Spectral analysis and discrete Fourier transform

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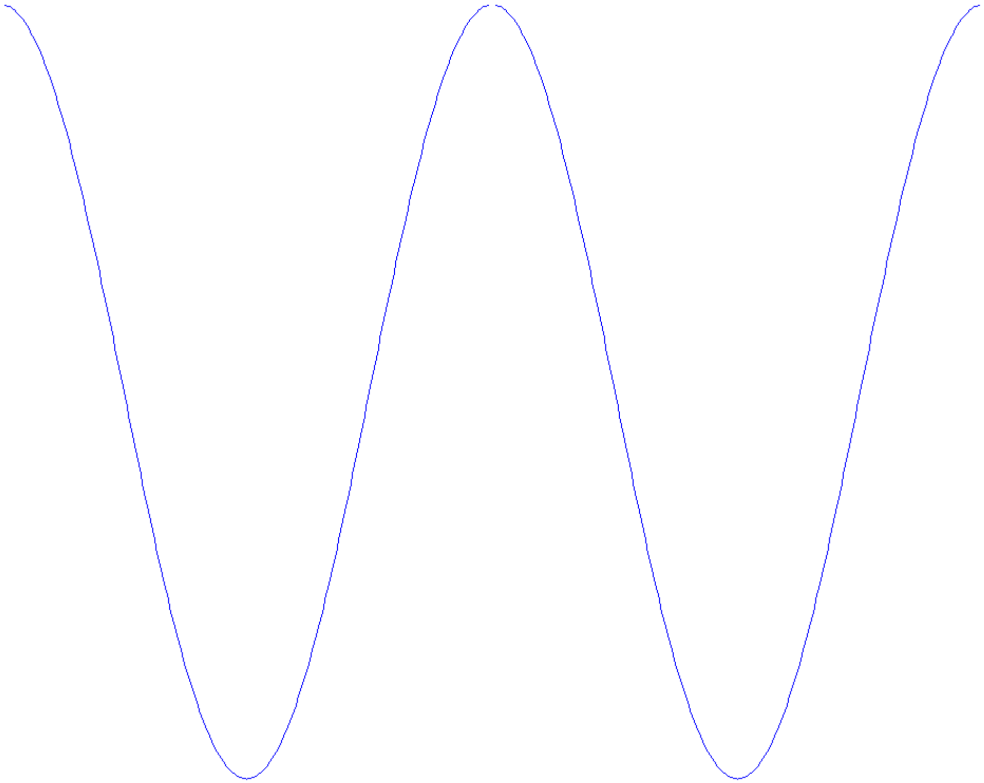
The cosine story …



$$f(x) = \cos(x)$$

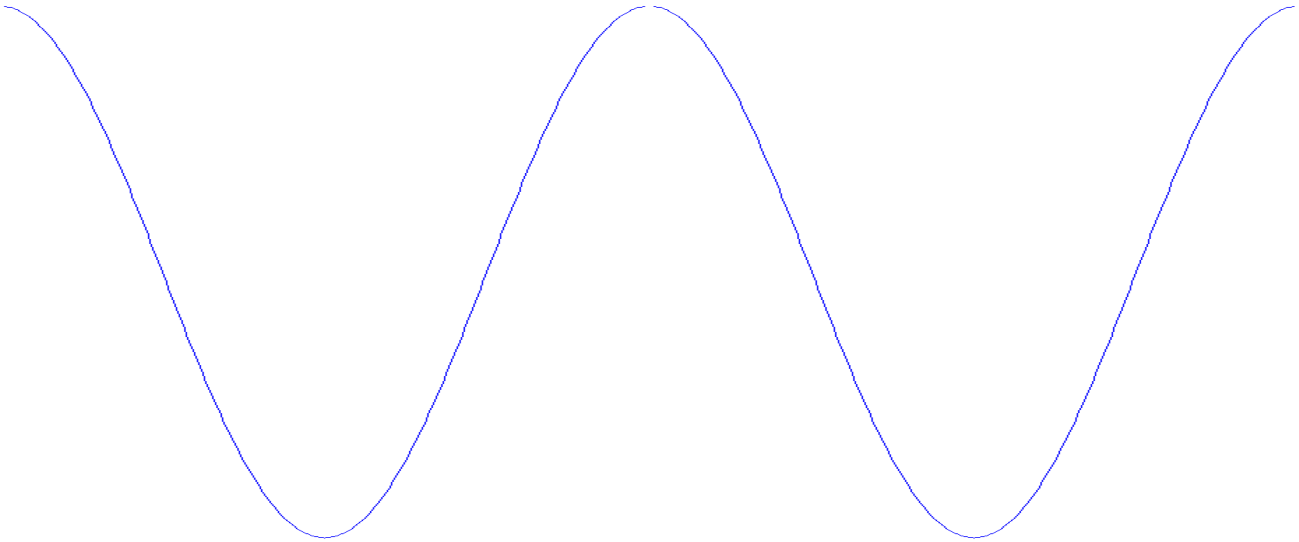
Argument of the cosine

- $0 \ldots 2\pi$
- and then every 2π

the period

Discrete time cosine

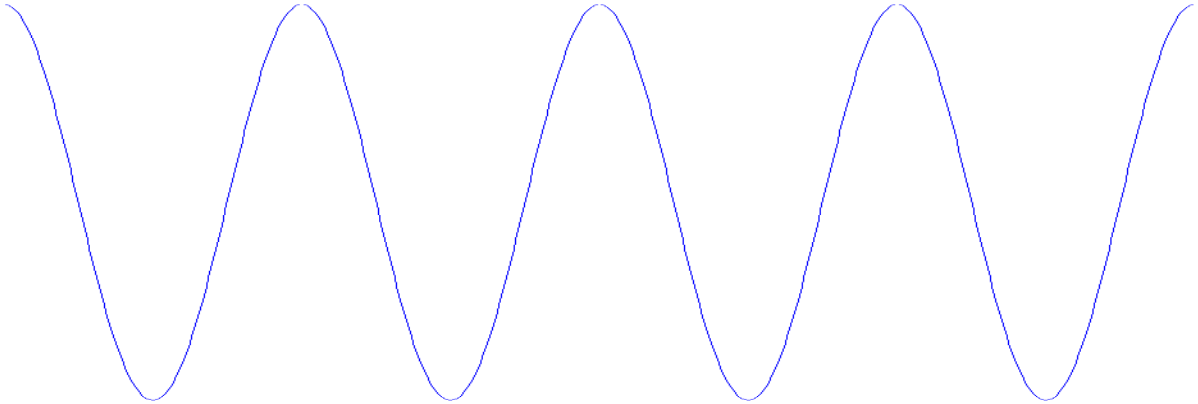
- Task #1: generate a cos, with 1 period per second on sampling frequency F_s



$$x[n] = \cos(\qquad n) \qquad\qquad x[n] = \cos(2\pi \frac{1}{8000} n)$$

Discrete time cosine

- Task #2: generate a cos, with 2 periods per second on sampling frequency F_s



$$x[n] = \cos(\qquad n)$$

$$x[n] = \cos(2\pi \frac{2}{8000} n)$$

Discrete time cosine

- Task #3: Generate a cosine that will do 440 periods per second on sampling frequency F_s - chamber „a“ 440Hz.

$$x[n] = \cos(\qquad n)$$

$$x[n] = \cos(2\pi \, \frac{440}{8000} \, n)$$

Check in Matlab

- Generate,
- Measure 1 period (by hand!)
- Compute period and frequency
- Play it !

DEMO 1 in Matlab

Normalized frequency

$$f = \frac{f_{skutecna}}{F_s}, \qquad f_{skutecna} = f\,F_s$$

- Unit ?
- Examples ?

