{reversal}(x)\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for an EPDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for an EPDA M_ε

Note: There exist these conversion:

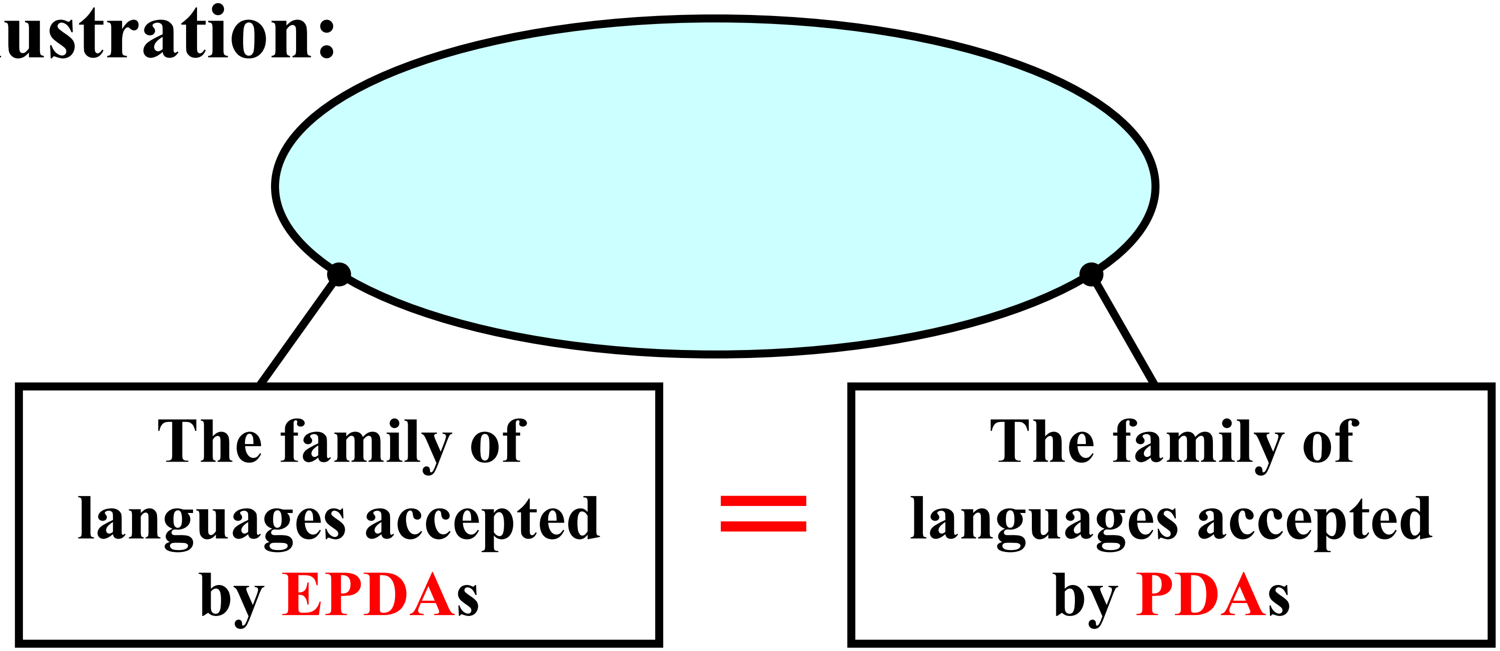


EPDAs and PDAs are Equivalent

Theorem: For every EPDA M, there is a PDA M', and $L(M)_f = L(M')_f$.

Proof: See page 419 in [Meduna: Automata and Languages]

Illustration:

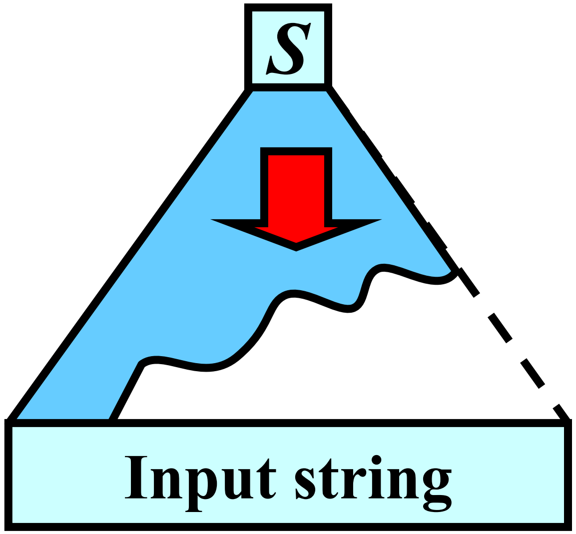


EPDAs and PDAs as Parsing Models for CFGs

Gist: **An EPDA or a PDA can simulate the construction of a derivation tree for a CFG**

