A convenient graph connectedness for digital imagery

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Abstract. In a simple undirected graph, we introduce a special connectedness induced by a set of paths of length 2. We focus on the 8-adjacency graph (with the vertex set $\mathbb{Z}^2$) and study the connectedness induced by a certain set of paths of length 2 in the graph. For this connectedness, we prove a digital Jordan curve theorem by determining the Jordan curves, i.e., the circles in the graph that separate $\mathbb{Z}^2$ into exactly two connected components. These Jordan curves are shown to have an advantage over those given by the Khalimsky topology on $\mathbb{Z}^2$.

Keywords: Simple undirected graph, connectedness, digital plane, Khalimsky topology, Jordan curve theorem.

1 Introduction

In our increasingly digital world, digital images become an integral part of our everyday life. They play an extremely important role in scientific data visualization and this is the main reason for studying them in this paper.

In digital geometry for two-dimensional (2D for short) computer imagery, we usually replace pixels of a computer screen by their center points so that the screen is then represented by a finite section of the digital plane $\mathbb{Z}^2$. But, instead of such a section, we work with the whole digital plane $\mathbb{Z}^2$. A 2D black and white digital image is then a finite subset of $\mathbb{Z}^2$ and its elements are called black points. The remaining elements of $\mathbb{Z}^2$, called white points, form the background of the image. One of the basic problems of 2D digital image analysis and processing is to find a convenient connectedness structure for the digital plane $\mathbb{Z}^2$. Since digital images are simply digital approximations of the real ones, a connectedness being convenient means that the digital plane provided with such a structure behaves in much the same way as the Euclidean plane. In particular, it is required that such a structure allows for a digital analogue of the Jordan curve theorem (recall that the classical Jordan curve theorem states that a Jordan, i.e., simple closed, curve in the Euclidean plane separates this plane into exactly two connected components). In digital images, digital Jordan curves represent borders of objects imaged and, therefore, play an important role in solving numerous problems such as pattern recognition, memory usage compression, image reconstruction, etc.

The classical, graph-theoretic approach to solving the problem of providing the digital plane with a convenient connectedness structure is based on using the well-known 4- and 8-adjacency graphs (see e.g. [7, 8, 12–14, 17]). A disadvantage of this approach is that neither of the two graphs itself allows for a digital Jordan curve theorem so that a combination of them has to be used. Therefore, in [3], a new, topological approach to the problem was proposed based on employing a single structure, the so-called called Khalimsky topology, to obtain a convenient connectedness in the digital plane $\mathbb{Z}^2$. The topological approach was then developed by many authors - see, e.g., [4–6, 9–11, 15, 16].
The Khalimsky topology has the property that its connectedness coincides with the connectedness in a simple undirected graph with the vertex set $\mathbb{Z}^2$, namely the connectedness graph of the topology. Thus, to equip the digital plane with a convenient connectedness structure, this graph, rather than the Khalimsky topology itself, may be used. A drawback of this approach is that Jordan curves in the (connectedness graph of the) Khalimsky topology may never turn at the acute angle $\frac{\pi}{4}$. It would, therefore, be useful to find some new, more convenient structures on $\mathbb{Z}^2$ that would allow Jordan curves to turn, at some points, at the acute angle $\frac{\pi}{4}$. In the present note, to obtain such a convenient structure, we employ the 8-adjacency graph with connectedness given by a certain set of paths of length 2 in the graph. For this connectedness, we prove a digital Jordan curve theorem to show that the graph with the set of paths provides a convenient structure on the digital plane for the study and processing of digital images.

2 Preliminaries

For the graph-theoretic concepts used see, for instance, [1]. By a *graph* we always mean an undirected simple graph without loops, hence and ordred pair (V, E) of sets where $E \subseteq \{\{a, b\};\ a, b \in V,\ a \neq b\}$. The elements of V are called *vertices* and those of E are called edges of the graph. If $\{a, b\} \in E$, then the vertices a and b are said to be *adjacent* and the edge $\{a, b\}$ is said to *join* the vertices a and b. For an arbitrary vertex $a \in V$, we denote by $E(a)$ the set of all vertices adjacent to a, i.e., $E(a) = \{b \in V;\ \{a, b\} \in E\}$. Clearly, $\{a, b\} \in E$ if and only if $b \in E(a)$ or, equivalently, $a \in E(b)$. Thus, the set E of edges of a graph may be given by determining the set $E(a)$ for every $a \in V$.
As usual, we graphically represent graphs by thinking of vertices as points and edges as line segments whose end points are just the vertices they join.
A graph (U, F) is called a *subgraph* of a graph (V, E) if $U \subseteq V$ and $F \subseteq E$. If, moreover, $F = E \cap \{\{a, b\};\ a, b \in U\}$, then (U, F) is said to be an *induced subgraph* of (V, E) being denoted briefly by U. A subgraph (U, F) of (V, E) is called a *factor* of (V, E) if $U = V$.
Recall that a *walk* in a graph (V, E) is a finite sequence $(a_i|\ i \leq n) = (a_0, a_1, ..., a_n)$, n a non-negative integer, of vertices such that $\{a_{i-1}, a_i\} \in E$ whenever $i \in \{1, 2, ...n\}$. If all vertices a_i, $i \in \{0, 1, ..., n\}$, are pairwise different, then the walk $(a_i|\ i \leq n)$ is said to be a *path* and the number

