Deep Pushdown Automata

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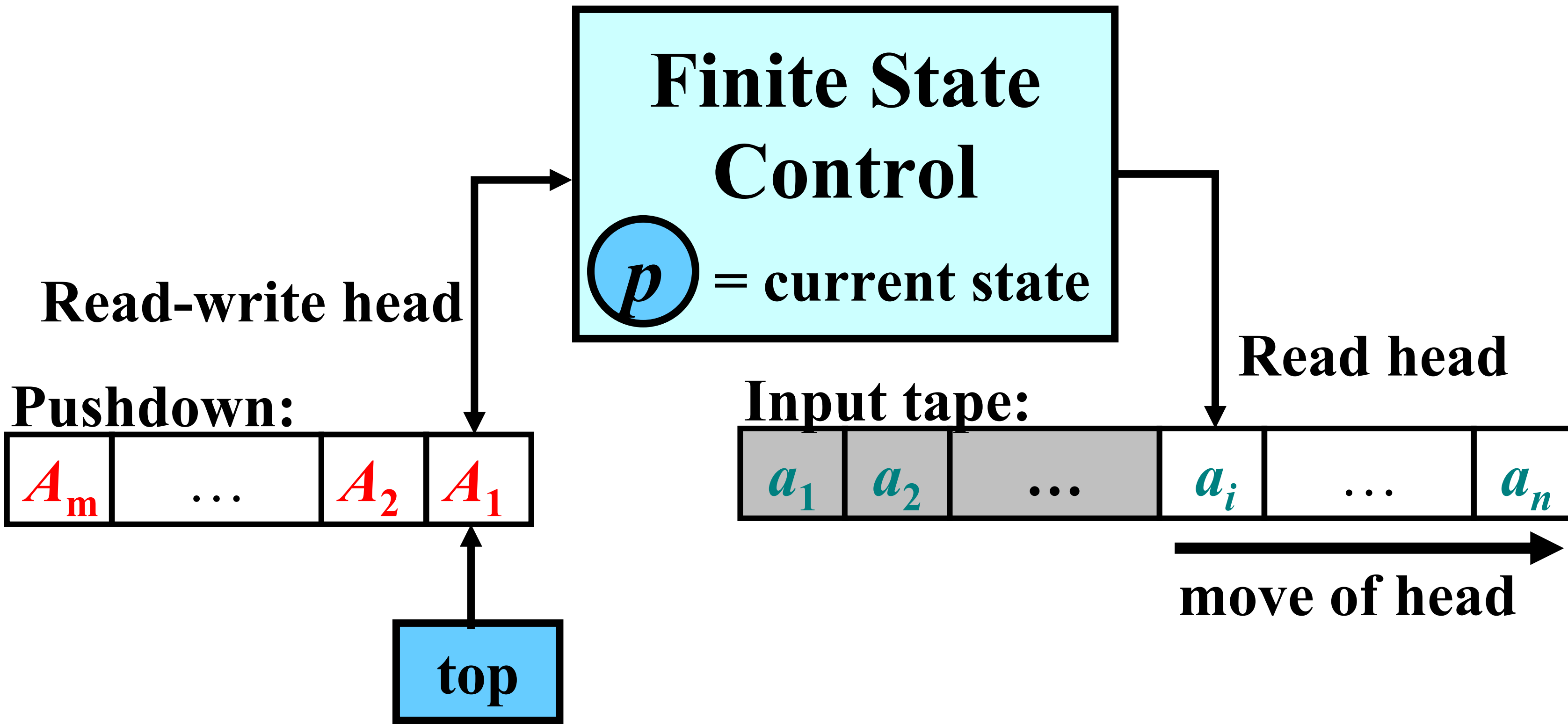
Based on

Meduna, A.: Deep Pushdown Automata,

***Acta Informatica*, 2006**

Pushdown Automaton (PDA)



Inspiration

- conversion of CFG to PDA that acts as a general top-down parser
  - if pd top = **input** symbol, **pop**
  - if pd top = **non-input** symbol, **expand**

PDA as a general Top-Down Parser

- Configuration:

$$(\textit{state}, \textit{input}, \textit{pd})$$

- **Pop:**

$$(q, ax, a\alpha) \ {}_{p}\!\Rightarrow (q, x, \alpha)$$

- **Expansion:**

$$(q, x, A\alpha) \ {}_{e}\!\Rightarrow (p, x, \beta\alpha)$$

by rule $qA \rightarrow p\beta$

Final state

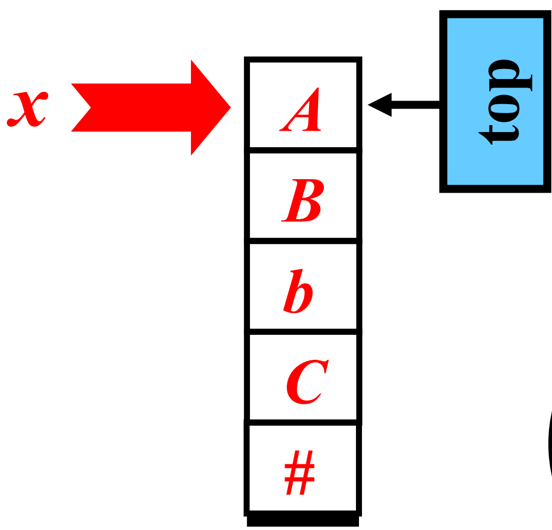
- Acceptance: $(s, x, S) \Rightarrow^{*} (f, \varepsilon, \varepsilon)$

