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A categorical approach to convergence: Compactness

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ABSTRACT

A convergence class is introduced on every object of a given category by using certain generalized nets for expressing the convergence. The obtained concrete category is then investigated whose objects are the pairs consisting of objects of the original category and convergence classes on them and whose morphisms are the morphisms of the original category that preserve the convergence. We define, in a natural way, separation and compactness of objects of the concrete category under investigation and study their behavior.

1. Introduction

The study of topological structures on (objects of) categories represents an important branch of categorical topology. It was initiated by D. Dikranjan and E. Giuli [4] who introduced and studied closure operators on categories. These operators were then investigated by a number of authors (see [6] and the references therein) who contributed to the development of the theory of categorical closure operators. In particular, some of these authors studied separation and compactness with respect to a categorical closure operator – see e.g. [2,3,5]. Later on, some more topological structures on categories were introduced and studied including convergence structures [16], neighborhood structures [10], and interior operators [20].

In the classical approach to the study of topological structures on (objects of) categories, categories with a given topological structure are considered and investigated. This approach is used also in [16] for the study of convergence on categories and related separation and compactness. Quite a different approach was used in [13–15] where concrete categories over *Set* were studied obtained by providing every set with a convergence and introducing continuous, i.e., convergence preserving maps. In the present paper, we use the approach of [13–15] but, instead of Set, an arbitrary category is considered (with no topological structure

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in general) and a convergence is newly defined for each of its objects. We obtain a concrete category over the original one whose objects are pairs consisting of objects of the original category and convergence on them and whose morphisms are the morphisms of the original category that preserve the convergence. Basic properties of the concrete category are investigated in [17]. In the present paper, we focus on the study of naturally defined convergence separation and compactness of objects of the concrete category obtained. We will show that the separation and compactness behave analogously to and even better than the usual topological separation and compactness.

2. Preliminaries

For the convenience of the reader, we repeat all the relevant definitions from [17], which makes our paper self-contained. For the categorical terminology used see [1] and [12].

Throughout the paper, we consider a category $\mathcal{K}$ with a terminal object $1_{\mathcal{K}}$. Further, we assume there is given a non-empty category $\mathcal{S}$ and a functor $\mathcal{F} : \mathcal{S} \to \mathcal{K}$.

Definition 1. For an arbitrary $\mathcal{K}$-object K, by an $\mathcal{F}$-*net* in K we understand any object of the comma category $(\mathcal{F} \downarrow K)$, i.e., any object $\mathcal{F}$-over K. Given a pair $\langle S, f\rangle, \langle T, g\rangle$ of $\mathcal{F}$-nets in K, $\langle S, f\rangle$ is said to be a *subnet* of $\langle T, g\rangle$ provided that there is a morphism from $\langle S, f\rangle$ to $\langle T, g\rangle$ in $(\mathcal{F} \downarrow K)$.

Example 1. Here and also in the other examples, we will denote by *Dir* the category of directed sets and cofinal maps and by $\underline{\omega}$ the subcategory of *Dir* whose only object is the least infinite ordinal ω and whose morphisms are the isotone injections.

(1) Let $\mathcal{K} = \mathit{Set}$, $\mathcal{S} = \mathit{Dir}$ and let $\mathcal{F} : \mathit{Dir} \to \mathit{Set}$ be the forgetful functor. Then $\mathcal{F}$-nets in a $\mathcal{K}$-object X and their subnets are the usual nets in the set X and their subnets (see e.g. [11]). If we replace *Dir* by $\underline{\omega}$, then we get the usual sequences and subsequences.

(2) Let $\mathcal{S}$ be a construct and let $\mathcal{F} : \mathcal{S} \to \mathit{Set}$ be the forgetful functor. Then $\mathcal{F}$-nets in a $\mathcal{K}$-object X and their subnets coincide with $\mathcal{S}$-nets in X and their subnets discussed in [13–15].

(3) The $\mathcal{B}$-nets in a set X and their $\mathcal{B}$-subnets from [21] are nothing but $\mathcal{F}$-nets in X and their subnets where $\mathcal{B}$ is a subconstruct of *Dir* and $\mathcal{F} : \mathcal{B} \to \mathit{Set}$ is the forgetful functor.