n is called the *length* of the path. Thus, also a single vertex is considered to be a path (of length 0). A *circle* in (V, E) is any walk $(a_i|\ i \leq n)$ with $n > 2$ such that $(a_i|\ i < n)$ is a path and $a_0 = a_n$. A subset $X \subseteq V$ is said to be *connected* if, for every pair $a, b \in X$, there is a path $(a_i|\ i \leq n)$ such that $a_0 = a$, $a_n = b$ and $a_i \in X$ for all $i \in \{0, 1, ..., n\}$. A maximal (with respect to set inclusion) connected subset of V is called a *component* of the graph (V, E).

A nonempty, finite and connected subset C of V is said to be a *simple closed curve* in (V, E) if the set $E(a) \cap C$ has two elements for every $a \in C$. Clearly, every simple closed curve is a circle. A simple closed curve in (V, E) is called a *Jordan curve* in (V, E) if it separates the set V into exactly two components, i.e., if the induced subgraph $V - C$ of (V, E) has exactly two components.

For every point $(x, y) \in \mathbb{Z}^2$, we put $A_4(x, y) = \{(x + i, y + j);\ i, j \in \{-1, 0, 1\},\ ij = 0,\ i+j \neq 0\}$ and $A_8(x, y) = A_4(x, y) \cup \{(x+i, y+j);\ i, j \in \{-1, 1\}\}$. The points of $A_4(x, y)$ and $A_8(x, y)$ are said to be 4-adjacent and 8-adjacent to (x, y), respectively. The graphs $(\mathbb{Z}^2, A_4)$ and $(\mathbb{Z}^2, A_8)$ are called the *4-adjacency graph* and *8-adjacency graph*, respectively, and are demonstrated in Figure 1.

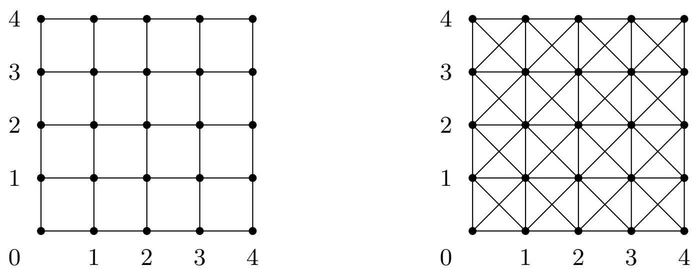


Fig. 1. Portions of the 4- and 8-adjacency graphs.

In digital image processing, the 4-adjacency and 8-adjacency graphs are the most frequently used structures on the digital plane. But, since the late 1980's, another structure on $\mathbb{Z}^2$ has been used too, namely the Khalimsky topology [3]. It is the product of two copies of the topology on $\mathbb{Z}$ given by the subbase $\{\{2k-1, 2k, 2k+1\};\ k \in \mathbb{Z}\}$ (for the basic concepts of general topology see [2]). Recall that, given a topology $\mathcal{T}$ on a set X, the *connectedness graph* of $\mathcal{T}$ is the graph with the vertex set X such that a pair of different points $x, y \in X$ is adjacent if and only if $\{x, y\}$ is a connected subset of the space (X, T). Since the Khalimsky topology is an Alexandroff topology (which means that the closure operator in the topology is completely additive), the connectedness in the Khalimsky topological space coincide with the connectedness in the connectedness graph of the Khalimsky topology. We will call the connectedness graph of the Khalimsky topology briefly the *Khalimsky graph.* The Khalimsky graph is the graph $(\mathbb{Z}^2, K)$ such that, for any $(x, y) \in \mathbb{Z}^2$,

$$K(x,y) = \begin{cases} A_8(x,y) \text{ if } x \text{ and } y \text{ have the same parity,} \\ A_4(x,y) \text{ if } x \text{ and } y \text{ have different parities.} \end{cases}$$

A portion of the Khalimsky graph is demonstrated in Figure 2. It is obvious that the graph is a factor of the 8-adjacency graph.