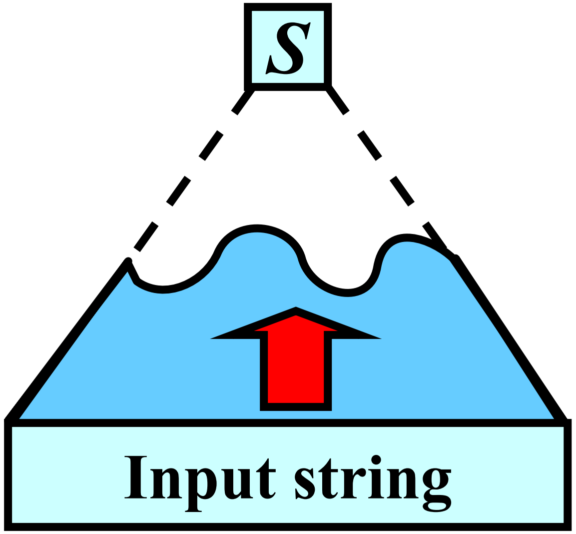
- **Two basic approaches:**

1) Top-Down Parsing



From *S* towards the input string

2) Bottom-Up Parsing

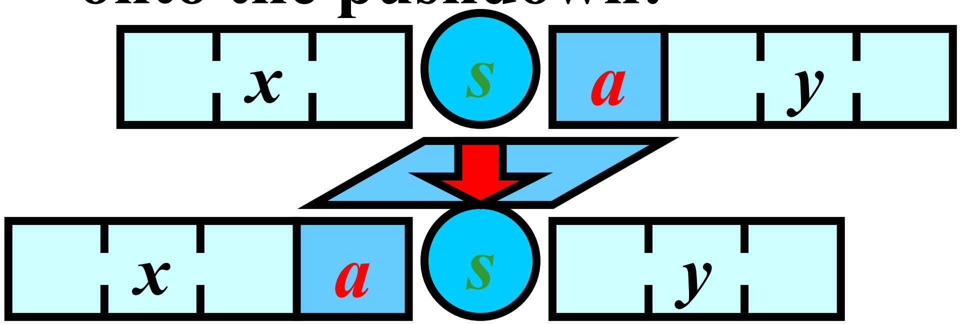


From the input string towards *S*

EPDAs as Models of Bottom-Up Parsers 1/2

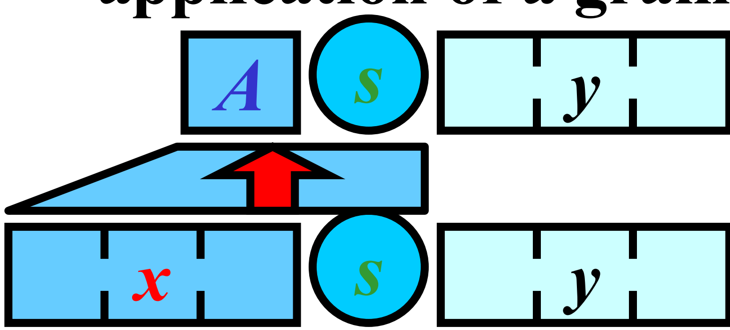
Gist: An EPDA *M* underlies a bottom-up parser

1) *M* contains *shift* rules that copy the input symbols onto the pushdown:



for every $a \in \Sigma$:
add $sa \rightarrow as$ to R;

2) *M* contains *reduction* rules that simulate the application of a grammatical rule in reverse:

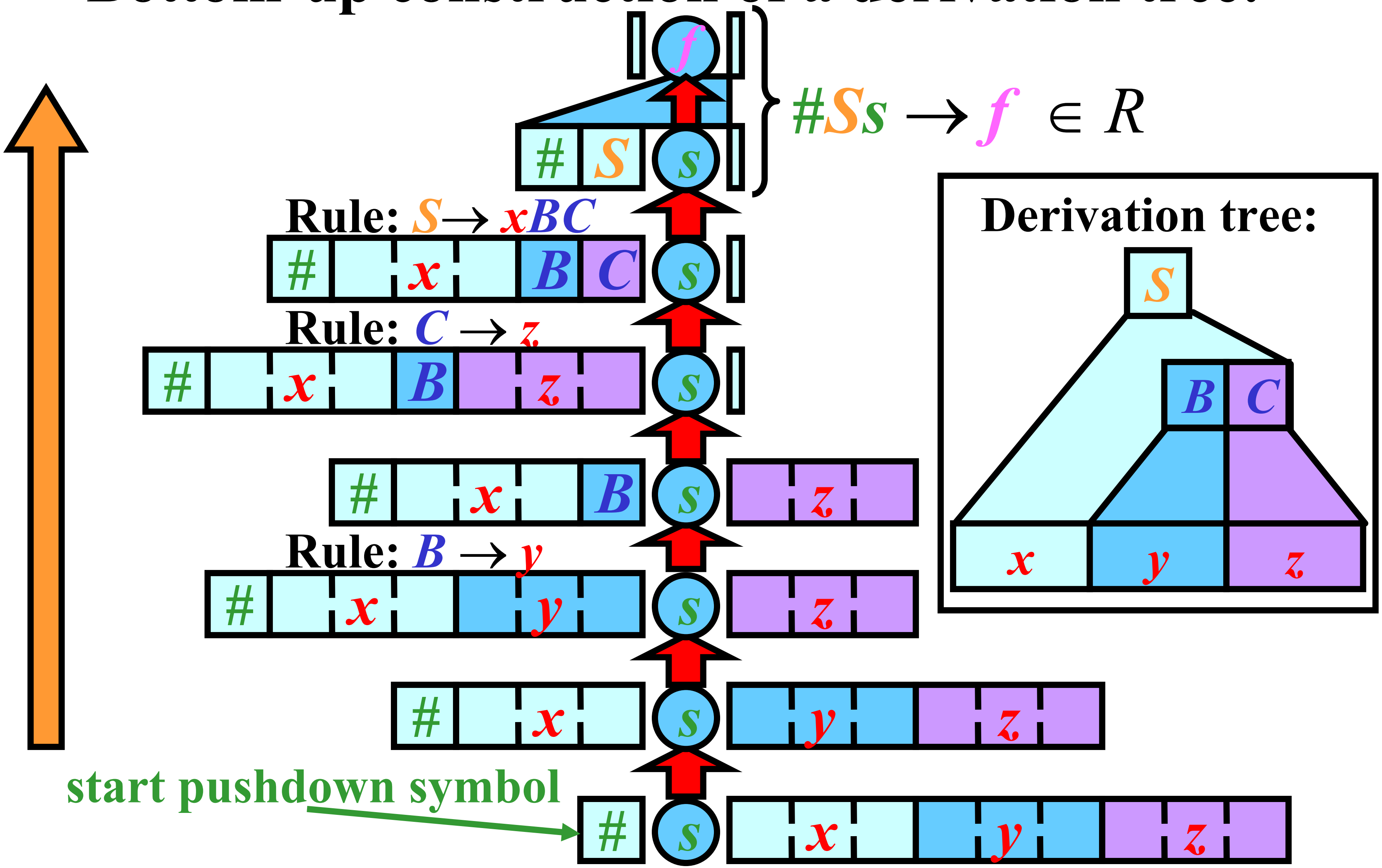


for every $A \rightarrow x \in P$ in G:
add $xs \rightarrow As$ to R;

3) *M* also contains the rule $\#Ss \rightarrow f$ that takes *M* to a final state f

EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



Algorithm: From CFG to EPDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** EPDA $M = (Q, \Sigma, \Gamma, R, s, \#, F)$; $L(G) = L(M)_f$

- **Method:**
- $Q := \{s, f\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T \cup \{\#\}$;
- Construction of R:
  - for every $a \in \Sigma$, add $sa \to as$ to R;
  - for every $A \to x \in P$, add $xs \to As$ to R;
  - add $\#Ss \to f$ to R;
- $F := \{f\}$;

From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

$N = \{S\}$, $T = \{(,)\}$, $P = \{S \to (S), S \to ()\}$

Objective: An EPDA M such that $L(G) = L(M)_f$

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$ where:

$Q = \{s, f\}$; $\Sigma = T = \{(,)\}$; $\Gamma = N \cup T \cup \{\#\} = \{S, (,), \#\}$