Deep Pushdown Automata: Fundamental Modification

- expansion may be performed **deeper in pd**

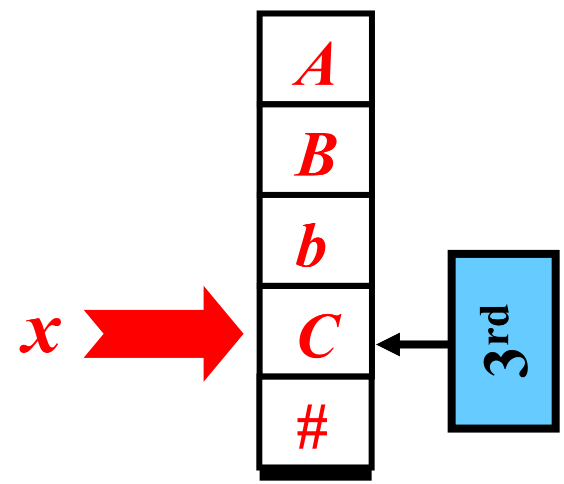
Standard Expansion

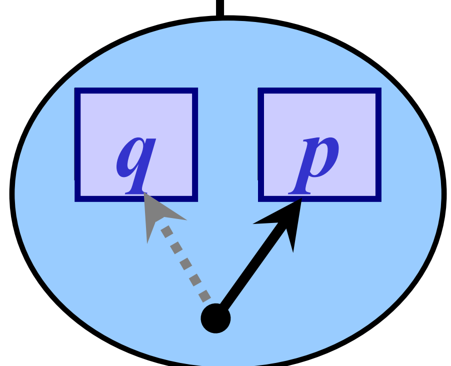
$qA \rightarrow px$



Deep Expansion

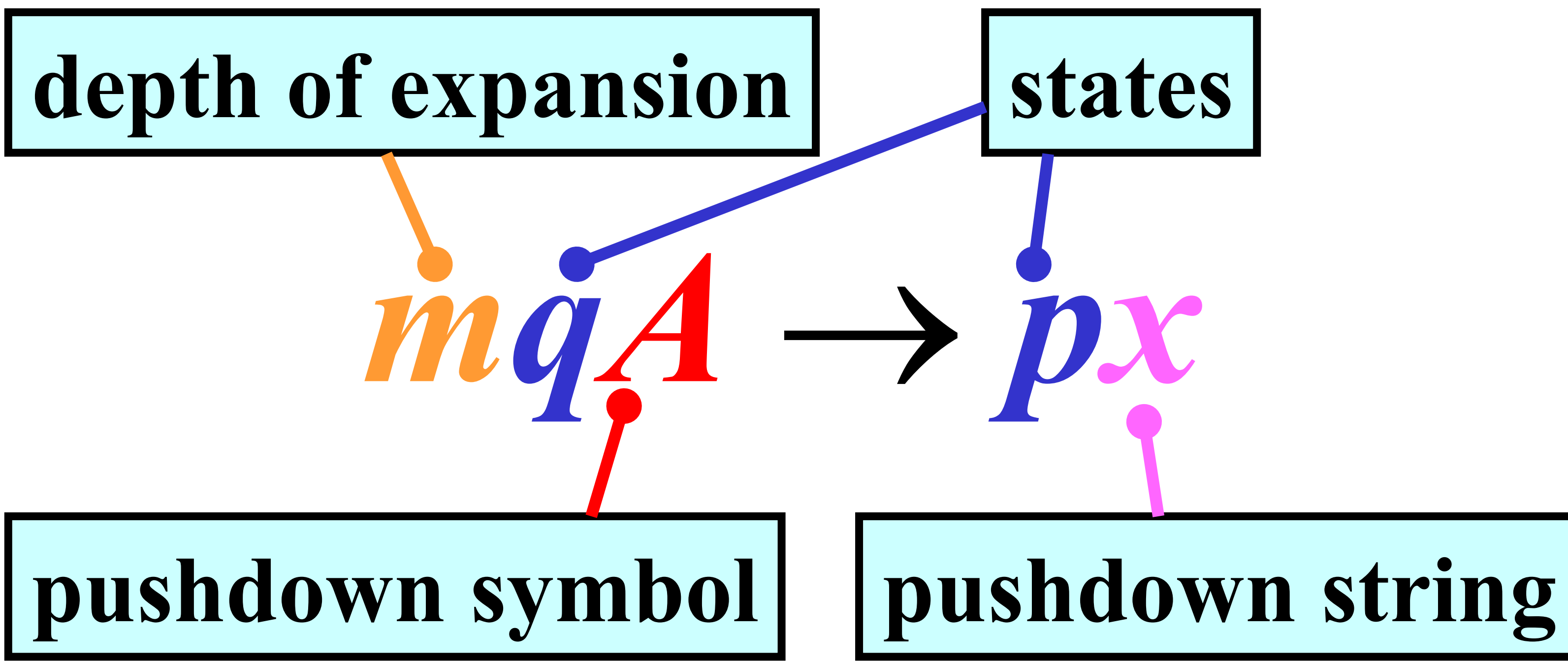
$3qC \rightarrow px$





Deep Pushdown Automata

- Same as Top-Down Parser except *deep expansions*
- *Expansion of depth m*:
    - the *m*th topmost non-input pd symbol is replaced with a string by rule



Expansion of Depth *m*

- ***Expansion of depth m***:

$$(q, w, uAz)\ {}_e\!\Rightarrow (p, w, uvz)$$

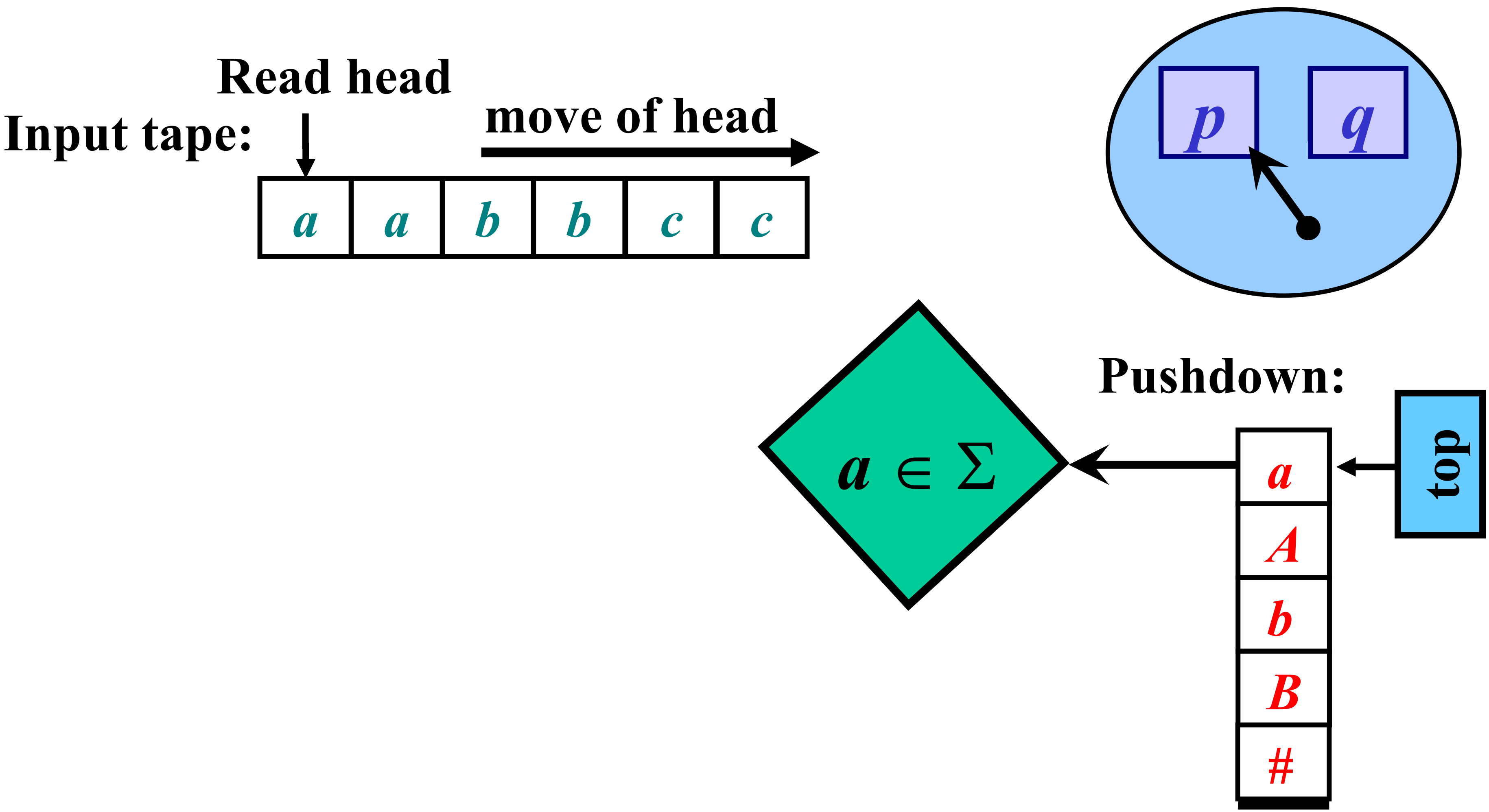
by ***rule of depth m***

$$[mqA \rightarrow pv],$$

where u contains m – **1** non-input symbols

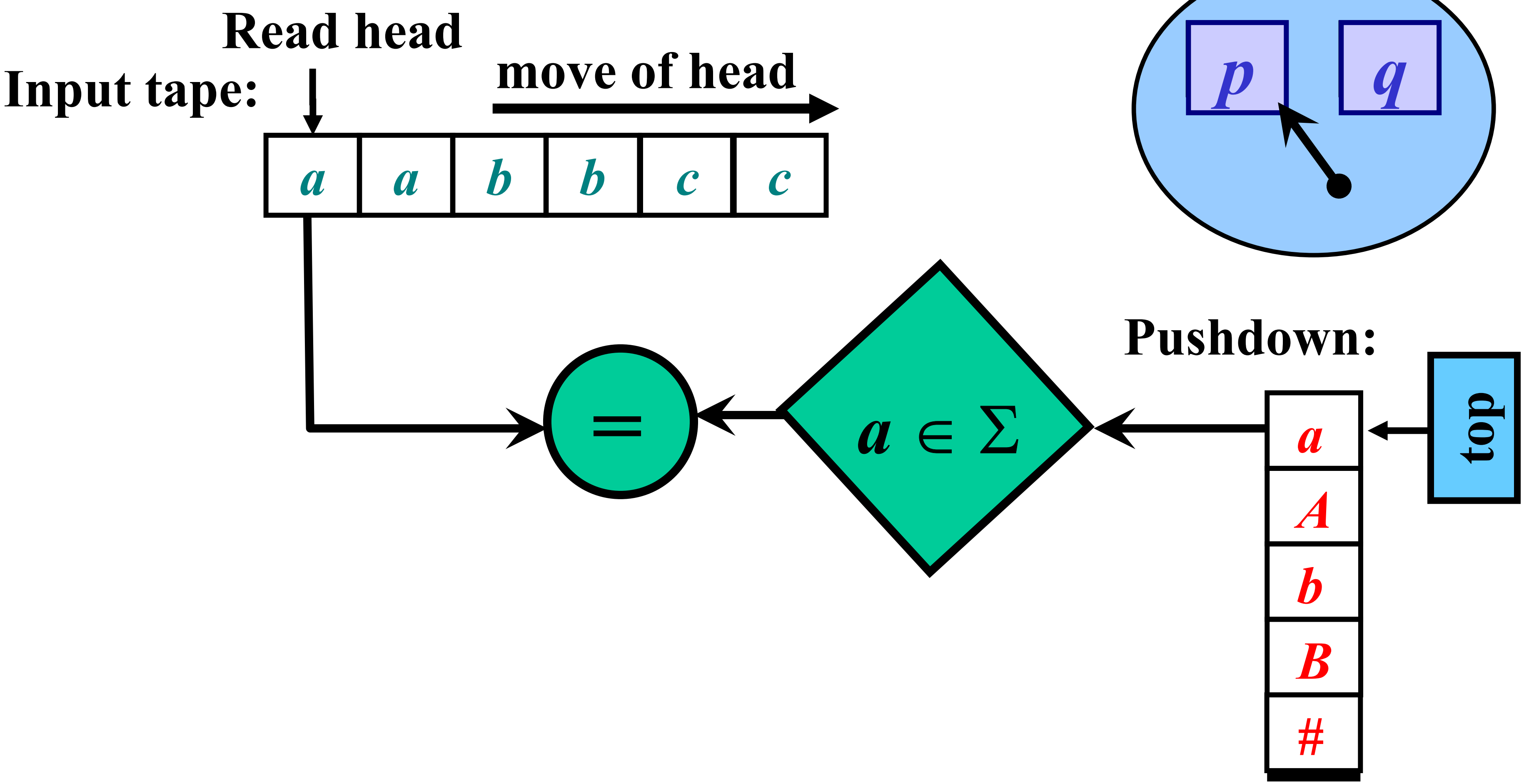
Pop: Illustration

Move: (p, *aabbcc*, *aAbB#*) $_p\Rightarrow$ (p, *abbcc*, *AbB#*)



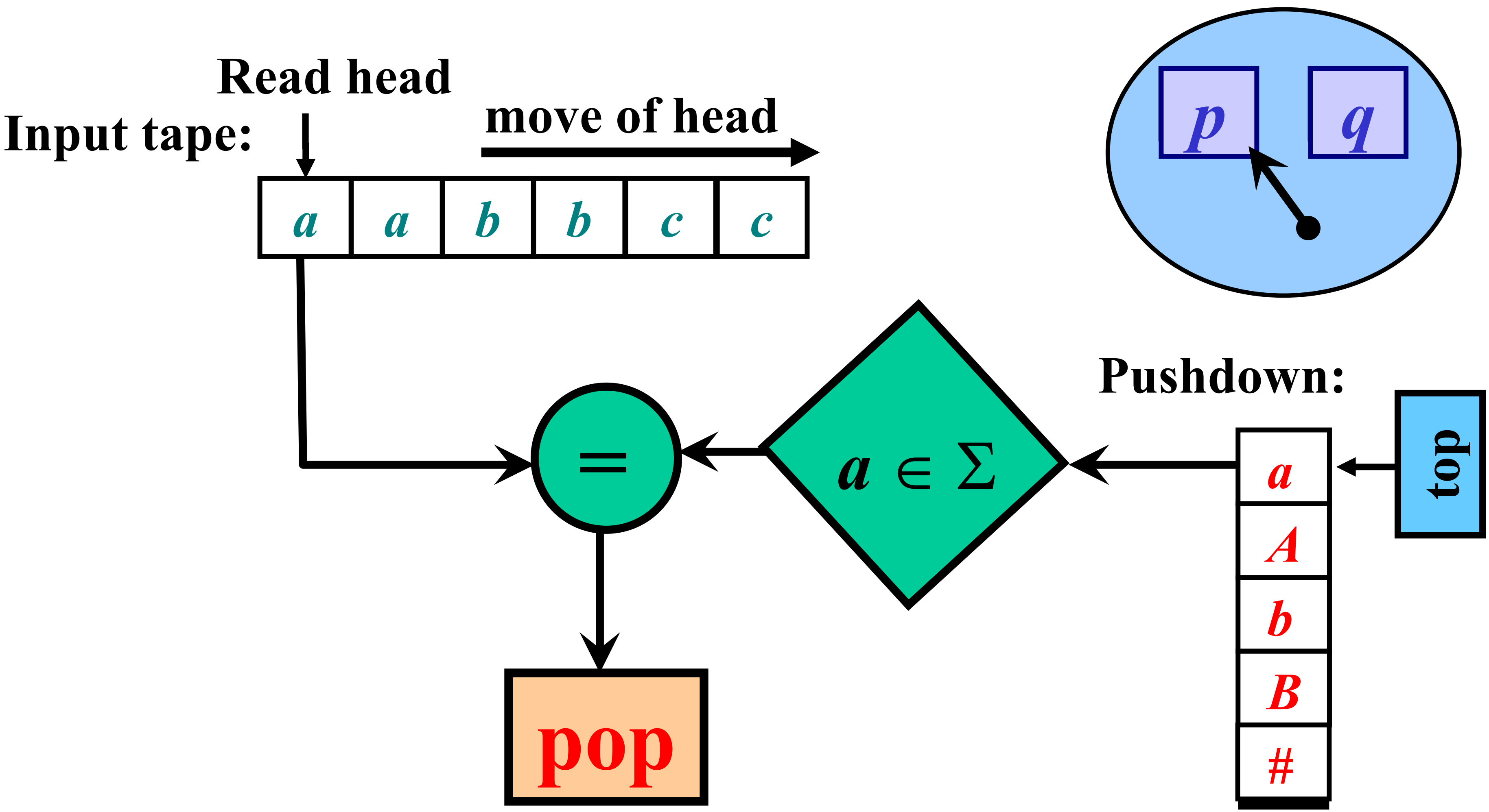
Pop: Illustration

Move: $(p, aabbcc, aAbB\#) \;_p\!\Rightarrow (p, abbcc, AbB\#)$



Pop: Illustration

Move: (p, $aabbcc$, $aAbB\#$) $_p\Rightarrow$ (p, $abbcc$, $AbB\#$)



Pop: Illustration

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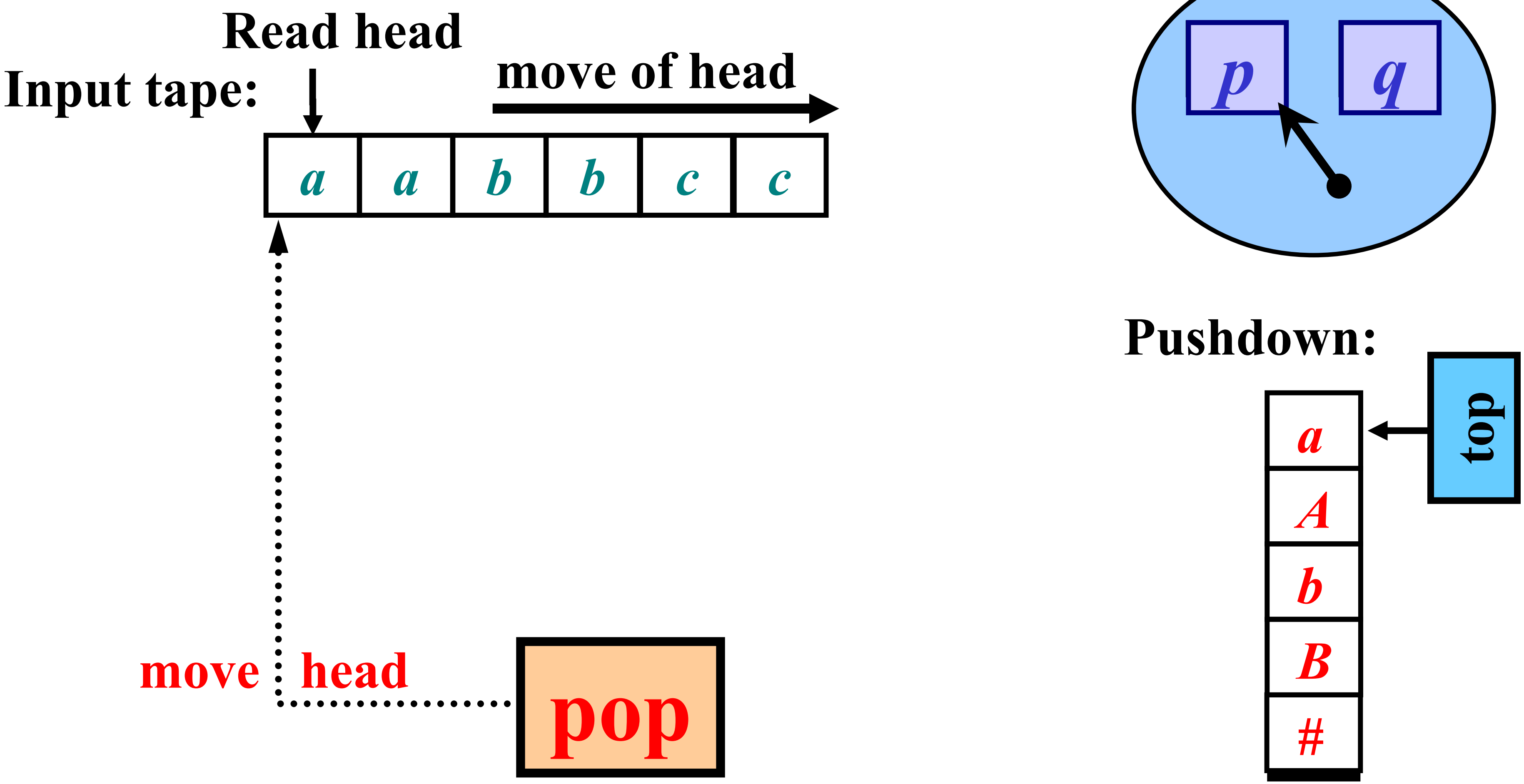