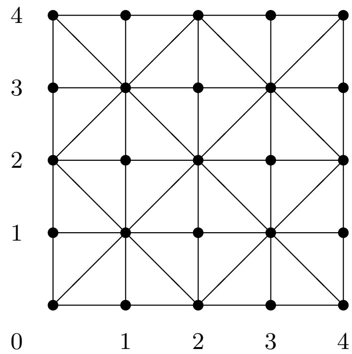


Fig. 2. A portion of the Khalimsky graph.

The famous Jordan curve theorem proved for the Khalimsky topology in [3] may be formulated as follows:

Theorem 1. *In the Khalimsky graph, every simple closed curve with at least four points is a Jordan curve.*

We denote by $(\mathbb{Z}^2, L)$ the factor of the Khalimsky graph $(\mathbb{Z}^2, K)$ given by $L = K - \bigcup\{\{(x,y),(z,t)\};\ (x,y) \in \mathbb{Z}^2,\ x \text{ and } y \text{ are odd and } (z,t) \in A_4(x,y)\}$. A portion of the graph $(\mathbb{Z}^2, L)$ demonstrated in Figure 3.

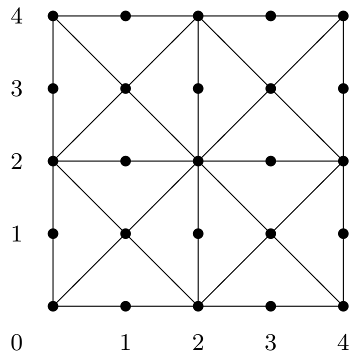


Fig. 3. A portion of the graph $(\mathbb{Z}^2, L)$.

The below corollary immediately follows from Theorem 1:

Corollary 1. *Every circle in the graph $(\mathbb{Z}^2, L)$ which does not turn, at any of its points, at the acute angle $\frac{\pi}{4}$ is a Jordan curve in the connectedness graph of the Khalimsky topology.*

It is readily verified that a simple closed curve (and thus also a Jordan curve) in the Khalimsky graph may never turn at the acute angle $\frac{\pi}{4}$. It could therefore be useful to replace the Khalimsky topology (Khalimsky graph) with some more convenient structure on $\mathbb{Z}^2$, another factor of the 8-adjacency graph, that would allow Jordan curves to turn at the acute angle $\frac{\pi}{4}$ at some points. And this is what we will do in the next section.

3 8-adjacency graph with a set of paths of length 2

In the 8-adjacency graph $(\mathbb{Z}^2, A_8)$, the set $A_8(x, y)$ provides the digital plane $\mathbb{Z}^2$ with a natural concept of neighborhood of any point $(x, y) \in \mathbb{Z}^2$. Therefore, it would be desirable to use the graph for structuring the digital plane. But the usual concept of connectedness in the 8-adjacency graph does not alow for a digital Jordan curve theorem. To solve this problem, we employ another concept of connectedness.

Definition 1. Let (V, E) be a graph, $\mathcal{B}$ a set of paths of length 2 in the graph, and n a nonnegative integer. A sequence $C = (c_i|\ i \le n)$ of elements of V is called a $\mathcal{B}$-*walk* if one of the the following three conditions is satisfied for every $i \in \{0, 1, ...n-1\}$:
(i) There exists $(a_0, a_1, a_2) \in \mathcal{B}$ such that $\{c_i, c_{i+1}\} = \{a_0, a_1\}$,
(ii) $i > 0$ and there exists $(a_0, a_1, a_2) \in \mathcal{B}$ such that $c_{i-1} = a_0$, $c_i = a_1$, and $c_{i+1} = a_2$,
(iii) $i < n-1$ and there exists $(a_0, a_1, a_2) \in \mathcal{B}$ such that $c_i = a_2$, $c_{i+1} = a_1$, and $c_{i+2} = a_0$.
A $\mathcal{B}$-walk $(c_i|\ i \le n)$ with the property that $n \ge 2$ and $c_i = c_j \Leftrightarrow \{i, j\} = \{0, n\}$ is said to be a $\mathcal{B}$-*circle*.