$$f = \frac{1}{Fs} \qquad f = 0 \qquad f = \frac{1}{2} \qquad f = 1$$

General cosine

$$x[n] = A\cos(2\pi f n + \phi)$$

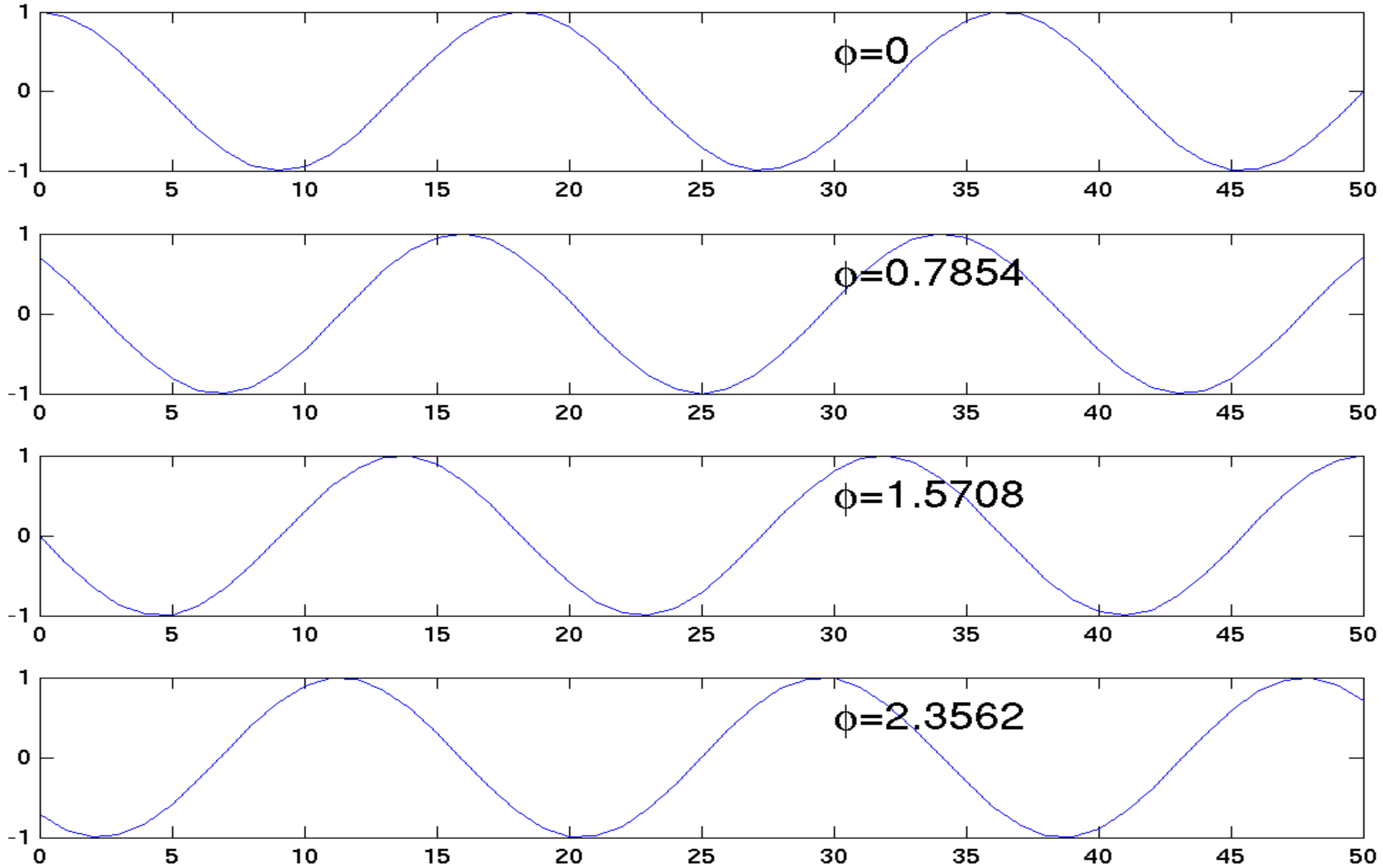
magnitude

normalized frequency

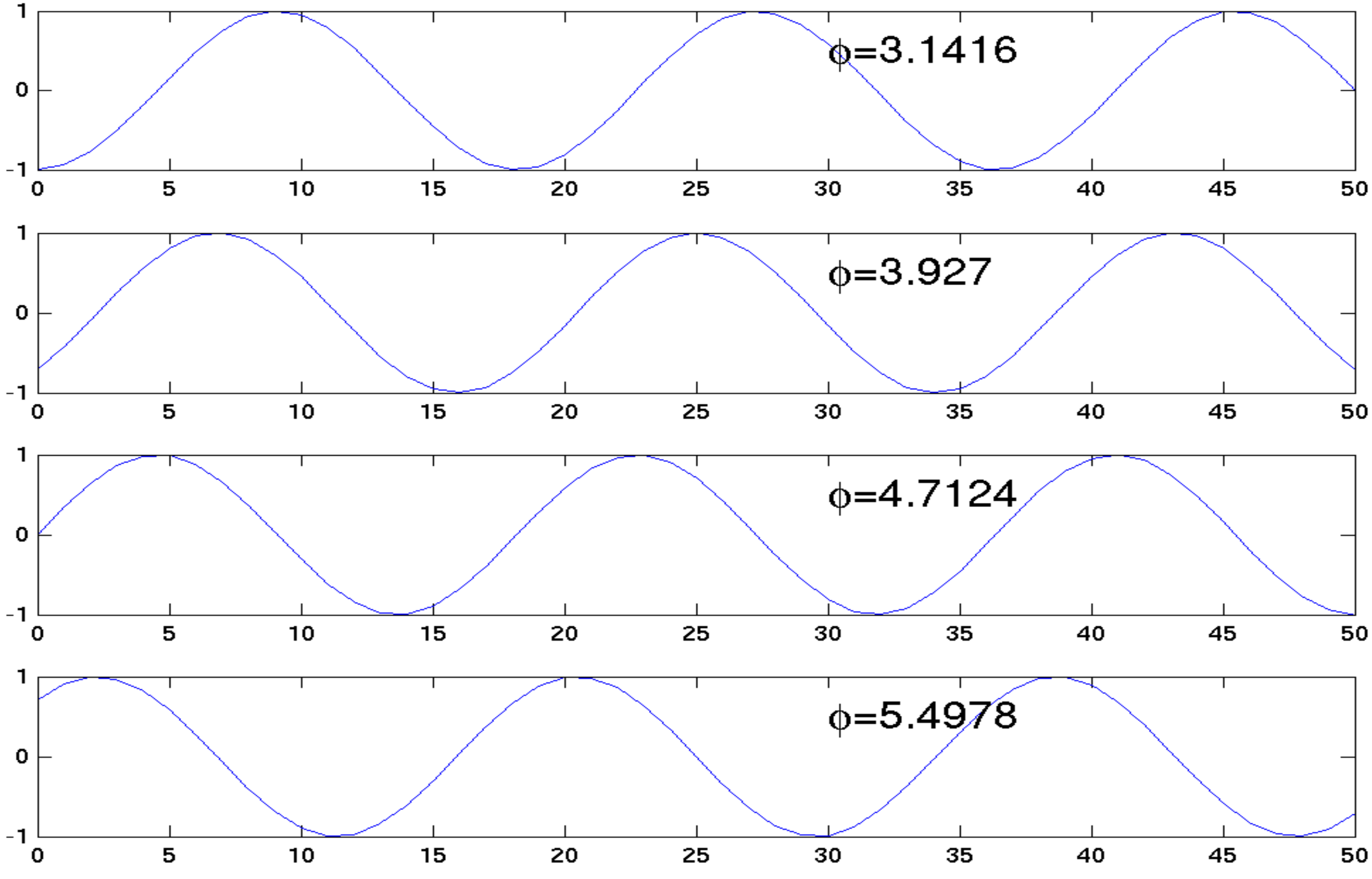
initial phase

- Units ?

Initial phase I.



Initial phase II.



A little song …

DEMO 2 in Matlab

- Lengths and durations of notes
- FUJ ☹

Real world signals

- Signal and spectrum a-open-string_16bit.wav (WS)
- Physics – see for example https://www.youtube.com/watch?v=BSIw5SgUirg (all vibration modes together)