(4) Let $\mathcal{K} = \mathit{Set}$, let $\mathcal{S}$ be the category of compact Hausdorff topological spaces (and continuous maps) and let $\mathcal{F} : \mathcal{S} \to \mathit{Set}$ be the forgetful functor. A quasi-topology [18] on a set X is nothing but a collection $(Q(S, X))_{S\in\mathcal{S}}$ where, for each $\mathcal{S}$-object S, $Q(S, X)$ is a set of $\mathcal{F}$-nets $\langle S, f\rangle$ in X satisfying certain given axioms.

For any $\mathcal{K}$-object K, we denote by $\cong$ the usual equivalence between subobjects of K (i.e., monomorphisms in $\mathcal{K}$ with the codomain K) and by K^* the class of all ($\cong$-equivalence classes of) points of K, i.e., subobjects of K whose domains are terminal objects.

Definition 2. Let K be a $\mathcal{K}$-object. A subclass $\pi \subseteq Ob(\mathcal{F} \downarrow K) \times K^*$ is said to be a *convergence class* on K if the following two conditions are satisfied:

(i) If $\langle S, f\rangle$ is an $\mathcal{F}$-net in K such that f factors through a point $x \in K^*$ (i.e., f is a constant morphism), then $(\langle S, f\rangle, x) \in \pi$.

(ii) If $(\langle S, f\rangle, x) \in \pi$, then $(\langle T, g\rangle, x) \in \pi$ for every subnet $\langle T, g\rangle$ of $\langle S, f\rangle$.

If π is a convergence class on a $\mathcal{K}$-object K, then we write $\langle S, f\rangle \xrightarrow{\pi} x$ instead of $(\langle S, f\rangle, x) \in \pi$ and say that $\langle S, f\rangle$ *converges* to x with respect to π. An $\mathcal{F}$-net $\langle S, f\rangle$ in K is said to be *convergent* w.r.t. π if there is a point $x \in K^*$ with $\langle S, f\rangle \xrightarrow{\pi} x$.

Let K, L be $\mathcal{K}$-objects and let π and ρ be convergence classes on K and L, respectively. A $\mathcal{K}$-morphism $\varphi : K \to L$ is said to be *continuous* (w.r.t. π and ρ) if $\langle S, f\rangle \xrightarrow{\pi} x$ implies $\langle S, \varphi \circ f\rangle \xrightarrow{\rho} \varphi \circ x$. We denote by $Conv_{\mathcal{F}}$ the category with objects the pairs (K, π) where K is a $\mathcal{K}$-object and π is a convergence class on K and with morphisms $\varphi : (K, \pi) \to (L, \rho)$ the continuous (w.r.t. π and ρ) $\mathcal{K}$-morphisms $\varphi : K \to L$. Note that the objects of $Conv_{\mathcal{F}}$ may not form a class so that, according to the terminology introduced in [1], $Conv_{\mathcal{F}}$ is a so-called quasicategory rather then a category. Since all the categorical concepts used may naturally be extended to quasicategories, we will avoid using the concept of a quasicategory here, i.e., we will call the quasicategory $Conv_{\mathcal{F}}$ simply a category. Similarly, (full) subquasicategories of $Conv_{\mathcal{F}}$ will be called (full) subcategories of $Conv_{\mathcal{F}}$ or briefly categories.

We denote by $Lim_{\mathcal{F}}$ the full subcategory of $Conv_{\mathcal{F}}$ whose objects are the pairs (K, π) where K is a $\mathcal{K}$-object and π is a convergence class on K satisfying the following (Urysohn) axiom:

For any $\mathcal{F}$-net $\langle S, f\rangle$ in K and any point $x \in K^*$, $\langle S, f\rangle \xrightarrow{\pi} x$ whenever every subnet of $\langle S, f\rangle$ has a subnet converging to x (w.r.t. π).

It may easily be seen that the categories $Conv_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$ are topological (over $\mathcal{K}$) and $Lim_{\mathcal{F}}$ is concretely reflective in $Conv_{\mathcal{F}}$. In [17], sufficient conditions are given under which these categories are cartesian closed.