Observe that, if $(x_0, x_1, ..., x_n)$ is a $\mathcal{B}$-walk, then $(x_n, x_{n-1}, ..., x_0)$ is a $\mathcal{B}$-walk, too (so that $\mathcal{B}$-walks are closed under reversion) and, if $(x_i|\ i \le m)$ and $(y_i|\ i \le p)$ are $\mathcal{B}$-walks with $x_m = y_0$, then, putting $z_i = x_i$ for all $i \le m$ and $z_i = y_{i-m}$ for all i with $m \le i \le m + p$, we get a $\mathcal{B}$-walk $(z_i|\ i \le m + p)$ (so that $\mathcal{B}$-walks are closed under composition).
Given a set $\mathcal{B}$ of paths of length 2 in a graph (V, E), a subset $A \subseteq V$ is said to be $\mathcal{B}$-*connected* if, for every pair $a, b \in A$, there is a $\mathcal{B}$-walk $(c_i|\ i \le n)$ such that $c_0 = a$, $c_n = b$ and $c_i \in A$ for all $i \in \{0, 1, ..., n\}$. A maximal (with respect to set inclusion) $\mathcal{B}$-connected subset of V is called a $\mathcal{B}$-*component* of (V, E).

Definition 2. Let $\mathcal{B}$ be a set of paths of length 2 in a graph (V, E). A nonempty, finite and $\mathcal{B}$-connected subset J of V is said to be a $\mathcal{B}$-*simple closed curve* if every element $(a_0, a_1, a_2) \in \mathcal{B}$ with $\{a_0, a_1\} \subseteq J$ satisfies $a_2 \in J$ and every $c \in J$ fulfills one of the following two conditions:
(1) There are exactly two elements $(a_0, a_1, a_2) \in \mathcal{B}$ satisfying both $\{a_0, a_1, a_2\} \subseteq J$ and $c \in \{a_0, a_2\}$ and there is no element $(b_0, b_1, b_2) \in \mathcal{B}$ satisfying both $\{b_0, b_1, b_2\} \subseteq J$ and $c = b_1$.

(2) There is exactly one element $(b_0, b_1, b_2) \in \mathcal{B}$ satisfying both $\{b_0, b_1, b_2\} \subseteq J$ and $c = b_1$ and there is no element $(a_0, a_1, a_2) \in \mathcal{B}$ satisfying both $\{a_0, a_1, a_2\} \subseteq J$ and $c \in \{a_0, a_2\}$.

Clearly, every $\mathcal{B}$-simple closed curve is a $\mathcal{B}$-circle.

Definition 3. Let $\mathcal{B}$ be a set of paths of length 2 in a graph (V, E). A $\mathcal{B}$-simple closed curve J is called a $\mathcal{B}$-*Jordan curve* if the subset $V - J \subseteq V$ consists (i.e., is the union) of exactly two $\mathcal{B}$-components.

From now on, $\mathcal{B}$ will denote the set of paths of length 2 in the 8-adjacency graph given as follows: For every $((x_i, y_i)|\ i \leq 2)$ such that $(x_i, y_i) \in \mathbb{Z}^2$ for every $i \leq 2$, $((x_i, y_i)|\ i \leq 2) \in \mathcal{B}$ if and only if one of the following eight conditions is satisfied:

(1) $x_0 = x_1 = x_2$ and there is $k \in \mathbb{Z}$ such that $y_i = 4k + i$ for all $i \leq 2$,
(2) $x_0 = x_1 = x_2$ and there is $k \in \mathbb{Z}$ such that $y_i = 4k - i$ for all $i \leq 2$,
(3) $y_0 = y_1 = y_2$ and there is $k \in \mathbb{Z}$ such that $x_i = 4k + i$ for all $i \leq 2$,
(4) $y_0 = y_1 = y_2$ and there is $k \in \mathbb{Z}$ such that $x_i = 4k - i$ for all $i \leq 2$,
(5) there is $k \in \mathbb{Z}$ such that $x_i = 4k + i$ for all $i \leq 2$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l + i$ for all $i \leq 2$,
(6) there is $k \in \mathbb{Z}$ such that $x_i = 4k + i$ for all $i \leq 2$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l - i$ for all $i \leq 2$,
(7) there is $k \in \mathbb{Z}$ such that $x_i = 4k - i$ for all $i \leq 2$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l + i$ for all $i \leq 2$,
(8) there is $k \in \mathbb{Z}$ such that $x_i = 4k - i$ for all $i \leq 2$ and there is $l \in \mathbb{Z}$ such that $y_i = 4l - i$ for all $i \leq 2$.