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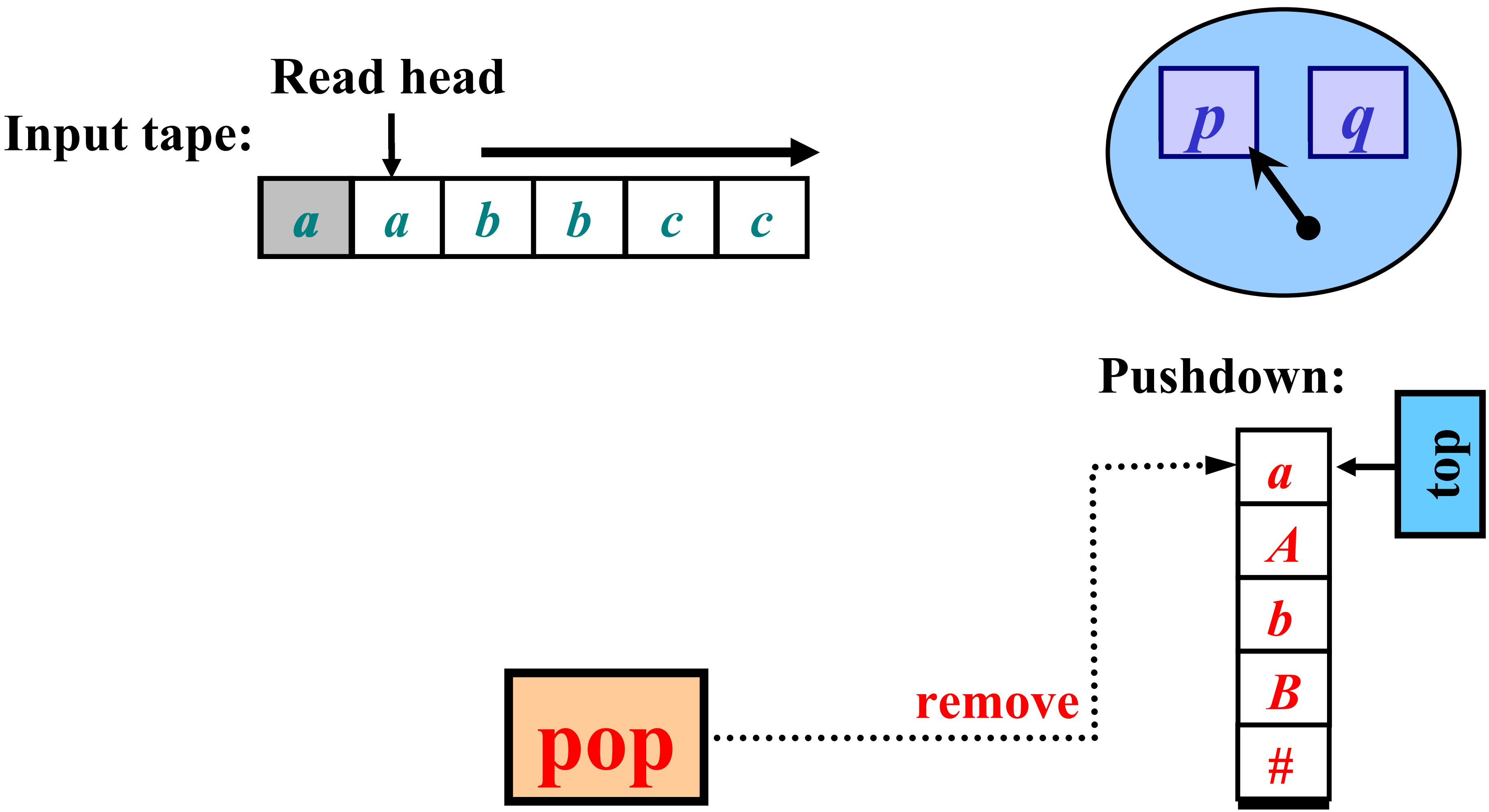
Move: $(p, aabbcc, aAbB\#) \;_p\Rightarrow (p, abbcc, AbB\#)$