Definition 3. A $Conv_{\mathcal{F}}$-object (K, π) is said to be

(a) *separated* if, for every $\mathcal{F}$-net $\langle S, f\rangle$ in K, from $\langle S, f\rangle \xrightarrow{\pi} x$ and $\langle S, f\rangle \xrightarrow{\pi} y$ it follows that $x \cong y$,
(b) *compact* if every $\mathcal{F}$-net in K has a subnet which is convergent w.r.t. π.

We denote by $HConv_{\mathcal{F}}$ and $HLim_{\mathcal{F}}$ the full subcategory of $Conv_{\mathcal{F}}$ whose objects are the separated objects of $Conv_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$, respectively.

Example 2. (1) Let $\mathcal{S} = \underline{\omega}$ and let $\mathcal{F} : \underline{\omega} \to Set$ be the forgetful functor. Then the objects of $HConv_{\mathcal{F}}$ and $HLim_{\mathcal{F}}$ are nothing but the well-known Fréchet $\mathcal{L}$-spaces and $\mathcal{L}^*$-spaces, respectively (see [9] and [19]). The convergence spaces studied in [7] are nothing but the compact objects of $HLim_{\mathcal{F}}$.

(2) In [8], the construct of L*-spaces and continuous maps is studied where L*-spaces are obtained from $\mathcal{L}^*$-spaces by replacing sequences with the usual nets. Thus, this construct coincides with the category $Lim_{\mathcal{F}}$ where $\mathcal{F} : Dir \to Set$ is the forgetful functor.

(3) The $\mathcal{B}$-convergence structures studied in [20] for special subcategories $\mathcal{B}$ of Dir are nothing but the objects of $Conv_{\mathcal{F}}$ where $\mathcal{F} : \mathcal{B} \to Set$ is the forgetful functor

(4) Let $\mathcal{S} = Dir$ or $\mathcal{S} = \underline{\omega}$ and let $\mathcal{F} : \mathcal{S} \to Set$ be the forgetful functor. Let $(X, \mathcal{O})$ be a topological space (given by the set $\mathcal{O}$ of open subsets). For an $\mathcal{F}$-net $\langle S, f\rangle$ in X, put $\langle S, f\rangle \xrightarrow{\pi} x$ if and only if, for every $A \in \mathcal{O}$ with $x \in A$, there exists $s_A \in \mathcal{F}S$ such that $f(s) \in A$ for every $s \in \mathcal{F}S$ with $s \geq s_A$. Then (X, π) is an object of $Lim_{\mathcal{F}}$ – we say that the convergence class π is associated with the topology $\mathcal{O}$. If $(X, \mathcal{O})$ is separated or compact, respectively, in the usual topological sense, then it is a separated or compact, respectively, object of $Lim_{\mathcal{F}}$ (and vice versa if $\mathcal{S} = Dir$).

(5) If $\mathcal{S}$ is a construct and $\mathcal{F} : \mathcal{S} \to Set$ the forgetful functor, then $Conv_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$ coincide with the categories $Conv_{\mathcal{S}}$ and $Lim_{\mathcal{S}}$, respectively, studied in [13–15].

Definition 4. An embedding $\varphi : (K, \pi) \to (L, \rho)$ in $Conv_{\mathcal{F}}$ (i.e., an initial morphism with a monomorphic underlying $\mathcal{K}$-morphism $\varphi : K \to L$) is said to be *closed* if from $\langle S, f\rangle \xrightarrow{\rho} x$ where f factors through φ it follows that also x factors through φ.

Clearly, every isomorphism in $Conv_{\mathcal{F}}$ is closed and closed embeddings in $Conv_{\mathcal{F}}$ are closed under composition.

Example 3. Let $\mathcal{F} : Dir \to Set$ or $\mathcal{F} : \underline{\omega} \to Set$ be the forgetful functor. Let $(X, \mathcal{O})$, $(Y, \mathcal{P})$ be topological spaces and let $\varphi : (X, \pi) \to (Y, \rho)$ be an embedding in $Lim_{\mathcal{F}}$ where π is associated with the topology $\mathcal{O}$ and ρ is associated with the topology $\mathcal{P}$. Then φ is closed if and only if $\varphi(X)$ is a closed or sequentially closed subset, respectively, of the topological space (Y, ρ).