A portion of $\mathcal{B}$ is shown in Figure 4. The paths of length 2, i.e., the ordered triples, belonging to $\mathcal{B}$ are represented by line segments oriented from first to last terms.

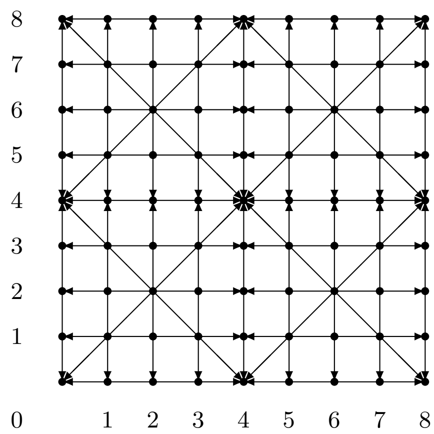


Fig. 4. A portion of the set $\mathcal{B}$.

Further, we denote by $(\mathbb{Z}^2, A)$ the factor of the 8-adjacency graph given as follows:

$$A(x,y) = \begin{cases} A_8(x,y) \text{ if } (x,y) = (4k, 4l),\ k,l \in \mathbb{Z}, \\ A_8(x,y) - A_4(x,y) \text{ if } (x,y) = (4k+2, 4l+2),\ k,l \in \mathbb{Z}, \\ \{(x-1,y),(x+1,y)\} \text{ if } (x,y) = (4k+i, 4l),\ k,l \in \mathbb{Z}, \\ \qquad i \in \{1,2,3\}, \\ \{(x,y-1),(x,y+1)\} \text{ if } (x,y) = (4k, 4l+i),\ k,l \in \mathbb{Z}, \\ \qquad i \in \{1,2,3\}, \\ \{(x-1,y-1),(x+1,y+1)\} \text{ if } (x,y) = (4k+i, 4l+i), \\ \qquad k,l \in \mathbb{Z},\ i \in \{-1,1\}, \\ \{(x-1,y+1),(x+1,y-1)\} \text{ if } (x,y) = (4k+i, 4l-i), \\ \qquad k,l \in \mathbb{Z},\ i \in \{-1,1\}, \\ \emptyset \text{ otherwise.} \end{cases}$$

A portion of the graph $(\mathbb{Z}^2, A)$ is demonstrated by Figure 5.

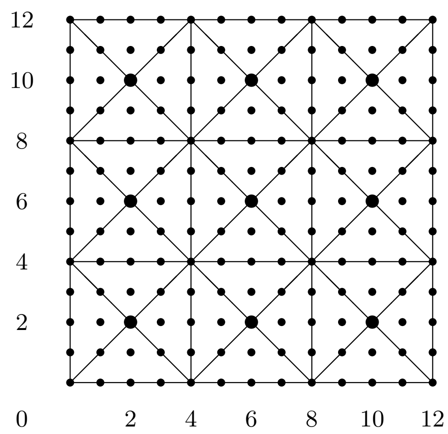


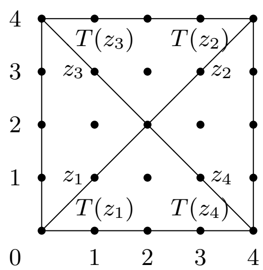
Fig. 5. A portion of the graph $(\mathbb{Z}^2, A)$.

Theorem 2. *Every circle in the graph $(\mathbb{Z}^2, A)$ that does not turn at any point $(4k+2, 4l+2)$, $k,l \in \mathbb{Z}$ (i.e, any point denoted by a bold dot in Figure 5) is a $\mathcal{B}$-Jordan curve.*

Proof. Clearly, every circle in the graph $(\mathbb{Z}^2, A)$ is a $\mathcal{B}$-simple closed curve. Let $z = (x,y) \in \mathbb{Z}^2$ be a point such that $x = 4k+p$ and $y = 4l+q$ for some $k,l,p,q \in \mathbb{Z}$ with $pq = \pm 1$. Then, we define the *fundamental triangle T(z)* to be the fifteen-point subset of $\mathbb{Z}^2$ given as follows:

$$T(z) = \begin{cases} \{(r,s) \in \mathbb{Z}^2; 4k \le r \le 4k+4,\ 4l \le s \le 4l+4k+4-r\} \\ \qquad\qquad \text{if } x = 4k+1 \text{ and } y = 4l+1 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ 4k \le r \le 4k+4,\ 4l \le s \le 4l+r-4k\} \\ \qquad\qquad \text{if } x = 4k+3 \text{ and } y = 4l+1 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ 4k \le r \le 4l+4,\ 4l+4k+4-r \le s \le 4l+4\} \\ \qquad\qquad \text{if } x = 4k+3 \text{ and } y = 4l+3 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ 4k \le r \le 4k+4,\ 4l+r-4k \le s \le 4l+4\} \\ \qquad\qquad \text{if } x = 4k+1 \text{ and } y = 4l+3 \text{ for some } k,l \in \mathbb{Z}. \end{cases}$$