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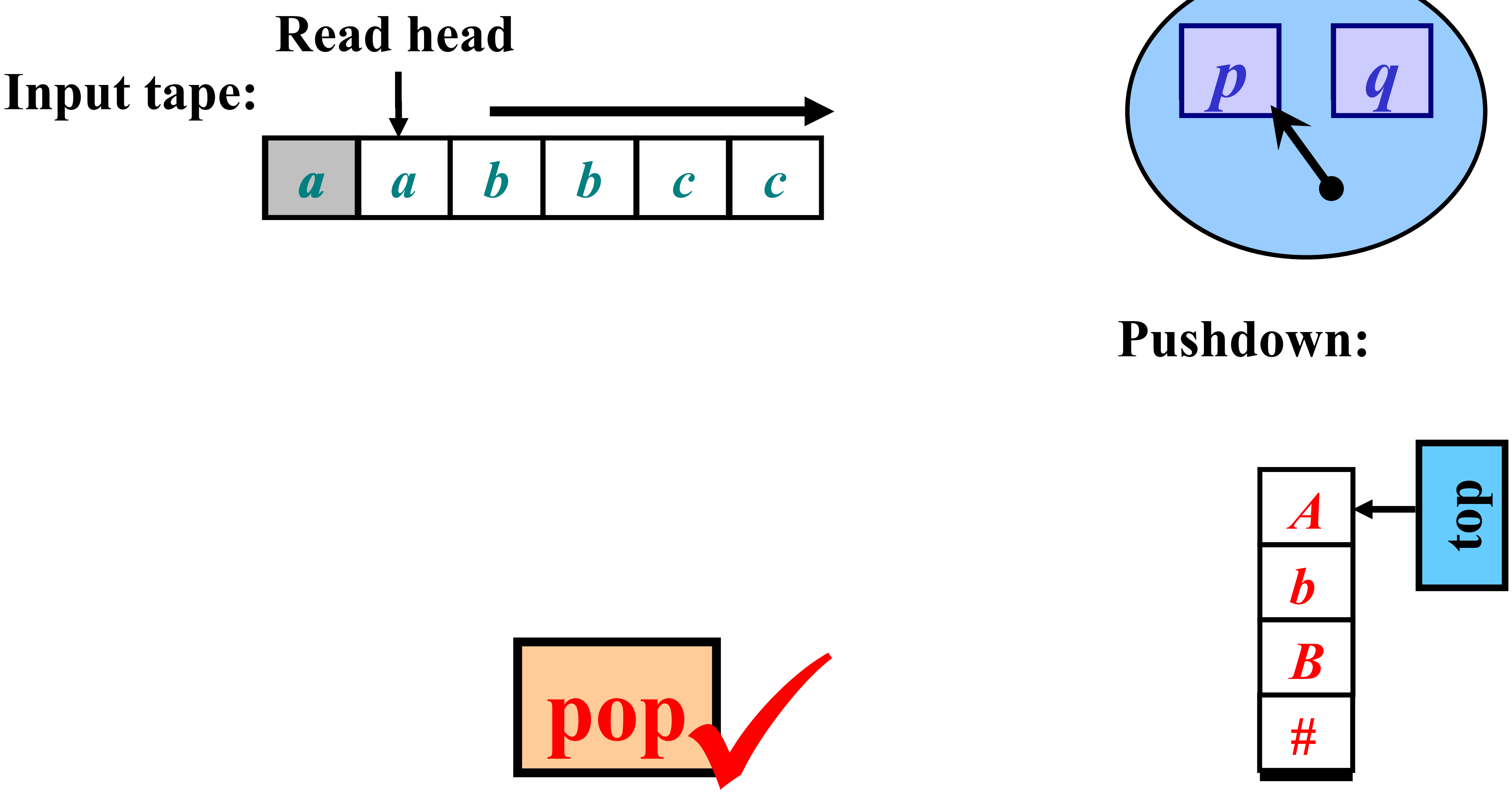
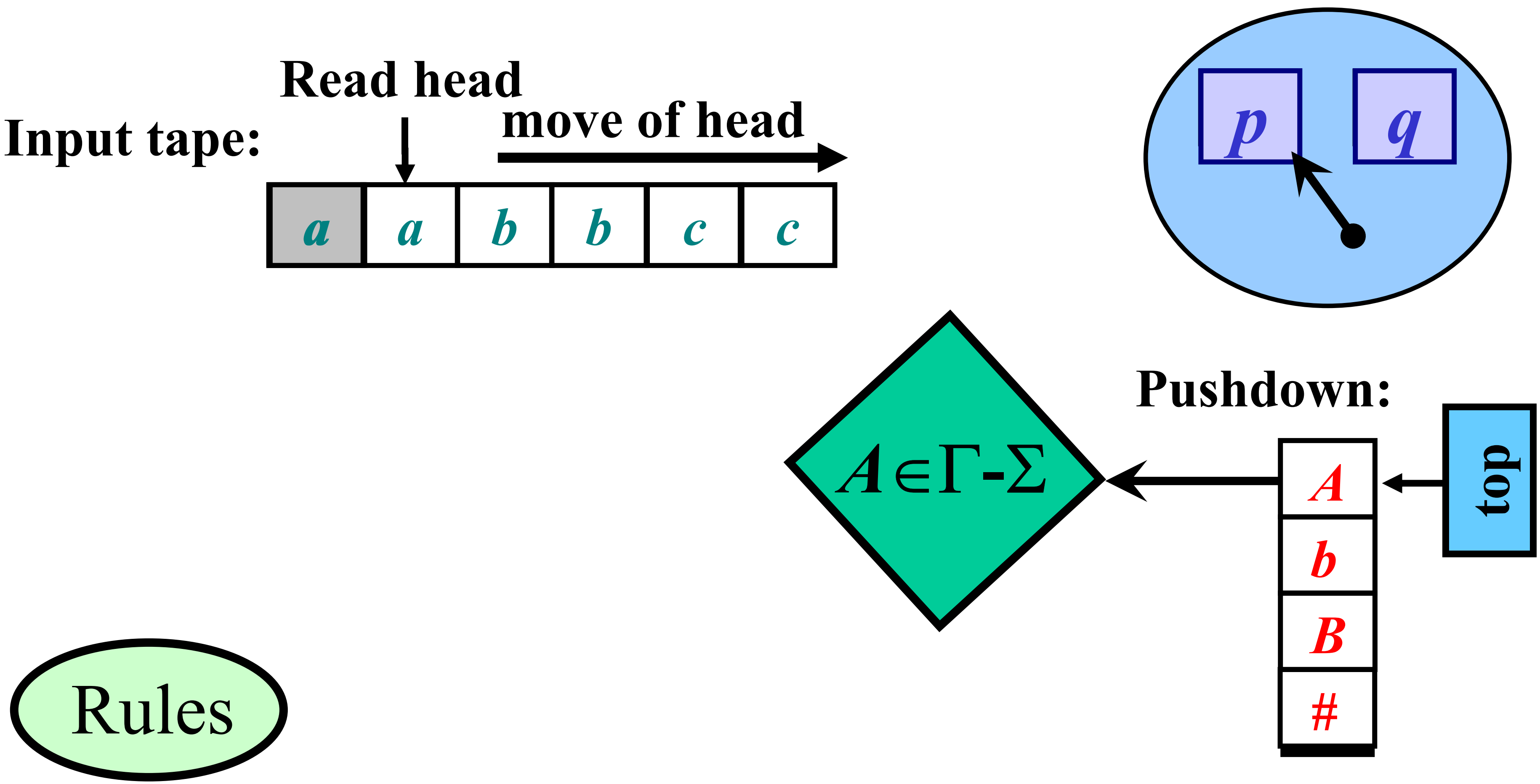


Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#)\ {}_e\!\Rightarrow (q, abbcc, AbBc\#)$ $[2pB \rightarrow qBc]$



Deep Expansion: Illustration

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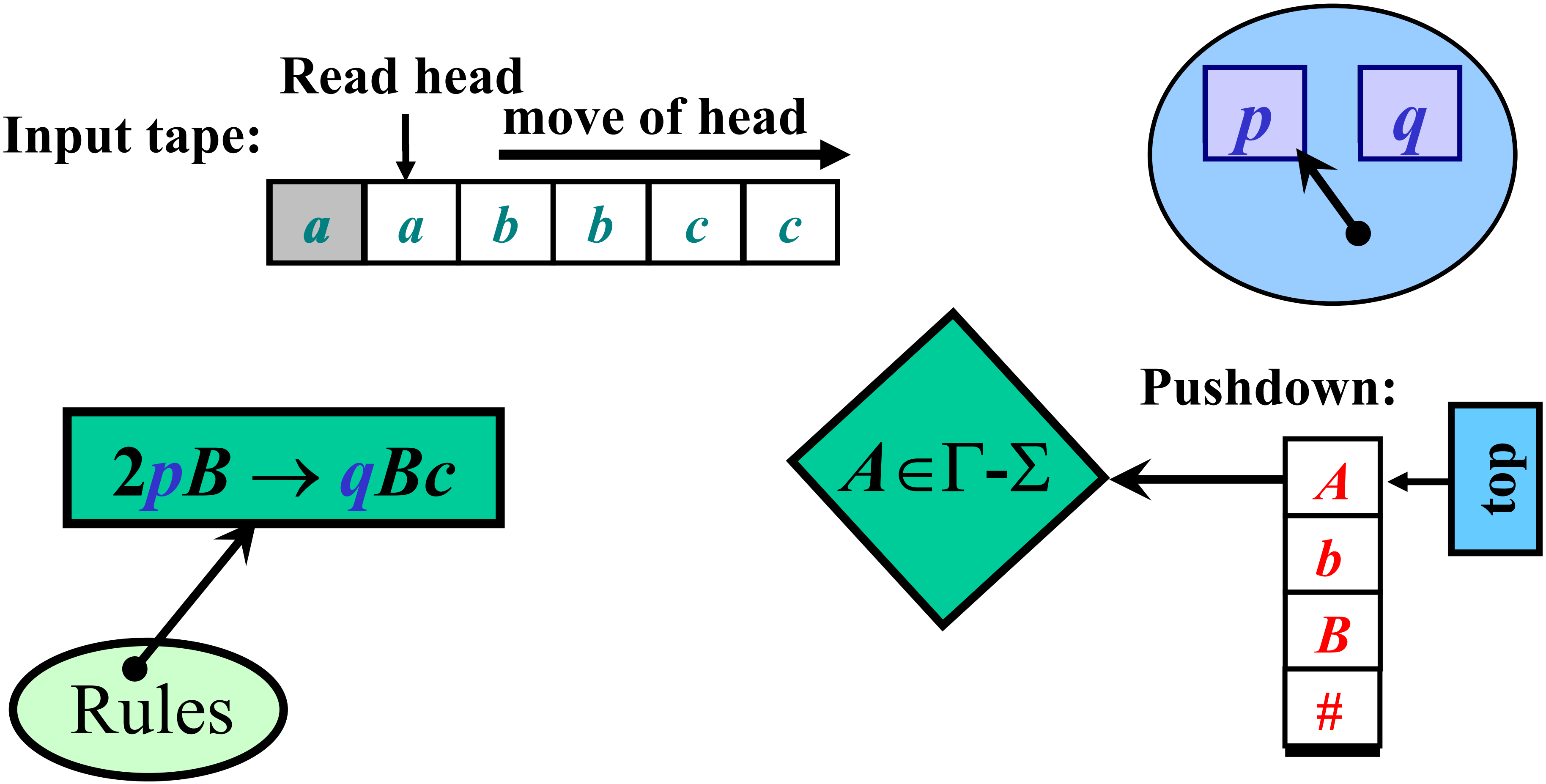