Let $\mathcal{K}$ have finite products and let (K_i, π_i), $i = 1, 2$, be $Conv_{\mathcal{K}}$-objects. We denote by $\pi_1 \times \pi_2$ the initial structure of the source $K_1 \times K_2 \to (K_i, \pi_i)$, $i = 1, 2$ (with respect to the forgetful functor $\mathcal{K} \to Conv_{\mathcal{K}}$, so that $(K_1 \times K_2, \pi_1 \times \pi_2)$ is the concrete product of (K_1, π_1) and (K_2, π_2)). Clearly, we have $\langle S, f\rangle \overset{\pi_1 \times \pi_2}{\to} x$ if and only if $\langle S, pr_i \circ f\rangle \overset{\pi_i}{\to} pr_i \circ x$ for $i = 1, 2$ (where pr_i denotes the i-th projection, $i = 1, 2$).

Theorem 1. *Let $\mathcal{K}$ have finite products. A $Conv_{\mathcal{F}}$-object (K, π) is separated if and only if the diagonal morphism $\Delta : (K, \pi) \to (K, \pi) \times (K, \pi)$ is closed.*

Proof. Let (K, π) be separated, let $\langle S, f\rangle$ be an $\mathcal{F}$-net in $K \times K$ with $\langle S, f\rangle \overset{\pi \times \pi}{\to} x$, and let f factor through Δ, say $f = \Delta \circ h$. Then $\langle S, pr_i \circ \Delta \circ h\rangle \overset{\pi_K}{\to} pr_i \circ x$ for $i = 1, 2$. Since $pr_1 \circ \Delta = pr_2 \circ \Delta = id_K$, we have $pr_1 \circ \Delta \circ h = pr_2 \circ \Delta \circ h = h$. Hence, $pr_1 \circ x \cong pr_2 \circ x$, which means $pr_1 \circ x = pr_2 \circ x$. Consequently, $x = \Delta \circ pr_1 \circ x (= \Delta \circ pr_2 \circ x)$. Therefore x factors through Δ, i.e., Δ is π-closed.

Conversely, let Δ be π-closed and let $\langle S, f\rangle$ be an $\mathcal{F}$-net in K with $\langle S, f\rangle \overset{\pi_K}{\to} x$ and $\langle S, f\rangle \overset{\pi_K}{\to} y$. Then $\langle S, [f, f]\rangle \overset{\pi_K \times K}{\to} z$ where $pr_1 \circ z = x$ and $pr_2 \circ z = y$. As $[f, f] = \Delta \circ f$, z factors through Δ, i.e., there exists a point $t \in K^*$ with $z = \Delta \circ t$. Consequently, $x = pr_1 \circ \Delta \circ t = pr_2 \circ \Delta \circ t = y$. Therefore, K is π-separated. □

Definition 5. A morphism $\varphi : (K, \pi) \to (L, \rho)$ in $Conv_{\mathcal{F}}$ is said to be *sublifting* if, for every $\mathcal{F}$-net $\langle S, f\rangle$ in K, $\langle S, \varphi \circ f\rangle \overset{\rho}{\to} y$ implies that there exists a subnet $\langle T, g\rangle$ of $\langle S, f\rangle$ such that $\langle T, g\rangle \overset{\pi}{\to} x$ where $x \in K^*$ is a point with $y \cong \varphi \circ x$.

Clearly, every isomorphism in $Conv_{\mathcal{K}}$ is sublifting and sublifting $Conv_{\mathcal{F}}$-morphisms are closed under composition.

The following assertion is evident:

Proposition 1. *If an embedding $\varphi : (K, \pi) \to (L, \rho)$ in $Conv_{\mathcal{F}}$ is sublifting, then it is closed.*

Example 4. Let $\mathcal{F} : Dir \to Set$ be the forgetful functor, let $(X, \mathcal{O})$ and $(Y, \mathcal{P})$ be topological spaces and let (X, π) and (Y, ρ) be the objects of $Lim_{\mathcal{F}}$ where π and ρ are the convergence classes associated with the topologies $\mathcal{O}$ and $\mathcal{P}$, respectively. A continuous map $\varphi : (X, \mathcal{O}) \to (Y, \mathcal{P})$ is closed (in the usual topological sense) if $\varphi : (X, \pi) \to (Y, \rho)$ is sublifting.