Graphically, every fundamental triangle $T(z)$ consists of fifteen points and forms a right triangle obtained from a 4×4-square by dividing it by a diagonal. More precisely, each of the two diagonals divides the square into just two fundamental triangles having a common hypotenuse coinciding with the diagonal. In every fundamental triangle $T(z)$, the point z is one of the three internal points of the triangle. The (four types of) fundamental triangles are demonstrated by the below figure:



Given a fundamental triangle, we speak about its sides - it is clear from the above picture which sets are understood to be the sides (note that each side consists of five points and that two different fundamental triangles may have at most one side in common).
Now, one can easily see that

(1) every fundamental triangle is $\mathcal{B}$-connected and so is every subset of $\mathbb{Z}^2$ obtained by subtracting, from a fundamental triangle, some of its sides.

Consequently,

(2) if S_1, S_2 are fundamental triangles having a common side D, then the set $(S_1 \cup S_2) - M$ is $\mathcal{B}$-connected whenever M is the union of some sides of S_1 or S_2 different from D.

It is also evident that,

(3) whenever S_1, S_2 are different fundamental triangles with a common side D and $X \subseteq S_1 \cup S_2$ is a $\mathcal{B}$-connected subset with $X \cap S_1 \neq \emptyset \neq X \cap S_2$, we have $X \cap D \neq \emptyset$.

We will show that, for every circle C in the graph $(\mathbb{Z}^2, A)$ which does not turn at any point $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, there are sequences $\mathcal{S}_F, \mathcal{S}_I$ of fundamental triangles, $\mathcal{S}_F$ finite and $\mathcal{S}_I$ infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the following two conditions are satisfied:

(a) Each member of $\mathcal{S}$, excluding the first one, has a common side with at least one of its predecessors.

(b) C is the union of those sides of fundamental triangles in $\mathcal{S}$ that are not shared by two different fundamental triangles of $\mathcal{S}$.

To this end, put $C_1 = C$ and let S_1^1 be an arbitrary fundamental triangle with $S_1^1 \cap C_1 \neq \emptyset$. For every $k \in \mathbb{Z}$, $1 \leq k$, if $S_1^1, S_2^1, ..., S_k^1$ are defined, let S_{k+1}^1 be a fundamental triangle with the following properties: $S_{k+1}^1 \cap C_1 \neq \emptyset$, S_{k+1}^1 has a side in common with S_k^1 which is not a subset of C_1 and $S_{k+1}^1 \neq S_i^1$ for all i, $1 \leq i \leq k$. Clearly, there will always be a (smallest) number $k \geq 1$ for which no such fundamental triangle S_{k+1}^1 exists. Denoting by k_1 this number, we have defined a sequence $(S_1^1, S_2^1, ..., S_{k_1}^1)$ of fundamental triangles. Let C_2 be the union of those sides of fundamental triangles in $(S_1^1, S_2^1, ..., S_{k_1}^1)$ that are disjoint from C_1 and not shared by two different fundamental triangles in $(S_1^1, S_2^1, ..., S_{k_1}^1)$. If $C_2 \neq \emptyset$, we construct a sequence $(S_1^2, S_2^2, ..., S_{k_2}^2)$ of fundamental triangles in a way similar to the one used for constructing of $(S_1^1, S_2^1, ..., S_{k_1}^1)$ by taking C_2 instead of C_1 (and obtaining k_2 in much the same way as we did k_1). Repeating this construction, we get sequences $(S_1^3, S_2^3, ..., S_{k_3}^3)$, $(S_1^4, S_2^4, ..., S_{k_4}^1)$, etc. We put $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, S_1^3, S_2^3, ..., S_{k_3}^3, ...)$ if $C_i \neq \emptyset$ for all $i \geq 1$ and $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, ..., S_1^l, S_2^l, ..., S_{k_l}^l)$ if $C_i \neq \emptyset$ for all i with $1 \leq i \leq l$ and $C_i = \emptyset$ for $i = l+1$.

