Efficient Inclusion Checking on Explicit and Semi-Symbolic Tree Automata

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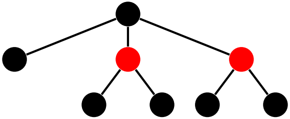
Outline

1. Tree Automata
2. Downward Inclusion Checking
3. Semi-Symbolic Encoding of Non-Deterministic TA
4. Conclusion

Trees

Very popular in computer science:

- data structures,
- computer network topologies,
- distributed protocols, . . .



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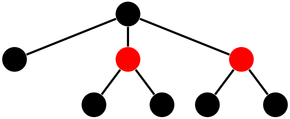

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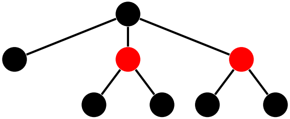

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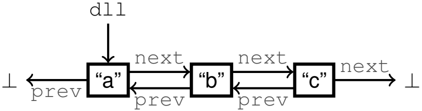

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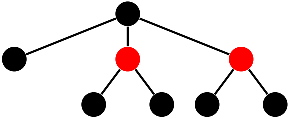

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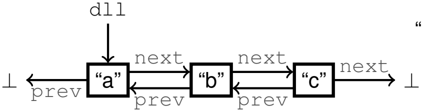

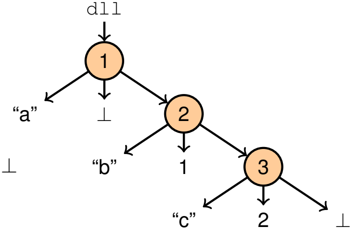

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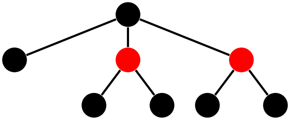

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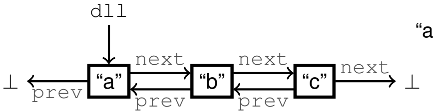

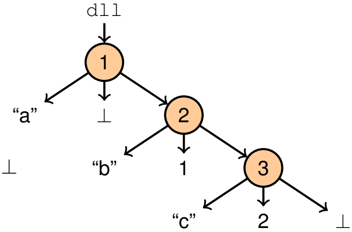

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Finite Tree Automaton (TA): $\mathcal{A} = (Q, \Sigma, \Delta, F)$

- extension of finite automaton to trees:
  - Q . . . set of states,
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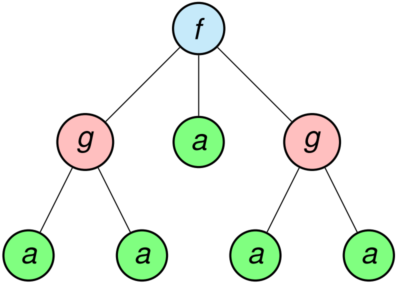

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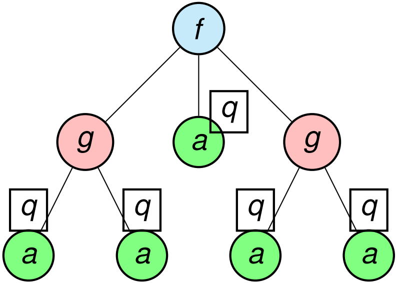

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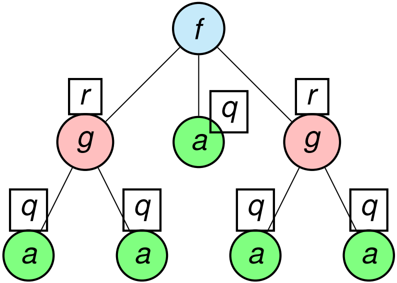

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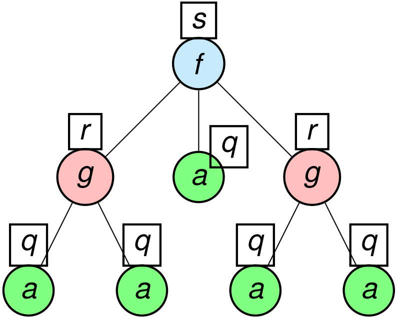

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- handling large alphabets (MSO, WSkS).