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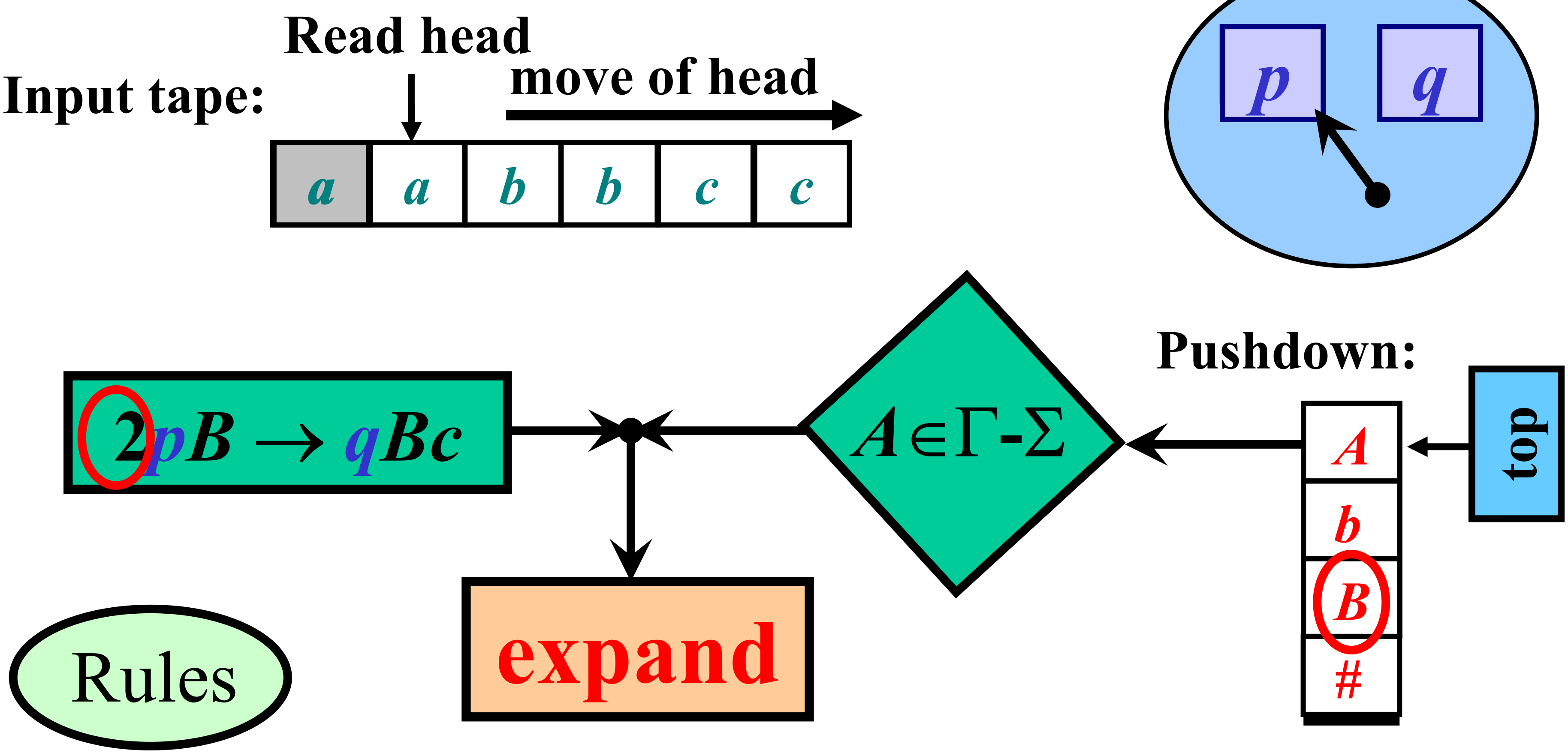
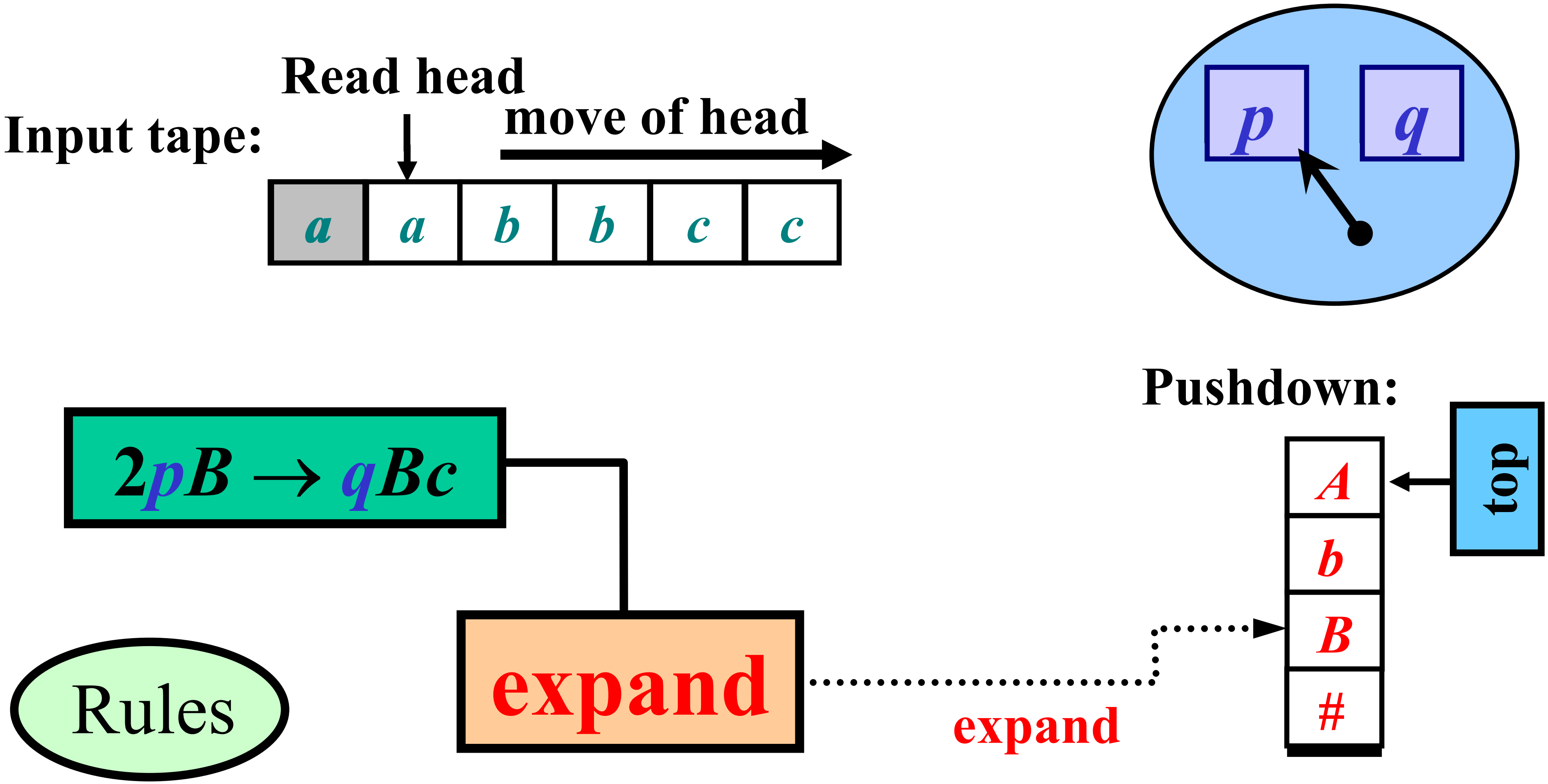


Deep Expansion: Illustration

Move: (p, $abbcc$, $AbB\#$) $_e\Rightarrow$ (q, $abbcc$, $AbBc\#$) [$2pB \rightarrow qBc$]