Further, let $S_1' = T(z)$ be a fundamental triangle such that $z \notin S$ whenever S is a member of $\mathcal{S}$. Having defined S_1', let $\mathcal{S}' = (S_1', S_2', ...)$ be a sequence of fundamental triangles defined analogously to $\mathcal{S}$ (by taking S_1' instead of S_1^1). Then, one of the sequences $\mathcal{S}$, $\mathcal{S}'$ is finite and the other is infinite. Indeed, $\mathcal{S}$ is finite (infinite) if and only if its first member equals such a fundamental triangle $T(z)$ for which $z = (k, l) \in \mathbb{Z}^2$ has the property that the cardinality of the set $\{(x, l) \in \mathbb{Z}^2;\ x > k\} \cap C$ is odd (even). The same is true for $\mathcal{S}'$. If we put $\{\mathcal{S}_F, \mathcal{S}_I\} = \{\mathcal{S}, \mathcal{S}'\}$ where $\mathcal{S}_F$ is finite and $\mathcal{S}_I$ is infinite, then the conditions (a) and (b) are clearly satisfied.

Given a circle C in the graph $(\mathbb{Z}^2, A)$ which does not turn at any point $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, let S_F and S_I denote the union of all members of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Then, $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$. Let $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$ be the sequences obtained from $\mathcal{S}_F$ and $\mathcal{S}_I$ by subtracting C from each member of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Let S_F^* and S_I^* denote the union of all members of $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$, respectively. Then, S_F^* and S_I^* are connected by (1) and (2) and it is clear that $S_F^* = S_F - C$ and $S_I^* = S_I - C$. So, S_F^* and S_I^* are $\mathcal{B}$-components of $\mathbb{Z}^2 - C$ by (3) ($S_F - C$ is called the *inside* component and $S_I - C$ is called the *outside* component). We have proved that every cycle in the graph shown in Figure 5 that does not turn at any point $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, is a $\mathcal{B}$-Jordan curve.

Example 1. Consider the set of points of $\mathbb{Z}^2$ demonstrated by Figure 6, which represents the (border of) letter K. This set is a circle in the graph $(\mathbb{Z}^2, A)$ that turns only at some of the vertices $(2k(n-1), 2l(n-1))$, $k, l \in \mathbb{Z}$, so that it is a $\mathcal{B}$-Jordan curve by Theorem 2. But, since the circle turns, at each of the four bold points, at the acute angle $\frac{\pi}{4}$, it is not a digital Jordan curve in the Khalimsky graph. For the circle to be a Jordan curve in the Khalimsky graph, it is necessary to remove, along with the four bold points, the four encircled points (because, otherwise, the circle would not even be a simple closed curve in the Khalimsky graph). But this would lead to a noticeable deformation of the image (note that the points represent centers of pixels) if the resolution of the computer screen used is not sufficiently high. This may be the case of some industrial monitors or displays.

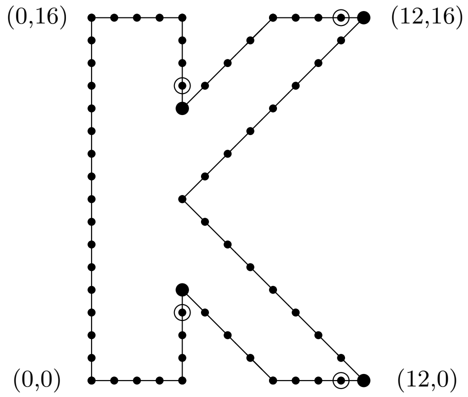


Fig. 6. A Jordan curve in $(\mathbb{Z}^2, u_3^2)$.

Remark 1. If we do not insist on structuring the digital plane by the 8-adjacency graph but admit structuring it by a factor of the graph, we may find a graph G with the vertex set $\mathbb{Z}^2$ having the property that every circle in the graph ($\mathbb{Z}^2$,A), not only a cycle that does not turn at any point $(4k+2, 4l+2)$, $k, l \in \mathbb{Z}$, is a Jordan curve in G (with respect to the natural connectedness in the graph G). Let us call graphs G with this property *sd-graphs.* The *sd*-graphs are studied in [18] where it is shown that the graph demonstrated in Figure 7 is a minimal (with respect to the set of edges) *sd*-graph. Note that this graph is even a factor of the Khalimsky graph.