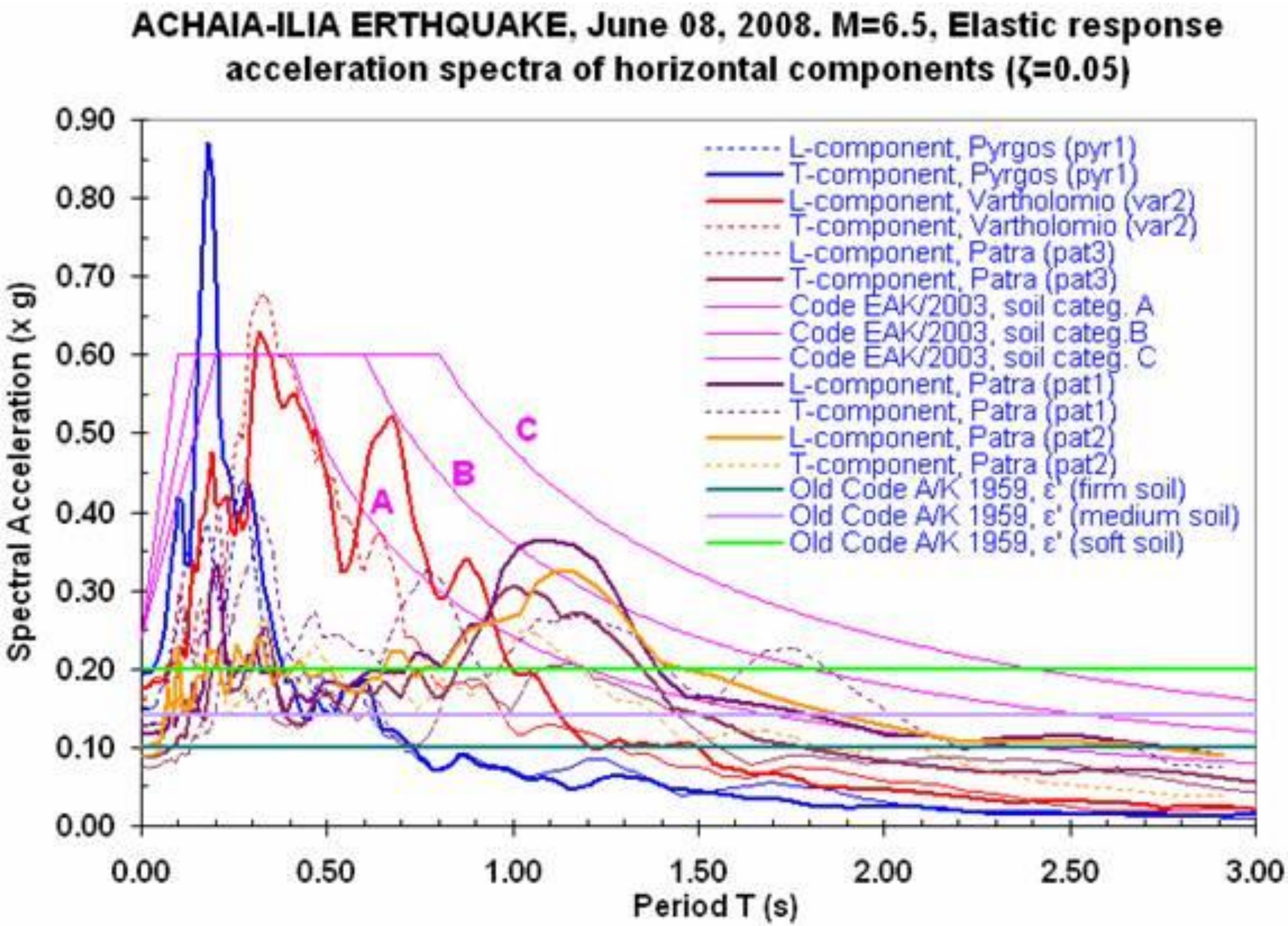
Real world signals

- Signal and spectrum fletna.wav (WS)
- Physics – see for example https://www.youtube.com/watch?v=KZ7intMz2Y4 (all vibration modes together)

Real world signals

- Signal and spectrum test.l16 (WS)
- Physics – see for example https://www.youtube.com/watch?v=y2okeYVclQo (flapping of vocal cords produces lots of harmonic frequencies …)

Seismology …



ACHAIA-ILIA ERTHQUAKE, June 08, 2008. M=6.5, Elastic response acceleration spectra of horizontal components (ζ=0.05)
Spectral Acceleration (x g)
Period T (s)
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
0.00
0.50
1.00
1.50
2.00
2.50
3.00
L-component, Pyrgos (pyr1)
T-component, Pyrgos (pyr1)
L-component, Vartholomio (var2)
T-component, Vartholomio (var2)
L-component, Patra (pat3)
T-component, Patra (pat3)
Code EAK/2003, soil categ. A
Code EAK/2003, soil categ.B
Code EAK/2003, soil categ. C
L-component, Patra (pat1)
T-component, Patra (pat1)
L-component, Patra (pat2)
T-component, Patra (pat2)
Old Code A/K 1959, ε' (firm soil)
Old Code A/K 1959, ε' (medium soil)
Old Code A/K 1959, ε' (soft soil)
A
B
C

Vibration analysis

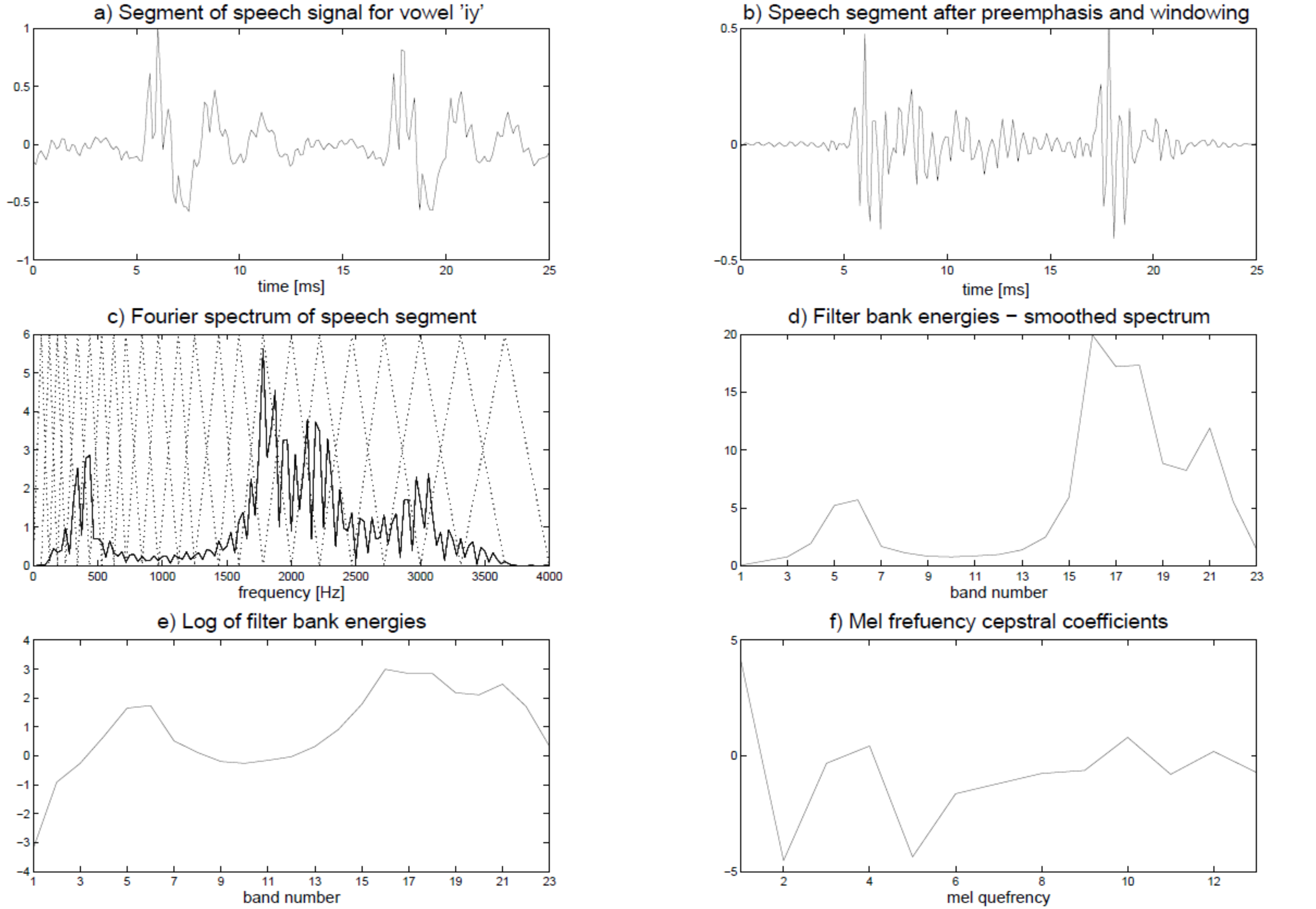
- http://www.dsi-hums.com/honeywell-zing-test/8500-c-plus/

Spectral analysis for what ? I.

- Visualize…

Spectral analysis for what ? II.

- Measure / detect / recognize



Spectral analysis for what ? III.

Filtering

$$y[n] = x[n] * h[n]$$

$$y[n] = F^{-1} [F(x[n]) F(h[n])]$$

Spectral analysis

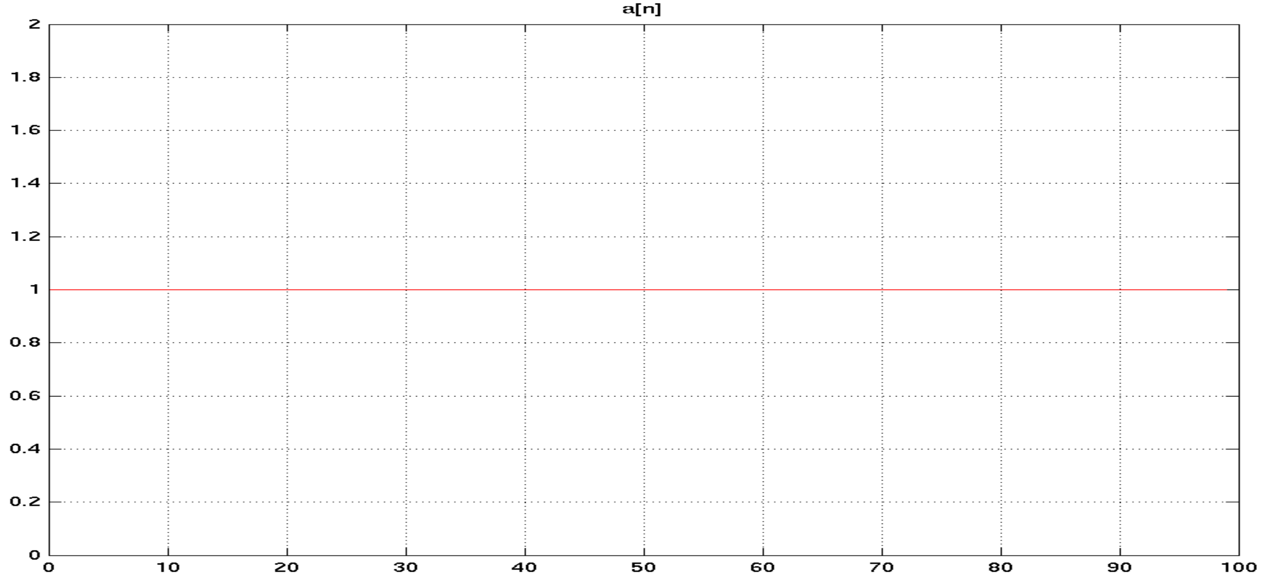
- Correlation
- Determination of similarity
- Projection to bases