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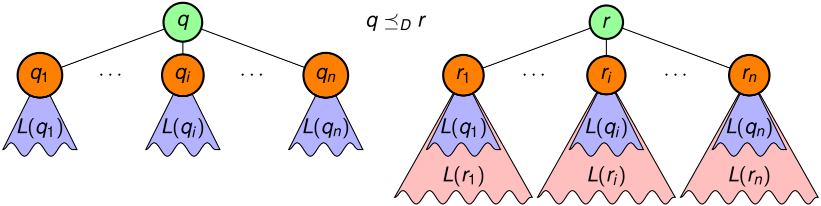

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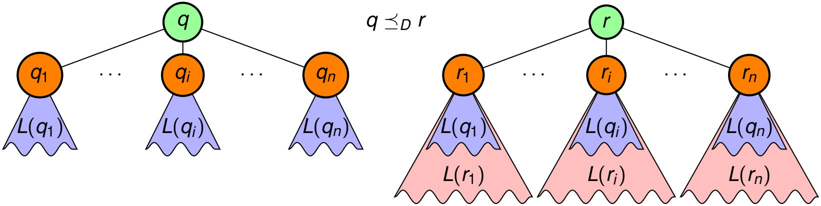

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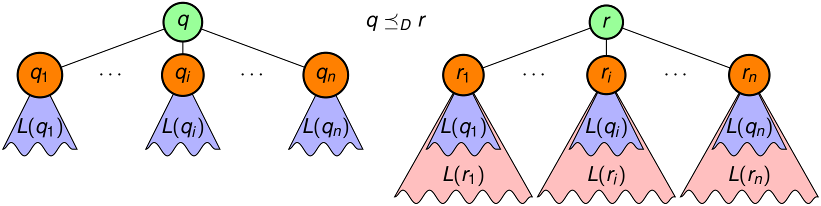

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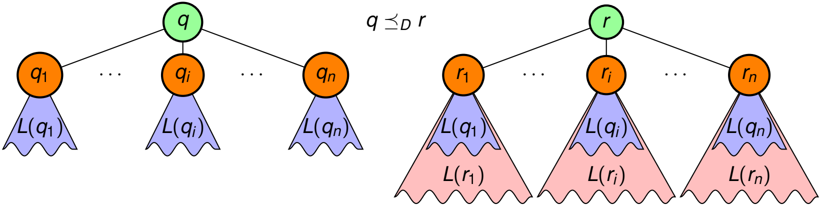

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  - upward simulation
    - not compatible with language inclusion,
    - but can be used to speed up exact checking

Approaches to Checking Tree Automata Inclusion

Exact: **EXPTIME-complete** . . .

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  - antichains[1]

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[1] M. De Wulf, L. Doyen, T. Henzinger, J.-F. Raskin. Antichains: A New Algorithm for Checking Universality of FA. CAV'06.

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  - antichains[1]
  - antichains combined with simulation[2,3]

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[2] L. Doyen, J.-F. Raskin. Antichain Algorithms for Finite Automata. TACAS'10.

[3] P. Abdulla, Y.-F. Chen, L. Holík, R. Mayr, T. Vojnar. When Simulation Meets Antichains. TACAS'10.

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Textbook algorithm for checking $\mathcal{L}(\mathcal{A}_S) \subseteq \mathcal{L}(\mathcal{A}_B)$ on TA:

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5. If you encounter (f, R), where $f \in F_S, R \cap F_B = \emptyset$, return false.
6. If no new pairs are found, return true.