3. Compactness

Theorem 2. *Let $\varphi : (K, \pi) \to (L, \rho)$ be a closed embedding in $Conv_{\mathcal{F}}$. If (L, ρ) is compact, then so is (K, π).*

Proof. Let (L, ρ) be compact and let $\langle S, f\rangle$ be an $\mathcal{F}$-net in K. Then $\langle S, \varphi \circ f\rangle$ is an $\mathcal{F}$-net in L, thus there exists a subnet $\langle T, g\rangle$ of $\langle S, \varphi \circ f\rangle$ and a point $y \in L^*$ such that $\langle T, g\rangle \overset{\rho}{\to} y$. Consequently, there is a $\mathcal{K}$-morphism $h : T \to S$ with $g = \varphi \circ f \circ h$. Since φ is closed, there is a point $x \in K$ with $y = \varphi \circ x$. We have $\langle T, f \circ h\rangle \overset{\pi}{\to} x$ because φ is an embedding. As $\langle T, f \circ h\rangle$ is a subnet of $\langle S, f\rangle$, the proof is complete. □

Theorem 3. *Let $\mathcal{K}$ have finite products. If (K, π) is compact, then the projection $p_L : (K, \pi) \times (L, \rho) \to (L, \rho)$ is sublifting for every object (L, ρ) of $Conv_{\mathcal{F}}$.*

Proof. Let (K,π) be compact and let $\langle S,f\rangle$ be an $\mathcal{F}$-net in $(K,\pi)\times(L,\rho)$ such that $\langle S, pr_L\circ f\rangle \overset{\rho}{\to} y$ ($y\in L^*$ a point). Then there exists a subnet $\langle T, pr_K\circ f\circ\varphi\rangle$ of $\langle S, pr_K\circ f\rangle$ such that $\langle T, pr_K\circ f\circ\varphi\rangle \overset{\pi}{\to} x$ (where $x\in K^*$ is a point). Since $\langle T, pr_L\circ g\circ\varphi\rangle \overset{\rho}{\to} y$, we have $\langle T, f\circ\varphi\rangle \overset{\pi\times\rho}{\to} z$ where $z\in(K\times L)^*$ is the point with $x = pr_K\circ z$ and $y = pr_L\circ z$. As $\langle T, f\circ\varphi\rangle$ is a subnet of $\langle S,f\rangle$, the proof is complete. □

Theorem 4. *Let $\varphi:(K,\pi)\to(L,\rho)$ be a $Conv_{\mathcal{F}}$-morphism. If (K,π) is compact and (L,ρ) is separated, then φ is sublifting.*

Proof. Let (K,π) be compact and (L,ρ) be separated. Let $\langle S,f\rangle$ be an $\mathcal{F}$-net in K with $\langle S,\varphi\circ f\rangle \overset{\rho}{\to} y$. Then there is a subnet $\langle T, f\circ s\rangle$ of $\langle S,f\rangle$ such that $\langle T,f\circ s\rangle \overset{\pi}{\to} x$ (where $x\in K^*$ is a point). Since φ is continuous, we have $\langle T,\varphi\circ f\circ s\rangle \overset{\rho}{\to} \varphi\circ x$. But $\langle T,\varphi\circ f\circ s\rangle \overset{\rho}{\to} y$ because $\langle T,\varphi\circ f\circ s\rangle$ is a subnet of $\langle S,\varphi\circ s\rangle$. Hence, $y\cong\varphi\circ x$ because (L,ρ) is separated. □

Proposition 1 and Theorem 4 result in

Corollary 1. *Let $\varphi:(K,\pi)\to(L,\rho)$ be an embedding in $Conv_{\mathcal{F}}$. If (K,π) is compact and (L,ρ) is separated, then φ is closed.*

For every $\mathcal{K}$-object K, we denote by $\widehat{K}$ the coproduct $K+1_{\mathcal{K}}$ in $\mathcal{K}$ (provided it exists).