Deep Expansion: Illustration

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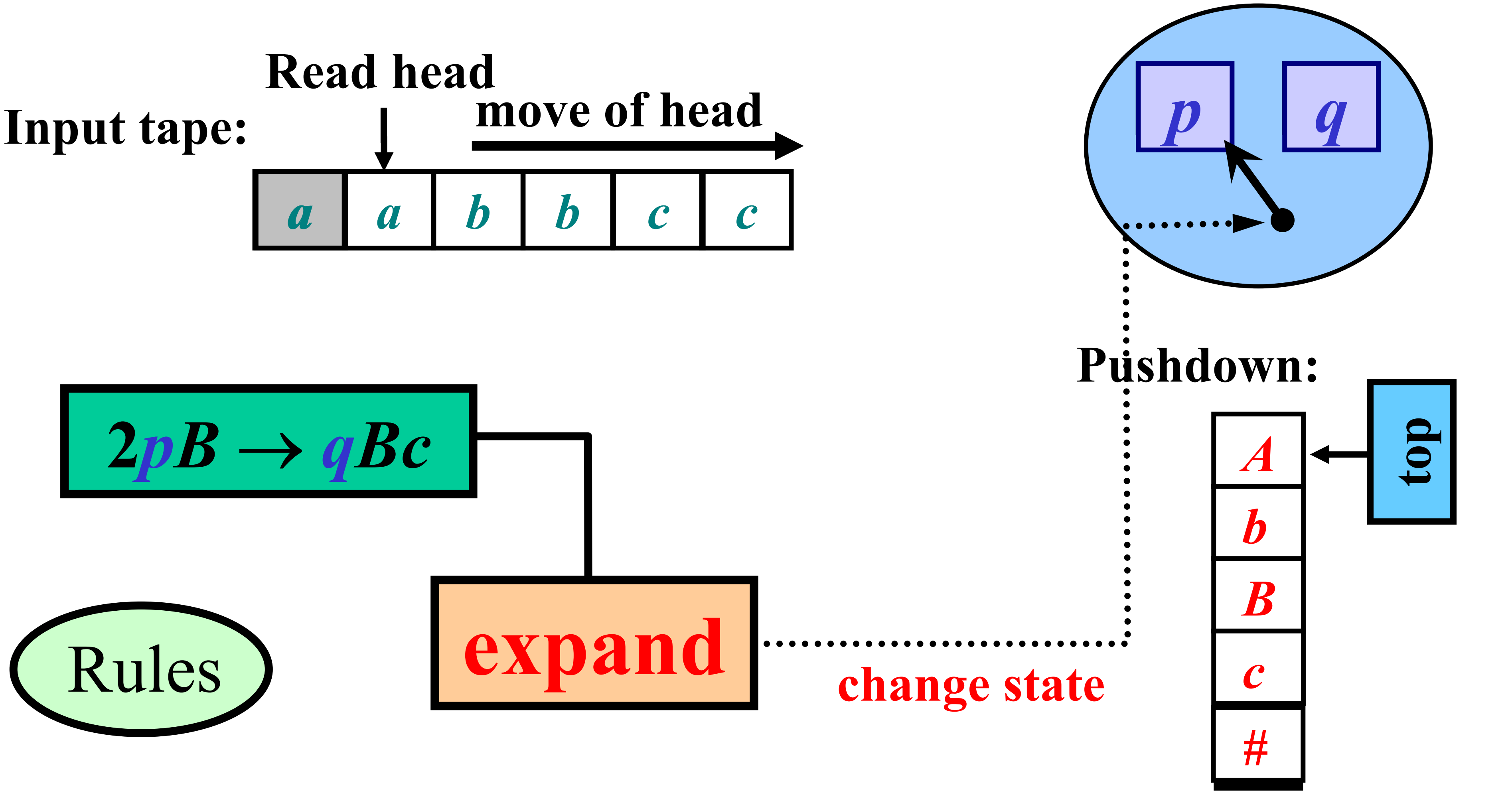
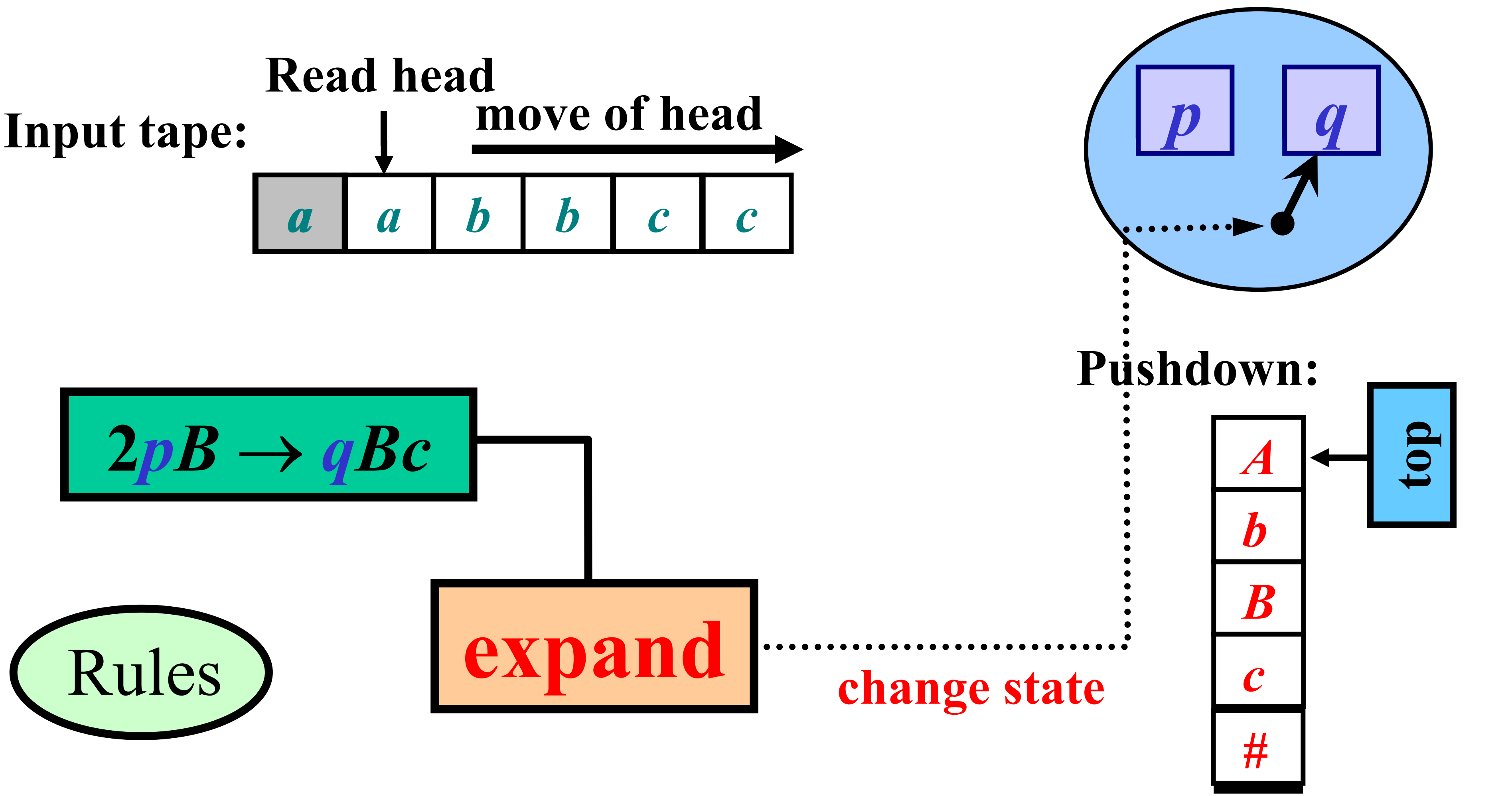


Deep Expansion: Illustration

Move: $(p, abbcc, AbB\#)\ _e\Rightarrow (q, abbcc, AbBc\#)\ [2pB \rightarrow qBc]$



Deep Expansion: Illustration

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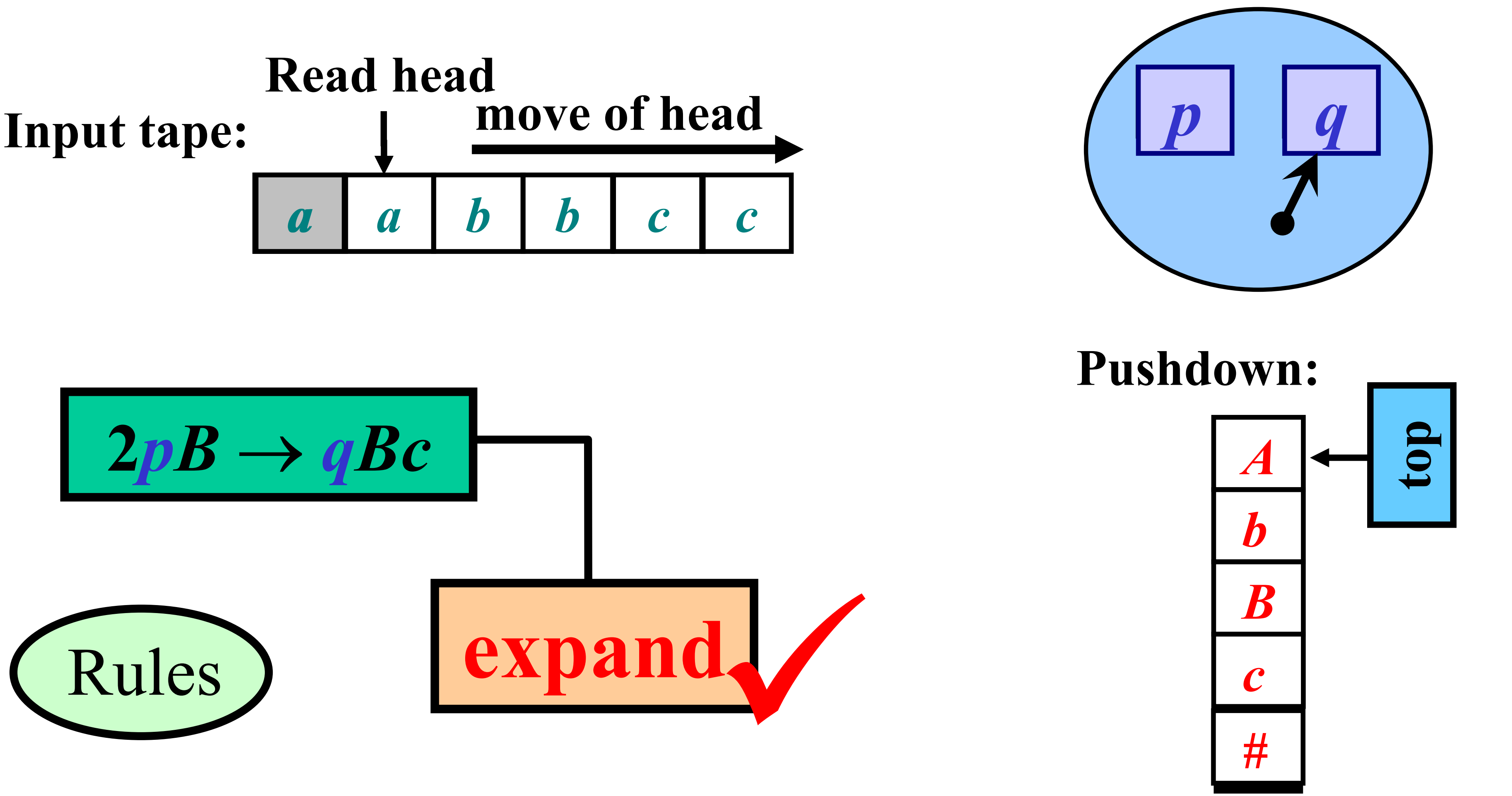