"(" $\in T$ "$)$" $\in T$ $S \to (S) \in P$ $S \to () \in P$

$R = \{s(\to (s,\ s) \to)s,\ (S)s \to Ss,\ ()s \to Ss,\ \#Ss \to f\}$

shift rules (for $s(\to (s$ and $s) \to)s$) **reduction rules** (for $(S)s \to Ss$ and $()s \to Ss$)

$F = \{f\}$

From CFG to EPDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$, where:

$Q = \{s, f\}$, $\Sigma = T = \{(,)\}$, $\Gamma = \{(,), S, \#\}$, $F = \{f\}$

$R = \{s(\rightarrow (s,\ s) \rightarrow)s,\ (S)s \rightarrow Ss,\ ()s \rightarrow Ss,\ \#Ss \rightarrow f\}$

Question: $(()) \in L(M)_f$?

\# s (( ))

Rule: $s(\rightarrow (s$

\# (s ( ))

(

Rule: $s(\rightarrow (s$

\# ((s))

((

Rule: $s) \rightarrow)s$

\# (( ) s)

(( )

Rule: $()s \rightarrow S$

\# (S s)

S

(( )

Rule: $s) \rightarrow)s$

\# (S) s

S

(( ))

Rule: $(S) \rightarrow S$

\# S s

S

S

(( ))

Rule: $\#Ss \rightarrow f$

Final state

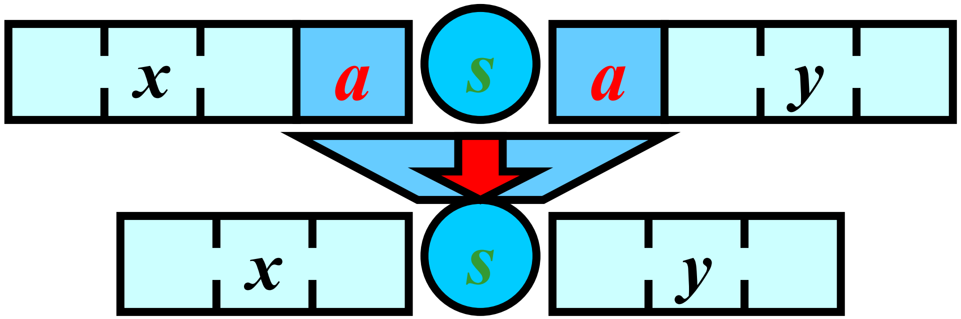
f

Answer: YES

PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA *M* underlies a top-down parser

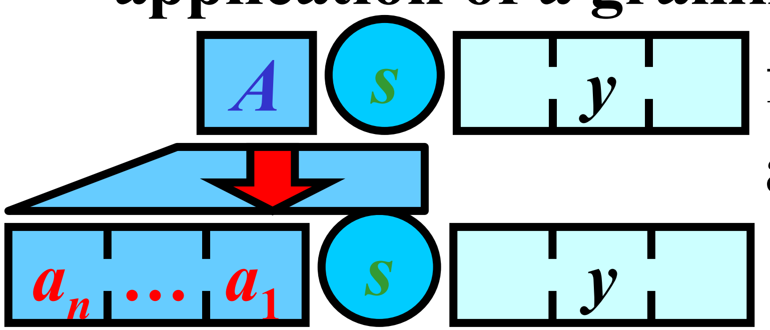
1) *M* contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:



for every $a \in \Sigma$:

add $asa \rightarrow s$ to R;

2) *M* contains *expansion* rules that simulate the application of a grammatical rule:



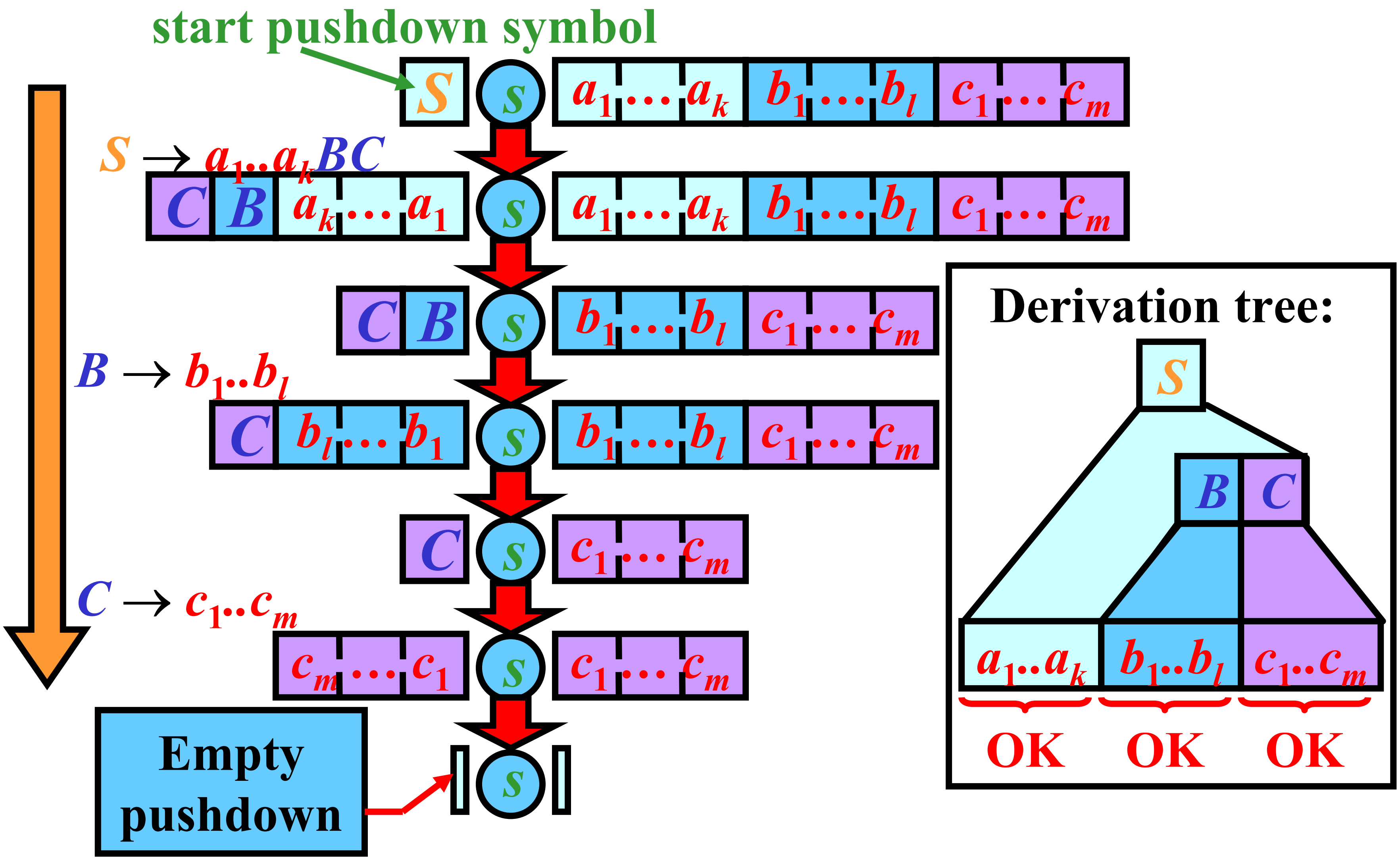
for every $A \rightarrow a_1 \ldots a_n \in P$ in G,

add $As \rightarrow \underbrace{a_n \ldots a_1}s$ to R;

= reversal($a_1 \ldots a_n$)

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



Algorithm: From CFG to PDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** PDA $M = (Q, \Sigma, \Gamma, R, s, S, F)$; $L(G) = L(M)_\varepsilon$

- **Method:**
- $Q := \{s\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T$;
- Construction of R:
  - for every $a \in \Sigma$, add $asa \rightarrow s$ to R;
  - for every $A \rightarrow x \in P$, add $As \rightarrow ys$ to R, where y = reversal(x);
- $F := \emptyset$;

From CFG to PDA: Example 1/2

- $G = (N, T, P, S)$, where:

$N = \{S\}$, $T = \{(,)\}$, $P = \{S \rightarrow (S), S \rightarrow ()\}$

Objective: An PDA M such that $L(G) = L(M)_\varepsilon$

$M = (Q, \Sigma, \Gamma, R, s, S, F)$ where:

$Q = \{s\}$; $\Sigma = T = \{(,)\}$; $\Gamma = N \cup T = \{S, (,)\}$

"(" $\in T$ ")" $\in T$ $S \rightarrow (S) \in P$ $S \rightarrow () \in P$

rev

rev

$R = \{(s(\rightarrow s,\ )s) \rightarrow s,\ Ss \rightarrow)S(s,\ Ss \rightarrow)(s\}$

popping rules **expansion rules**

$F = \emptyset$

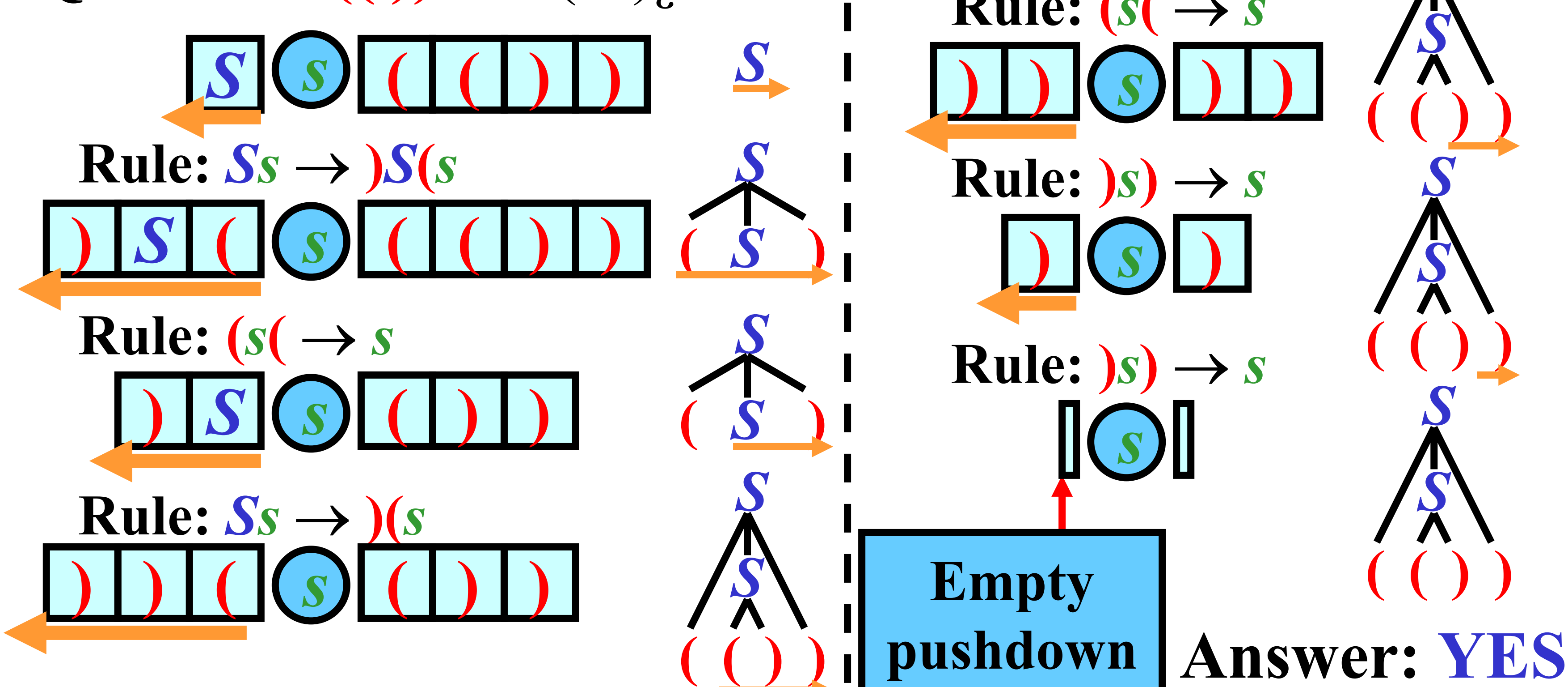
From CFG to PDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, S, F)$, where:

$Q = \{s\}$, $\Sigma = T = \{(,)\}$, $\Gamma = \{(,), S\}$, $F = \emptyset$

$P = \{(s(\rightarrow s,\quad)s) \rightarrow s,\quad Ss \rightarrow)S(s,\quad Ss \rightarrow)(s\ \}$

Question: $(())\in L(M)_\varepsilon$?



Answer: YES

Models for Context-free Languages

Theorem: For every CFG G, there is an PDA M such that $L(G) = L(M)_\varepsilon$.

Proof: See the previous algorithm.

Theorem: For every PDA M, there is a CFG G such that $L(M)_\varepsilon = L(G)$.

Proof: See page 486 in [Meduna: Automata and Languages]

Conclusion: The fundamental models for context-free languages are

1) Context-free grammars 2) Pushdown automata