Definition 6. Let (K,π) be a $Conv_{\mathcal{F}}$-object. An embedding $i:(K,\pi)\to(\widehat{K},\rho)$ in $Conv_{\mathcal{F}}$ is said to be a *one-point compactification* of (K,π) if the following two conditions are satisfied:

1° $i: K\to K+1_{\mathcal{K}}$ is the canonical injection.
2° $\langle S,g\rangle \overset{\rho}{\to} y$ if and only if
  (a) there exist an $\mathcal{F}$-net $\langle S,f\rangle$ in K and a point $x\in K^*$ such that $g=i\circ f$, $y=i\circ x$ and $\langle S,f\rangle \overset{\pi}{\to} x$, or
  (b) y factors through the canonical injection $j:1_{\mathcal{K}}\to K+1_{\mathcal{K}}$ and there is no subnet $\langle T,h\rangle$ of $\langle S,g\rangle$ for which there exists a convergent (w.r.t. π) $\mathcal{F}$-net $\langle T,f\rangle$ in K such that $h=i\circ f$.

Remark 1. Let $i:(K,\pi)\to(\widehat{K},\rho)$ be a one-point compactification of a $Conv_{\mathcal{F}}$-object (K,π). Then it is evident that $(\widehat{K},\rho)$ is a compact $\mathcal{K}$-object and that (K,π) is separated if and only if $(\widehat{K},\rho)$ is separated. It is also evident that a one-point compactification of a $Conv_{\mathcal{F}}$-object (if it exists) is unique up to (compositions with) isomorphisms.

Example 5. (1) Let $\mathcal{F}:\underline{\omega}\to Set$ or $\mathcal{F}: Dir\to Set$ be the forgetful functor, let $(X,\mathcal{O})$ be a topological space and let (X,π) be the $Lim_{\mathcal{F}}$-object where π is the convergence class associated with the topology $\mathcal{O}$. Then (X,π) need not have any one-point compactification. On the other hand, if (X,π) has a one-point compactification $(\widehat{X},\rho)$, then ρ is associated with a topology $\mathcal{P}$ on $\widehat{X}$ such that $(\widehat{X},\mathcal{P})$ is the Alexandroff one-point compactification of $(X,\mathcal{O})$.

(2) Let $\mathcal{F}:\underline{\omega}\to Set$ be the forgetful functor. If (X,π) is a $HConv_{\mathcal{F}}$-object (i.e., a Fréchet $\mathcal{L}$-space), then it has a one-point compactification.

(3) If $\mathcal{S}$ is a construct and $\mathcal{F}:\mathcal{S}\to Set$ the forgetful functor, then the one-point compactifications of $Conv_{\mathcal{F}}$-objects coincide with their one-point compactifications studied in [13] and [15].

Proposition 2. *A $Conv_{\mathcal{F}}$-object is compact if and only if its one-point compactification is closed.*

Proof. Let $i:(K,\pi)\to(\widehat{K},\rho)$ be a one-point compactification of a $Conv_{\mathcal{F}}$-object (K,π). If i is closed, then (K,π) is compact by Theorem 2. To prove the converse implication, let (K,π) be compact. Let $\langle S,g\rangle \overset{\rho}{\to} y$

where y factors through i, i.e., there exists an $\mathcal{K}$-morphism $f : \mathcal{F}S \to K$ with $g = i \circ f$. Then $\langle S, f\rangle$ is an $\mathcal{F}$-net in K, thus there exists a subnet $\langle T, h\rangle$ of $\langle S, f\rangle$ which is convergent w.r.t. π. As $\langle T, i \circ h\rangle$ is a subnet of $\langle S, g\rangle$, the condition (b) of the axiom 2° from Definition 6 is not satisfied. Consequently, the condition (a) is valid, which means that there is a point $x \in K^*$ with $y = i \circ x$. Therefore, i is closed. □

Corollary 1 and Theorem 2 result in

Corollary 2. *Let (K, π) be a $HConv_{\mathcal{F}}$-object which has a one-point compactification $i : (K, \pi) \to (\widehat{K}, \rho)$. Then (K, π) is compact if and only if every embedding in $HConv_{\mathcal{F}}$ with the domain (K, π) is closed.*