} **The same !**

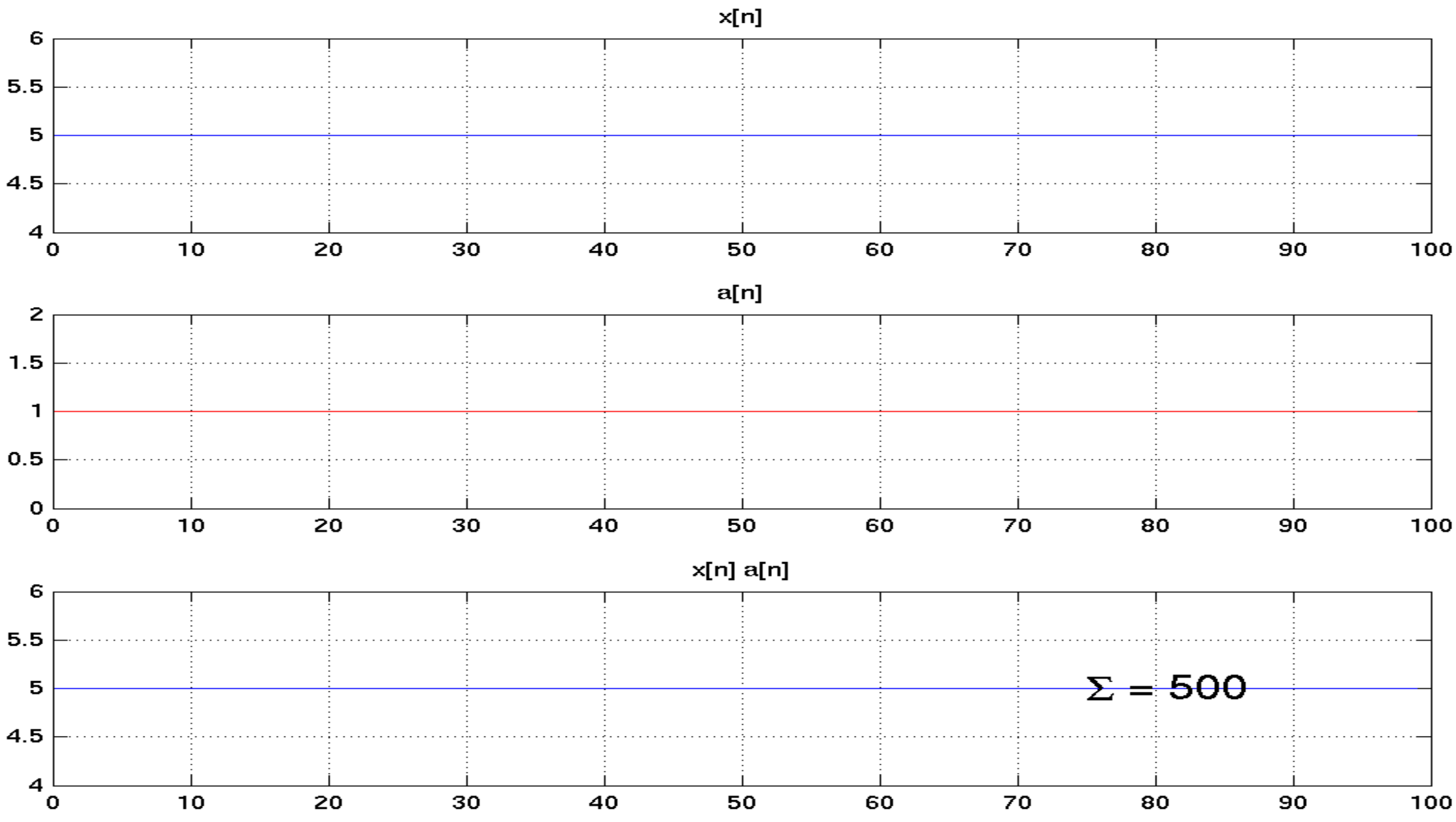
$$c = \sum_{n=0}^{N-1} x[n]a[n]$$

Examples of analysis

- Signal with *N=100* samples
- Let’s begin with a D.C. signal …

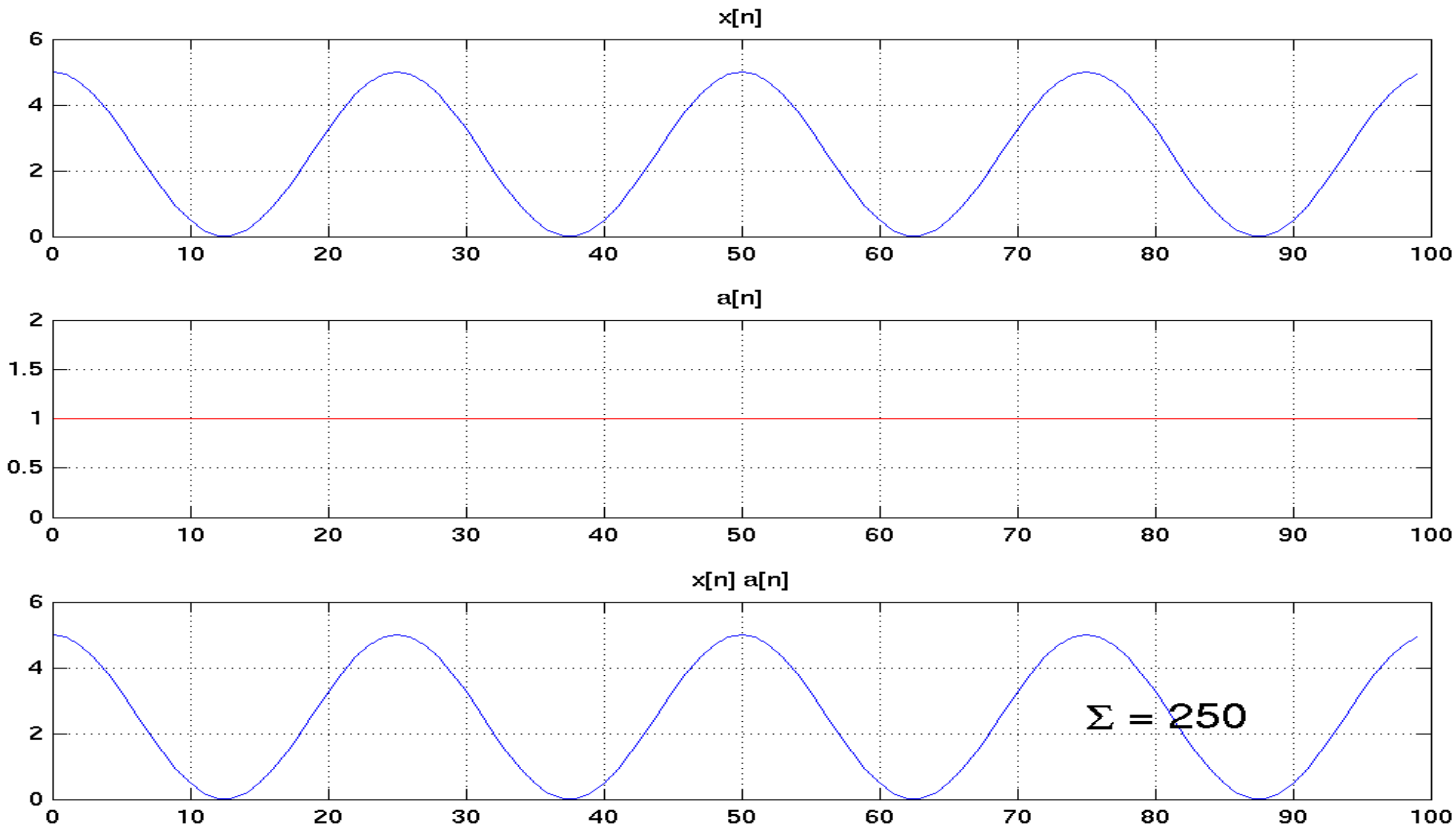


Another D.C.