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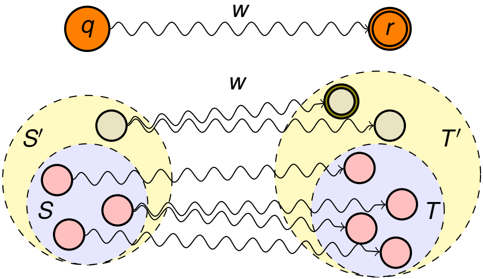

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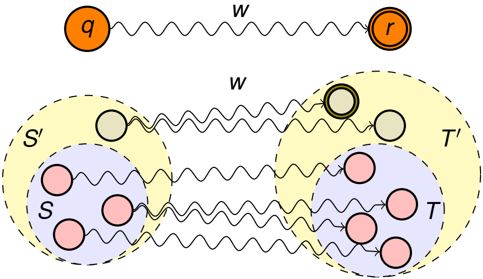

2. use simulation to furter prune the searched space.

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- Upward simulation → hard to compute and too strong. ☹
- Not compatible with downward simulation (easy & rich). ☹

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[4]

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- inspired by XML Schema containment checking[4],

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- uses antichains and downward simulation.

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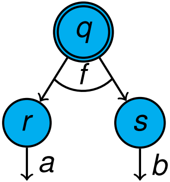
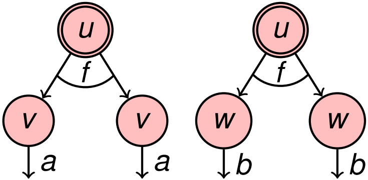


$\mathcal{A}_B$

$\underline{u} \xrightarrow{f} (v, v)$
$\underline{u} \xrightarrow{f} (w, w)$
$v \xrightarrow{a}$
$w \xrightarrow{b}$



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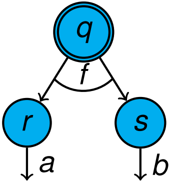

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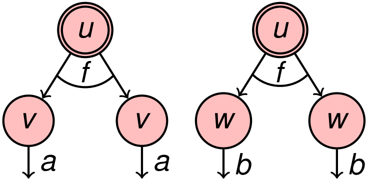

$\mathcal{L}(q) \subseteq \mathcal{L}(u)$ if and only if

$$\mathcal{L}(r) \times \mathcal{L}(s) \subseteq (\mathcal{L}(v) \times \mathcal{L}(v)) \cup (\mathcal{L}(w) \times \mathcal{L}(w))$$

(language inclusion of tuples!)

Checking language inclusion of tuples

Note that in general

$$(\mathcal{L}(v_1)\times\mathcal{L}(v_2))\cup(\mathcal{L}(w_1)\times\mathcal{L}(w_2)) \neq (\mathcal{L}(v_1)\cup\mathcal{L}(w_1))\times(\mathcal{L}(v_2)\cup\mathcal{L}(w_2))$$

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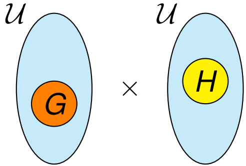

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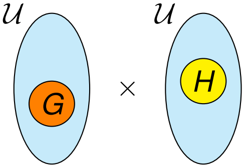

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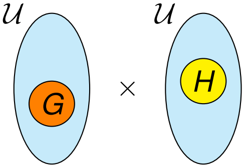

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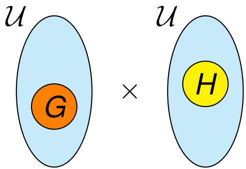

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Using distributive laws, this becomes

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Basic Downward Inclusion Checking Algorithm

- DFS, maintain a workset W of product states (q_S, P_B).

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  - for all clauses …
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- Sooner or later, the DFS either
  - reaches a leaf, or
  - reaches a pair (q_S, P_B) which is already in W.

Optimised Downward Inclusion Checking Algorithm

Optimisations:

1. It is possible to maintain a cache *NN* of pairs (q_S, P_B) for which $\mathcal{L}(q_S) \not\subseteq \mathcal{L}(P_B)$ has been shown and prune the search.