Theorem 5. *Let (K, π) be a $Conv_{\mathcal{F}}$-object which has a one-point compactification $i : (K, \pi) \to (\widehat{K}, \rho)$ and let there exist the product $K \times K$ in $\mathcal{K}$. If the projection $pr_{\widehat{K}} : (K, \pi) \times (\widehat{K}\rho) \to (\widehat{K}\rho)$ is sublifting, then (K, π) is compact.*

Proof. Let the projection $pr_{\widehat{K}} : (K, \pi) \times (\widehat{K}, \rho) \to (\widehat{K}, \rho)$ be sublifting and suppose that (K, π) is not compact. Then there exists an $\mathcal{F}$-net $\langle S, f\rangle$ in K no subnet of which is convergent w.r.t. π. Consequently, $\langle S, i \circ f\rangle \xrightarrow{\rho} y$ where $y \in \widehat{K}^*$ factors through the injection $j : 1_{\mathcal{K}} \to \widehat{K}$. Let $\Delta : K \to K \times K$ be the diagonal morphism in $\mathcal{K}$. Since the $\mathcal{K}$-monomorphism i fulfills $i = pr_{\widehat{K}} \circ (id_K \times i) \circ \Delta$, we have $\langle S, pr_{\widehat{K}} \circ (id_K \times i) \circ \Delta \circ f\rangle \xrightarrow{\rho} y$. As $pr_{\widehat{K}}$ is sublifting, there exists a subnet $\langle T, (id_K \times i) \circ \Delta \circ f \circ s\rangle$ of $\langle S, (id_K \times i) \circ \Delta \circ f\rangle$ which converges (w.r.t. $\pi \times \rho$) to a point $x \in (K \times \widehat{K})^*$ with $y \cong pr_{\widehat{K}} \circ x$. Let $pr_K : K \times \widehat{K} \to K$ be the projection. Since $pr_{\widehat{K}} \circ (id_K \times i) \circ \Delta = i \circ pr_1 \circ \Delta = i \circ pr_2 \circ \Delta = i \circ pr_K \circ (id_K \times i) \circ \Delta$ (where pr_1 and $pr_2 : K \times K \to K$ are the first and second projections, respectively), we have $i \circ f \circ s = pr_{\widehat{K}} \circ (id_K \times i) \circ \Delta \circ f \circ s = i \circ pr_K \circ (id_K \times i) \circ \Delta \circ f \circ s$. It follows that $\langle T, i \circ f \circ s\rangle \xrightarrow{\rho} i \circ pr_K \circ x$, hence $\langle T, f \circ s\rangle \xrightarrow{\pi} pr_K \circ x$. But this is a contradiction because $\langle T, f \circ s\rangle$ is a subnet of $\langle S, f\rangle$. □

Theorems 3 and 5 result in

Corollary 3. *Let $\mathcal{K}$ have finite products and let (K, π) be a $Conv_{\mathcal{F}}$-object having a one-point compactification. Then (K, π) is compact if and only if the projection $pr_L : (K, \pi) \times (L, \rho) \to (L, \rho)$ is sublifting for every $Conv_{\mathcal{F}}$-object (L, ρ).*

Theorem 6. *Let $HConv_{\mathcal{F}}$ be amnestic and let (K, π) be a $HLim_{\mathcal{F}}$-object. If (K, π) is compact, then it is a maximal element of the fibre of K in $HConv_{\mathcal{F}}$.*