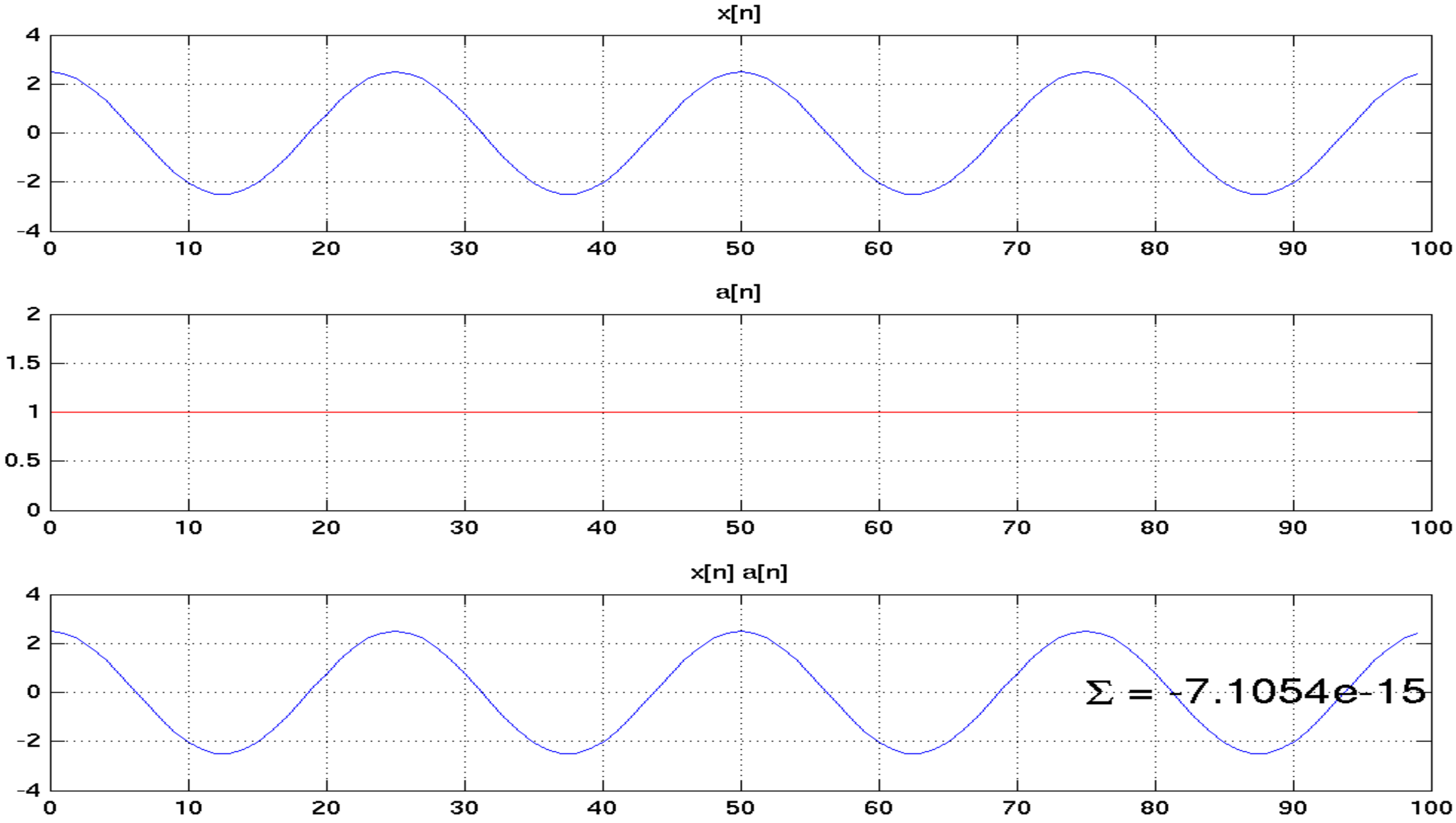


x[n]
6
5.5
5
4.5
4
0
10
20
30
40
50
60
70
80
90
100
a[n]
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90
100
x[n] a[n]
6
5.5
5
4.5
4
0
10
20
30
40
50
60
70
80
90
100
Σ = 500