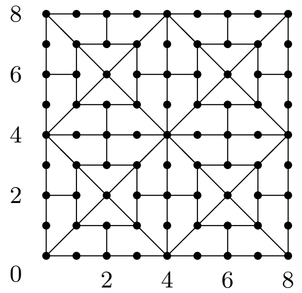


Fig. 7. A portion of a minimal *sd*-graph.

4 Conclusions

We have found a structure on the digital plane $\mathbb{Z}^2$, the graph $(\mathbb{Z}^2, A_8)$ together with the set $\mathcal{B}$ of paths of length 2, which provides the plane with a connectedness allowing for a digital analogue of the Jordan curve theorem (Theorem 2). This means that the graph $(\mathbb{Z}^2, A_8)$ together with the set $\mathcal{B}$ may be used as a background structure on the digital plane for the study and processing of digital images. An advantage of the $\mathcal{B}$-Jordan curves in the graph $(\mathbb{Z}^2, A)$ over the Jordan curves in then Khalimsky plane is that they may turn, at some points, under the acute angle $\frac{\pi}{4}$. Hence, the graph $(\mathbb{Z}^2, A_8)$ endowed with the set $\mathcal{B}$ provides a variety of Jordan curves richer than the one provided by the Khalimsky topology. Thus, the graph offers a convenient alternative to the topology. Since Jordan curves represent borders of objects in digital images, the structure on $\mathbb{Z}^2$ given by the graph $(\mathbb{Z}^2, A_8)$ with the set $\mathcal{B}$ may be used in digital image processing for solving problems related to boundaries such as pattern recognition, boundary detection, contour filling, data compression, etc.

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References

1. J.A. Bondy, U.S.R. Murty, *Graph Theory* (Springer, 2008)
2. Engelking, R.: General Topology. Państwowe Wydawnictwo Naukowe, Warszawa (1977)
3. Khalimsky, E.D., Kopperman, R., Meyer, P.R.: Computer graphics and connected topologies on finite ordered sets. Topology Appl. **36**, 1–17 (1990)
4. Khalimsky, E.D, Kopperman, R., Meyer, P.R.: Boundaries in digital plane. J. Appl. Math. Stochast. Anal. , 27–55 (1990)
5. Kiselman, C.O. Digital Jordan curve theorems. In: Borgefors, G., Nyström, I., Baja, G.S. (eds.) Discrete Geometry for Computer Imagery, Lect. Notes Comput. Sci., vol.1953, pp. 46–56. Springer, Heidelberg (2000)
6. T.Y. Kong, T.Y., Kopperman, R., Meyer, P.R.: A topological approach to digital topology. Amer. Math. Monthly **98**, 902–917 (1991)
7. Kong, T.Y, Roscoe, W.: A theory of binary digital pictures. Comput. Vision Graphics Image Process. **32**, 221–243 (1985).
8. Kong, T.Y, Rosenfeld, A.: Digital topology: Introduction and survey. Comput. Vision Graphics Image Process. **48**, 357–393 (1989)
9. Kopperman, R., Meyer, P.R., Wilson, R.G.: A Jordan surface theorem for three-dimensional digital space. Discrete Comput. Geom. **6**, 155–161 (1991)
10. Melin, E.: Digital surfaces and boundaries in Khalimsky spaces. J. Math. Imaging and Vision **28**, 169–177 (2007)
11. Melin, E.: Continuous digitization in Khalimsky spaces, J. Approx. Theory**150**, 96–116 (2008)
12. Rosenfeld, A.: Connectivity in digital pictures. J. Assoc. Comput. Math. **17**, 146–160 (1970)
13. Rosenfeld, A.: Digital topology. Amer. Math. Monthly **86**, 621–630 (1979).
14. Rosenfeld, A.: Picture Languages. Academic Press, New York (1979)
15. Šlapal, J.: Jordan curve theorems with respect to certain pretopologies on $\mathbb{Z}^2$. In: Brlek, S., Reutenauer, Ch., Provencal, X. (eds.) Discrete Geometry for Computer Imagery. Lect. Notes Comput. Sci., vol. 5810, pp. 252–262. Springer, Heidelberg (2009)
16. Šlapal, J.: A Jordan curve theorem in the digital plane. In: Aggarwal, J.K. at al. (eds.) Combinatorial Image Analysis. Lect. Notes Comput. Sci., vol. 6636, pp. 120–131. Springer, Heidelberg (2011)
17. Šlapal, J.: Graphs with a path partition for structuring digital spaces, Inform. Sciences **233**, 305–312 (2013)
18. Šlapal, J.: Convenient adjacencies on $\mathbb{Z}^2$. Filomat **28**, 305–312 (2014)