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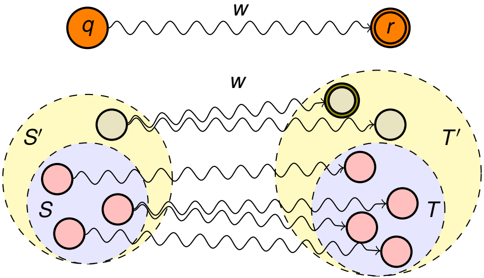

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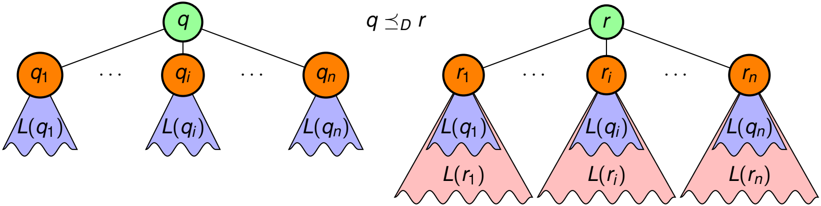

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5. Even further, if $\exists s \in S : q \preceq_D s$, then surely $\mathcal{L}(q) \subseteq \mathcal{L}(S)$.

Experiments

Size	50–250	400–600
Pairs	323	64
Timeout	20 s	60 s
`Up`	31.21 %	9.38 %
`Up+s`	0.00 %	0.00 %
`Down`	53.50 %	39.06 %
`Down+s`	15.29 %	51.56 %
Avg up	1.71	0.34
Avg down	3.55	46.56

a)

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`Down+s`	18.18 %	79.69 %
Avg up	1.33	9.92
Avg down	3.60	2116.29

b)

a) Comparison of methods (w/ simulation computation time).

b) Comparison of methods (w/o simulation computation time).

Semi-Symbolic TA

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- decision procedures of some logics: WSkS, MSO.

Current approach:

- use the MONA tree automata package (MTBDD-based)
- But only deterministic automata supported →
  - often runs out of reasonable memory or time.

Dual representation

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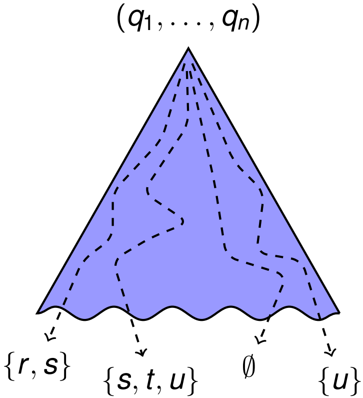
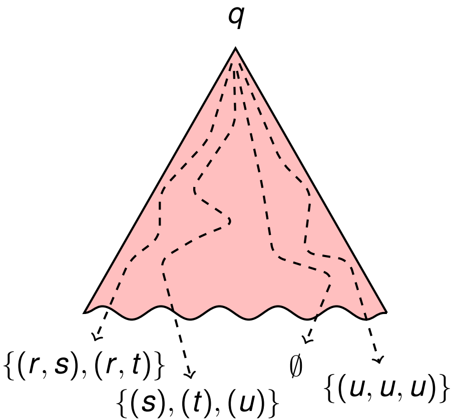
- Bottom-up:
- Top-down:





Bottom-up : inspired by MONA, but has sets of states in leaves.

Top-down : sets of state tuples in leaves.

Semi-Symbolic Encoding of Non-Deterministic TA

Algorithms for

- union,
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  - based on M. Henzinger, T. Henzinger, and P. Kopke's algorithm.

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Experiments:

- Use of CUDD to implement MTBDDs.
- $\sim$ 8500 times faster downward inclusion checking than explicit representation for tested automata with large alphabets.

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- …that makes use of antichains and downward simulation.
- A new symbolic encoding of non-deterministic tree automata proposed.

Future work

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- Improve the symbolic downward simulation algorithm.
- Create a tree automata package replacing MONA.

Thank you for your attention.

Questions?