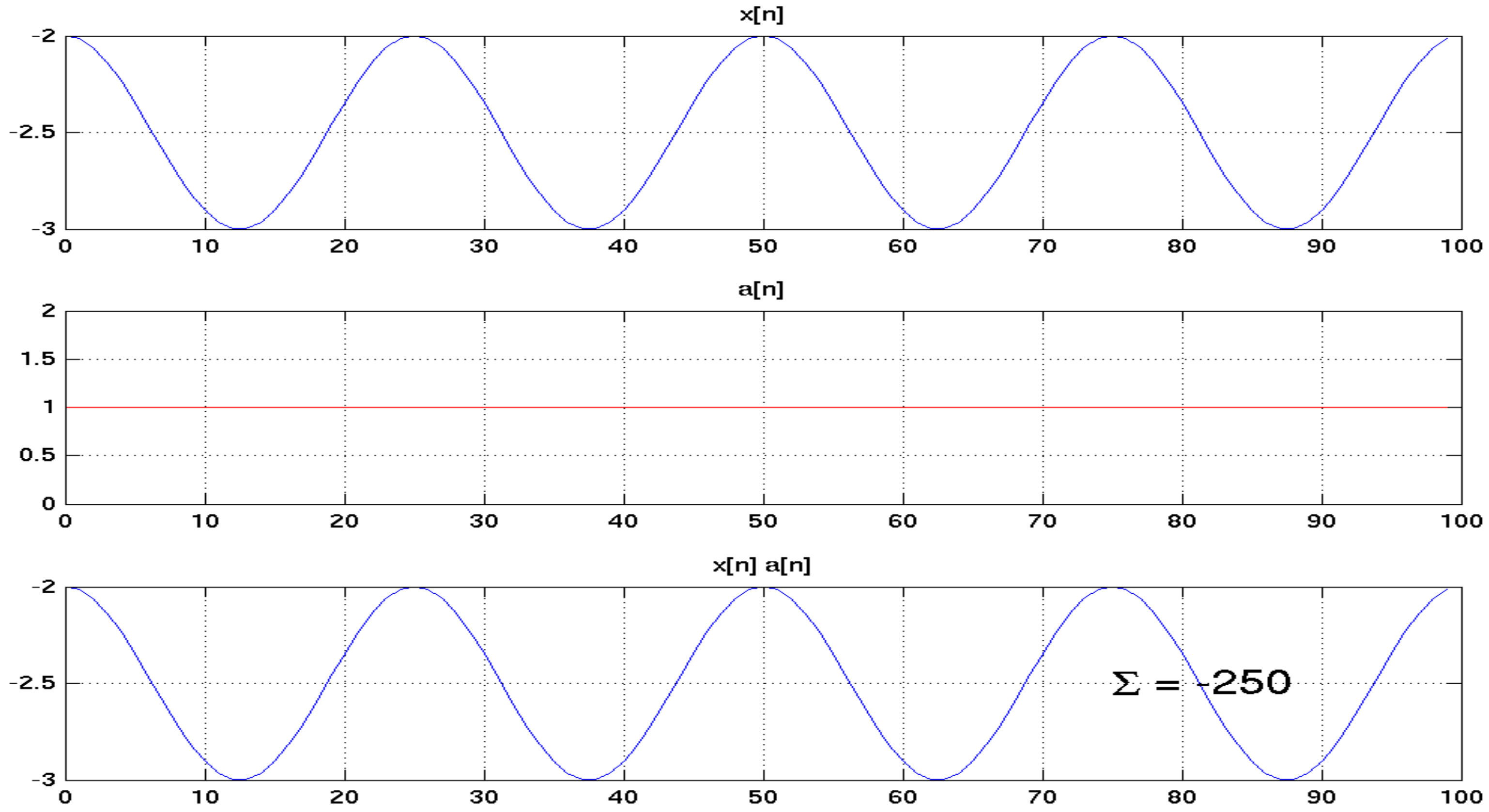
Cosine with a D.C. component



Cosine around zero

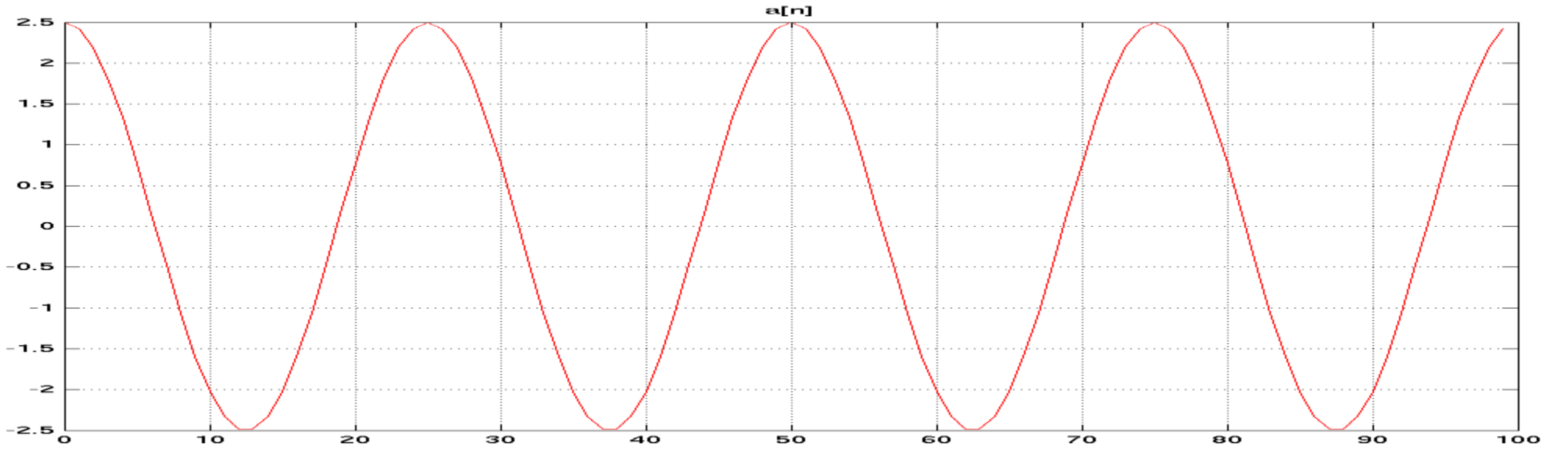


Cosine with a minus D.C. component

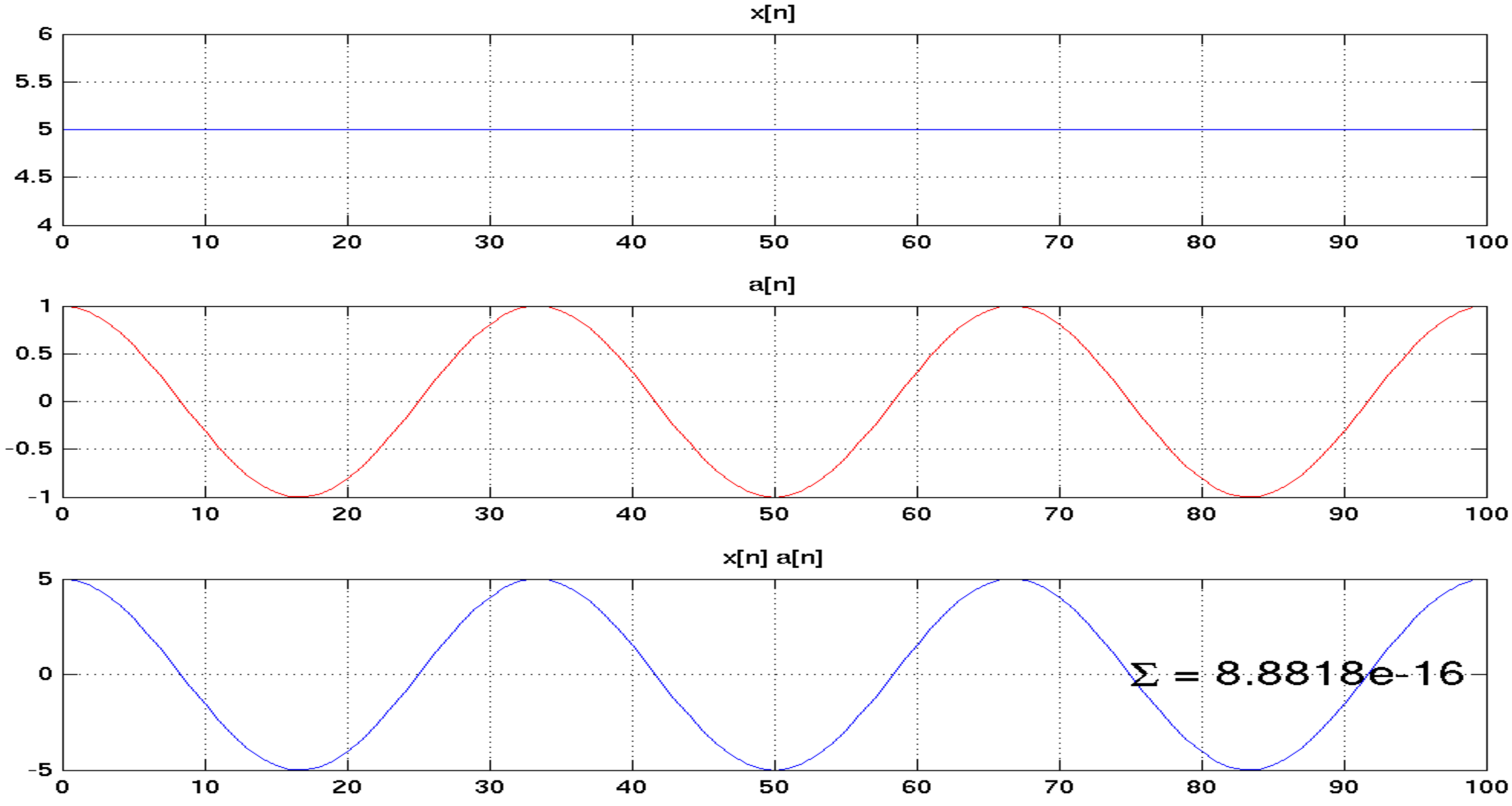


Now with something else

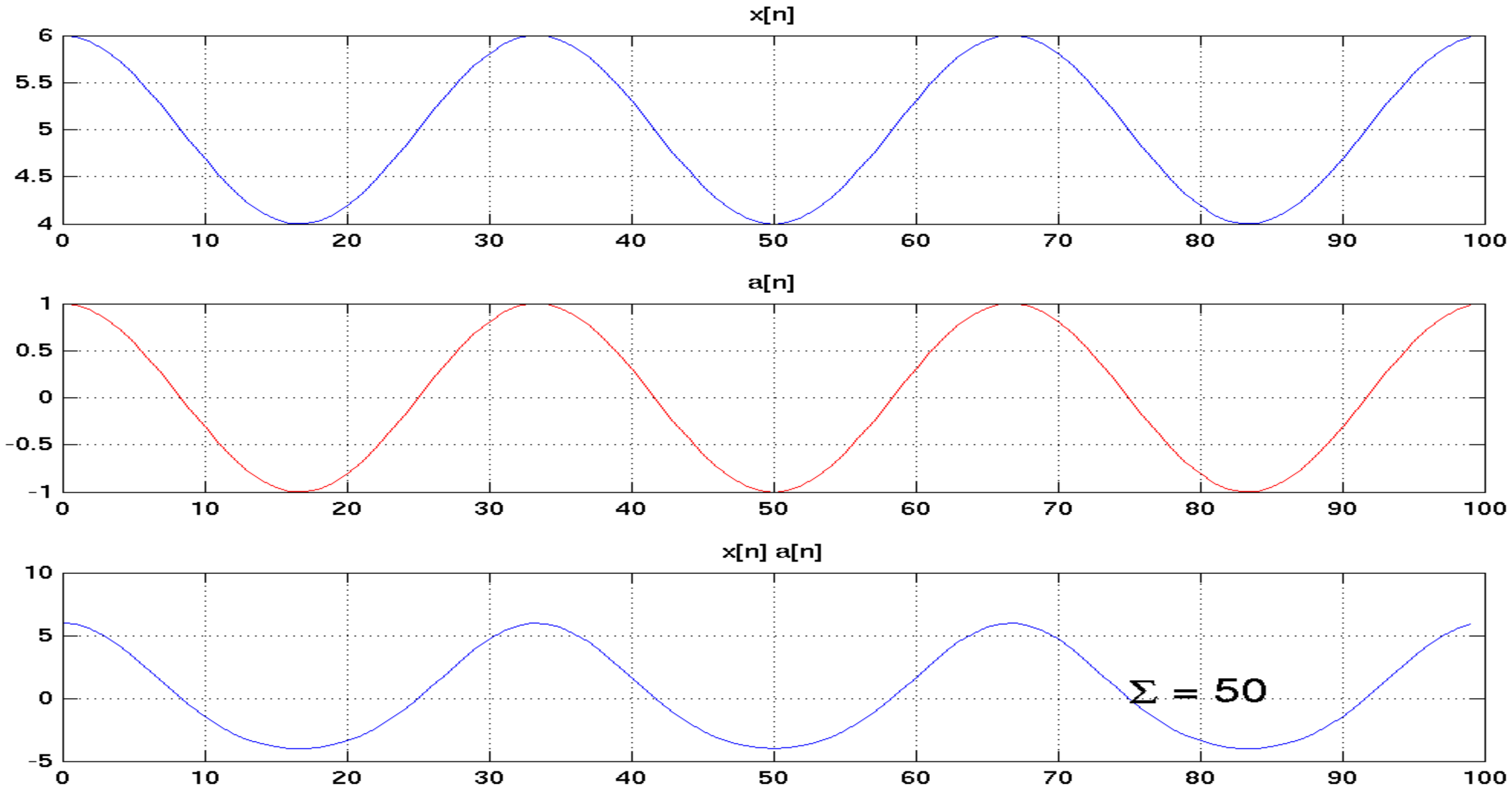
- The analyzing signal will do 3 periods in 100 samples.
- Generating it ?