Example: Deep PDA

Deep PDA M:

[1]. $1sS \rightarrow qAB$
[2]. $1qA \rightarrow paAb$
[3]. $1qA \rightarrow fab$
[4]. $2pB \rightarrow qBc$
[5]. $1fB \rightarrow fc$

M accepts $aabbcc$:

$(s, aabbcc, S\#)$
${}_e\Rightarrow (q, aabbcc, AB\#)$ [1]
${}_e\Rightarrow (p, aabbcc, aAbB\#)$ [2]
${}_p\Rightarrow (p, abbcc, AbB\#)$
${}_e\Rightarrow (q, abbcc, AbBc\#)$ [4]
${}_e\Rightarrow (f, abbcc, abbBc\#)$ [3]
${}_p\Rightarrow (f, bbcc, bbBc\#)$
${}_p\Rightarrow^2 (f, cc, Bc\#)$
${}_e\Rightarrow (f, cc, cc\#)$ [5]
${}_p\Rightarrow (f, c, c\#)$
${}_p\Rightarrow (f, \varepsilon, \#)$

$$L(M) = \{a^n b^n c^n: n \geq 1\} \in PD_2$$

Definition 1/3

A deep pushdown automaton is a 7-tuple

$$M = (Q, \Sigma, \Gamma, R, s, S, F), \text{ where}$$

- Q – *states*,
- $\Sigma \subseteq \Gamma$ – *input alphabet*,
- Γ – *pushdown alphabet*, *bottom symbol* $\# \in \Gamma - \Sigma$
- R – *finite set of rules* of the form

$$mqA \rightarrow pw \quad \text{or} \quad mq\# \rightarrow pv\#$$

- $s \in Q$ – *start state*
- $S \in \Gamma$ – *start pushdown symbol*
- $F \subseteq Q$ – *final states*

Definition 2/3

- if an input symbol is on pd top, ***M*** **pops** the pd as

$$(q, au, az)\ {}_{p}\Rightarrow (q, u, z), \quad a \in \Sigma$$

- no explicit rule needed in R

- if a non-input symbol is on pd top, ***M*** **expands** the pd as

$$(q, w, uAz)\ {}_{e}\Rightarrow (p, w, uvz)\ [mqA \to pv],$$

where u contains $m - 1$ non-input symbols

Definition 3/3

- $\boldsymbol{M}$ is ***of depth n***, denotated by ${}_{\boldsymbol{n}}\boldsymbol{M}$, if n is the minimal positive integer such that each of M's rules is of depth n or less.

- Language accepted by ${}_nM$, $L({}_nM)$, is defined as

$$L({}_nM) = \{\boldsymbol{w} \in \Sigma^*: (\boldsymbol{s}, \boldsymbol{w}, \boldsymbol{S}\#) \Rightarrow^* (\boldsymbol{f}, \varepsilon, \#) \text{ in } {}_nM \text{ with } \boldsymbol{f} \in F\}.$$

Main Result and its Proof

- PD_n – the language family defined by DeepPDAs of depth n.

Theorem: $PD_n \subset PD_{n+1}$, for all $n \geq 1$.

Proof (Sketch):

- State grammars (Kasai, 1970) are needed in the proof
- State grammar is a modification of CFG based on rules of the form

$$(q, A) \rightarrow (p, v)$$

Proof 1/6: State Grammar

- ***State grammar*** $G = (V, W, T, P, S)$
  - V – total alphabet, W – states, $T \subseteq V$ – terminals,
  - P – set of rules of the form $(q, A) \to (p, v)$
  - $S \in (V - T)$ – start symbol,

- ***Configuration*** – (q, x)
- ***Derivation step***:

$$(q, uAz) \Rightarrow (p, uvz) \; [(q, A) \to (p, v)]$$

and for every nonterminal B in u, P contains no rule with (q, B) on the left-hand side

Proof 2/6: n-limited Step

- ***n-limited derivation step***:
  each derivation step within the first $\boldsymbol{n}$ non-terminals

$$(q, uAz)\ {}_n\!\Rightarrow (p, uvz) \text{ and}$$

uA has $\boldsymbol{n}$ or fewer non-terminals

- ***n-limited state language:***

$$L(G, n) = \{w \in T^*: (q, S)\ {}_n\!\Rightarrow^* (p, w)\}$$

- ST_n – the family of n-limited state languages

Proof 3/6: Example

State Grammar G:

[1]. $(1, S) \rightarrow (2, AC)$

[2]. $(2, A) \rightarrow (3, aAb)$

[3]. $(2, A) \rightarrow (4, ab)$

[4]. $(3, C) \rightarrow (2, Cc)$

[5]. $(4, C) \rightarrow (4, c)$

$W = \{1, 2, 3, 4\}$

G generates $aabbcc$:

$(S, 1) \Rightarrow (AC, 2)$ [1]

$\Rightarrow (aAbC, 3)$ [2]

$\Rightarrow (aAbCc, 2)$ [4]

$\Rightarrow (aabbCc, 4)$ [3]

$\Rightarrow (aabbcc, 4)$ [5]

$$L(G) = \{a^n b^n c^n : n \geq 1\} \in ST_2$$

Proof 4/6: $\boldsymbol{PD_n} \subseteq \boldsymbol{ST_n}$, $n \geq 1$

- G simulates the application of $\boldsymbol{ipA} \rightarrow \boldsymbol{qy} \in R$:
  - make a left-to-right scan of the pd until the $\boldsymbol{i}$th occurence of a non-terminal
  - if $X_{\boldsymbol{i}} = \boldsymbol{A}$, then replace $\boldsymbol{A}$ with $\boldsymbol{y}$ and return to the beginning of the sentential form
  - rightmost symbol is always a special $\boldsymbol{a'}$, and G completes the simulation by changing $\boldsymbol{a'}$ to $\boldsymbol{a}$

Proof 5/6: $\boldsymbol{ST_n} \subseteq \boldsymbol{PD_n}$, $n \geq 1$

- ${}_nM$ simulates the n-limited derivations of G in pd:
  - always records the first $\boldsymbol{n}$ non-terminals from the current sentential form of G in its state
  - fewer than $\boldsymbol{n}$ non-terminals are extended by #s
  - reads the string, empties pd, enters $\$ \in F$

Proof 6/6: $PD_n \subset PD_{n+1}$, $n \geq 1$

1) As $PD_n \subseteq ST_n$ and $ST_n \subseteq PD_n$ for all $n \geq 1$, $ST_n = PD_n$.

2) Kasai (1970): $ST_n \subset ST_{n+1}$, for all $n \geq 1$.

For all $n \geq 1$, $PD_n = ST_n \subset ST_{n+1} = PD_{n+1}$

Q. E. D.

Open Problem Areas

- Determinism
- Rules of form $mqA \rightarrow p\varepsilon$

Discussion and End

Any questions, please?