Proof. Let (K, π) be compact and let (K, ρ) be a $HConv_{\mathcal{F}}$-object such that $(K, \pi) \leq (K, \rho)$ (where $\leq$ denotes the partial order of the fibre of K in $HConv_{\mathcal{F}}$). Let $\langle S, f\rangle \xrightarrow{\rho} y$ and let $\langle U, h\rangle$ be an arbitrary subnet of $\langle S, f\rangle$. As (K, π) is compact, there is a subnet $\langle V, p\rangle$ of $\langle U, h\rangle$ such that $\langle V, p\rangle \xrightarrow{\pi} x$ (where $x \in K^*$ is a point). Consequently, $\langle V, p\rangle \xrightarrow{\rho} x$. Since $\langle V, p\rangle$ is a subnet of $\langle S, f\rangle$, we have $\langle V, p\rangle \xrightarrow{\rho} y$. Hence, $x \cong y$ (because (K, ρ) is separated) and, consequently, $\langle S, f\rangle \xrightarrow{\pi} y$ (because (K, π) is a $Lim_{\mathcal{F}}$-object). Therefore, id_K is a continuous $\mathcal{K}$-morphism from (K, ρ) to (K, π), i.e., $(K, \rho) \leq (K, \pi)$. Thus, $(K, \rho) = (K, \pi)$, which proves the statement. □

Theorem 7. *Let (K, π) be a $Conv_{\mathcal{F}}$-object with $K^* \neq \emptyset$. If (K, π) is a maximal element of the fibre of K in $Conv_{\mathcal{F}}$, then (K, π) is compact.*

Proof. Let (K, π) be a maximal element of the fibre of K in $Conv_{\mathcal{F}}$ and suppose that (K, π) is not compact. Then there exists an $\mathcal{F}$-net $\langle T, g\rangle$ in K no subnet of which is convergent w.r.t. π. Let $a \in K^*$ be a point and let (K, ρ) be the $Conv_{\mathcal{F}}$-object with $\langle S, f\rangle \xrightarrow{\pi} x$ if and only if either $\langle S, f\rangle \xrightarrow{\pi} x$, or $x = a$ and $\langle S, f\rangle$ is a subnet of $\langle T, g\rangle$. Then (K, ρ) is a $Conv_{\mathcal{F}}$-object and the identity $\mathcal{K}$-morphism id_K is continuous w.r.t. π and ρ. Thus, $(K, \pi) \le (K, \rho)$ and, since clearly $(K, \pi) \ne (K, \rho)$, we get a contradiction. □

Clearly, Theorem 7 remains valid if $Conv_{\mathcal{F}}$ is replaced by $HConv_{\mathcal{F}}$ (because, in the proof of Theorem 7, (K, ρ) is separated whenever (K, π) is separated). Therefore, Theorems 6 and 7 result in

Corollary 4. *Let $HConv_{\mathcal{F}}$ be amnestic and let (K, π) be a $HLim_{\mathcal{F}}$-object with $K^* \ne \emptyset$. Then (K, π) is compact if and only if it is a maximal element of the fibre of K in $HConv_{\mathcal{F}}$.*

The following, Tychonoff's theorem for objects of $Conv_{\mathcal{F}}$ is an immediate consequence of Theorem 3.9 from [16]. If compared with the Tychonoff's theorem with respect to a closure operator (see [3]), its assumptions are much simpler:

Theorem 8. *If the product of a family $(K_i)_{i\in I}$ of compact $Conv_{\mathcal{F}}$-objects exists, then it is compact provided that each I-indexed sink in $Conv_{\mathcal{F}}$ has a natural source.*

Remark 2. Our results show that convergence separation and convergence compactness, i.e., the introduced separation and compactness of objects of $Conv_{\mathcal{F}}$, preserve (analogues of) all basic properties of the classical separation and compactness of topological spaces. Moreover, Theorem 7 represents the results on convergence compactness that are not true for the classical topological compactness. It is well known that, for topological compactness, the corresponding analogue of the condition in Corollary 4 (which was proved to be equivalent to convergence compactness) is only necessary in general.

Most of the examples presented by the paper are related to topological convergence, i.e., based on the forgetful functors $\mathcal{F} : \underline{\omega} \to Set$ and $\mathcal{F} : Dir \to Set$. There are, of course, numerous examples obtained by considering other functors (indeed, every functor from a non-empty category to a category with a terminal object gives rise to such examples). But, in most of these examples, the convergence will no longer have the topological properties one expects of a convergence. Anyway, the categorical convergence discussed may be viewed as a common generalization of the sequential convergence and the net convergence (note that $\underline{\omega}$ is not a full subcategory of Dir). It also provides new cartesian closed categories as shown in [17].

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