DC ...

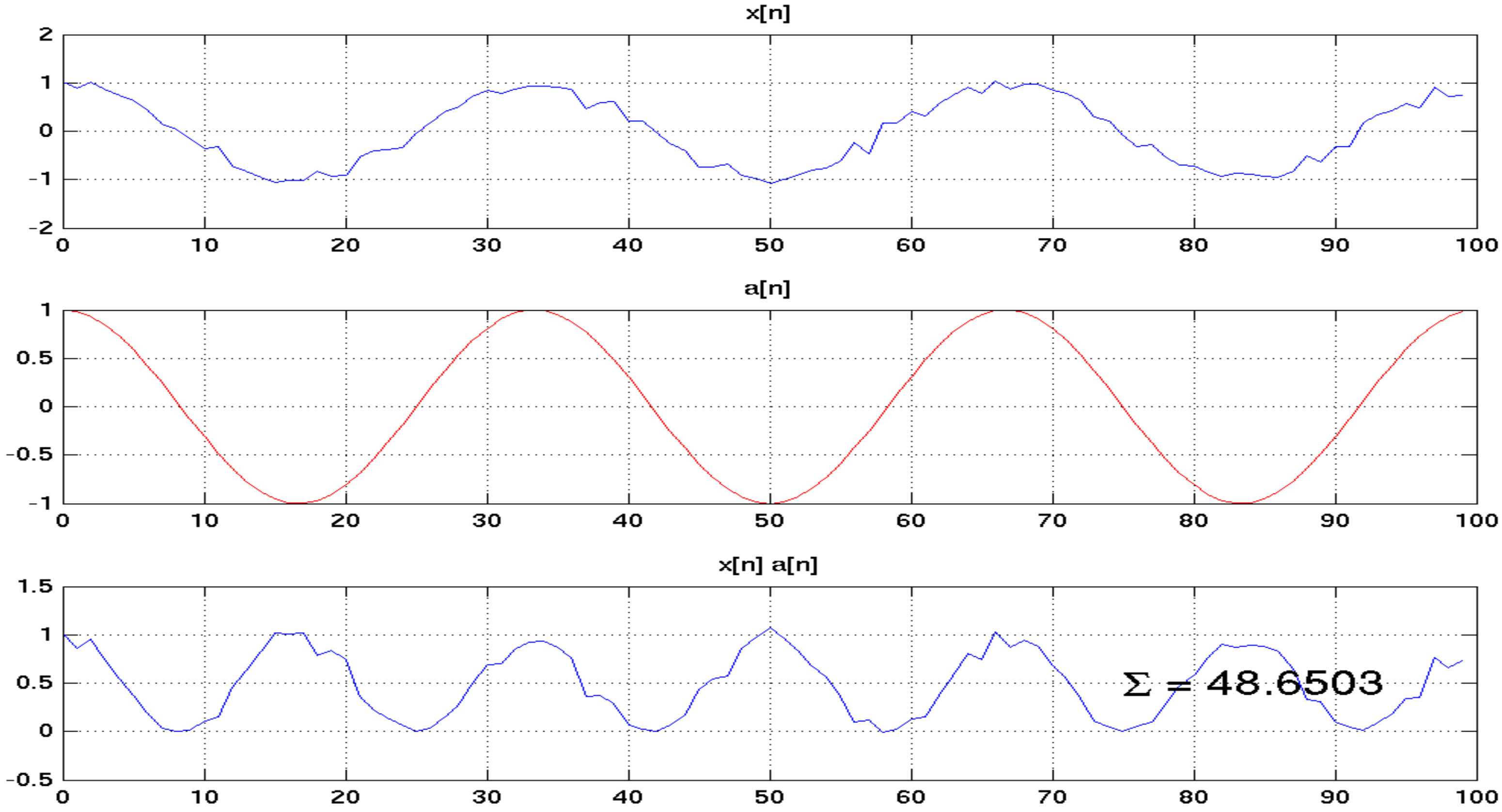

x[n]
6
5.5
5
4.5
4
0 10 20 30 40 50 60 70 80 90 100
a[n]
1
0.5
0
-0.5
-1
0 10 20 30 40 50 60 70 80 90 100
x[n] a[n]
5
0
-5
0 10 20 30 40 50 60 70 80 90 100
Σ = 8.8818e-16

Another (the same) cosine

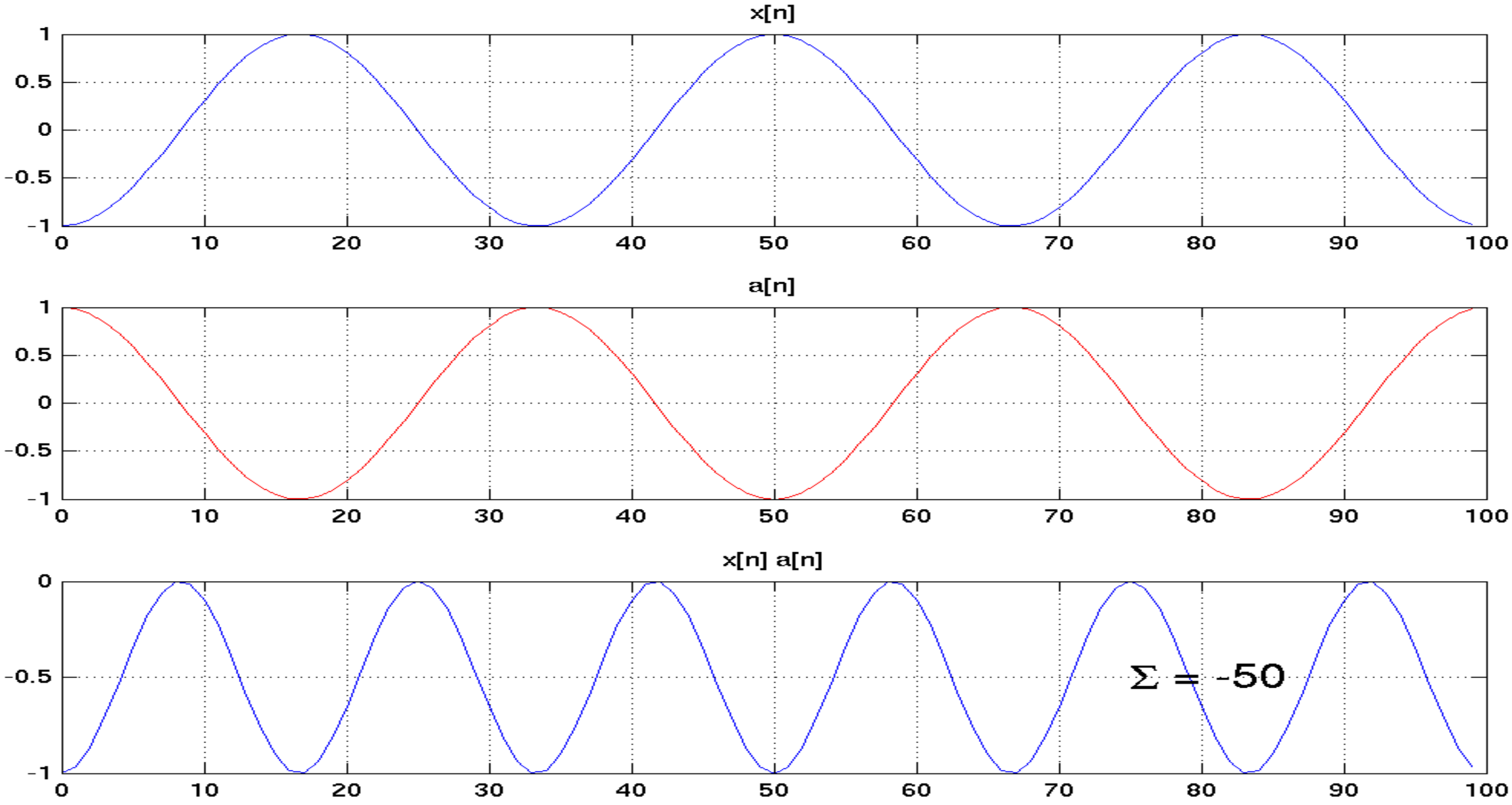


- Will the D.C. have any influence ?

Cosine with noise



Cosine with minus sign



What do the coefficients tell us

- **Big positive** – correlation, similarity, this frequency IS in the analyzed signal.
- **Big negative** – anti-correlation, similar, but in the inverse sense, the frequency IS in the analyzed signal with minus sign.
- **Small / zero** – no correlation, no similarity, the frequency is not there or just a little.

Let’s analyze something more complicated

- **WS: signal.wav**
- Basic period 100 samples
  - How much is this in Hz ?
- Lots of harmonics colored by speech sound “a”
  - Geeks, see spec_matlab.m

Not one but whole group of cosines

- **DEMO 3 in Matlab**
- Until which frequency should we run ?

$$a_0[n] = \cos(2\pi \frac{0}{N} n)$$

$$a_1[n] = \cos(2\pi \frac{1}{N} n)$$

$$a_2[n] = \cos(2\pi \frac{2}{N} n)$$

$$\cdots$$

$$a_{\frac{N}{2}}[n] = \cos(2\pi \frac{\frac{N}{2}}{N} n)$$

Analysis by all this

$$c_0 = \sum_{n=0}^{N-1} a_0[n]x[n]$$

$$c_1 = \sum_{n=0}^{N-1} a_1[n]x[n]$$

$$c_2 = \sum_{n=0}^{N-1} a_2[n]x[n]$$

$$\cdots$$

$$c_{\frac{N}{2}} = \sum_{n=0}^{N-1} a_{\frac{N}{2}}[n]x[n]$$

$$\mathbf{c} = \mathbf{A}\mathbf{x}$$

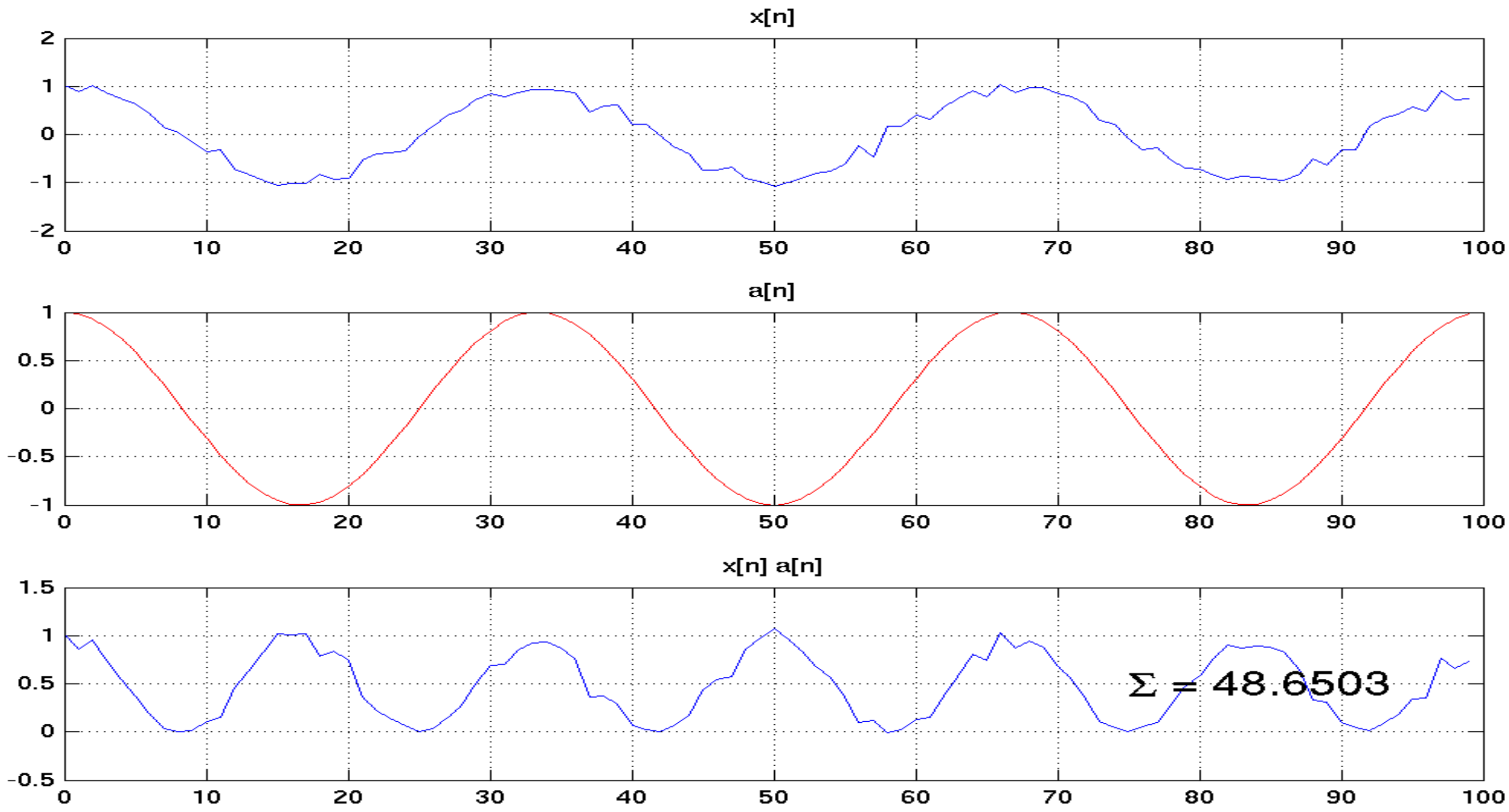
The results and re-synthesis

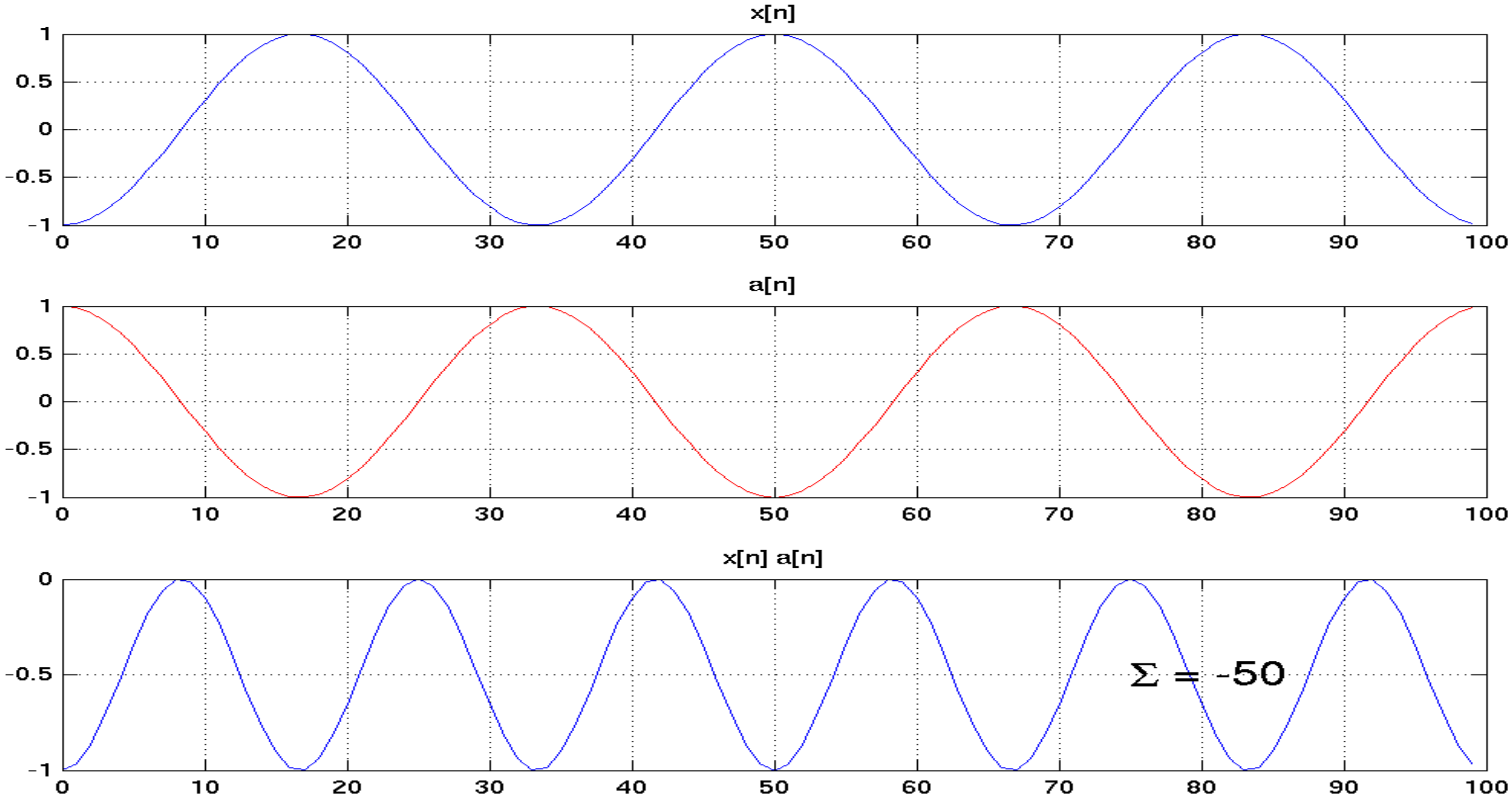
- **still DEMO 3 …**
  - Plot of the results
  - Or of their absolute values
  - Synthesis from the coefficients…

$$xs[n] = c_0 + c_1 \cos(2\pi \frac{1}{N} n) + c_2 \cos(2\pi \frac{2}{N} n) + \ldots + c_{\frac{N}{2}} \cos(2\pi \frac{\frac{N}{2}}{N} n)$$

- **HM HM … ☹**

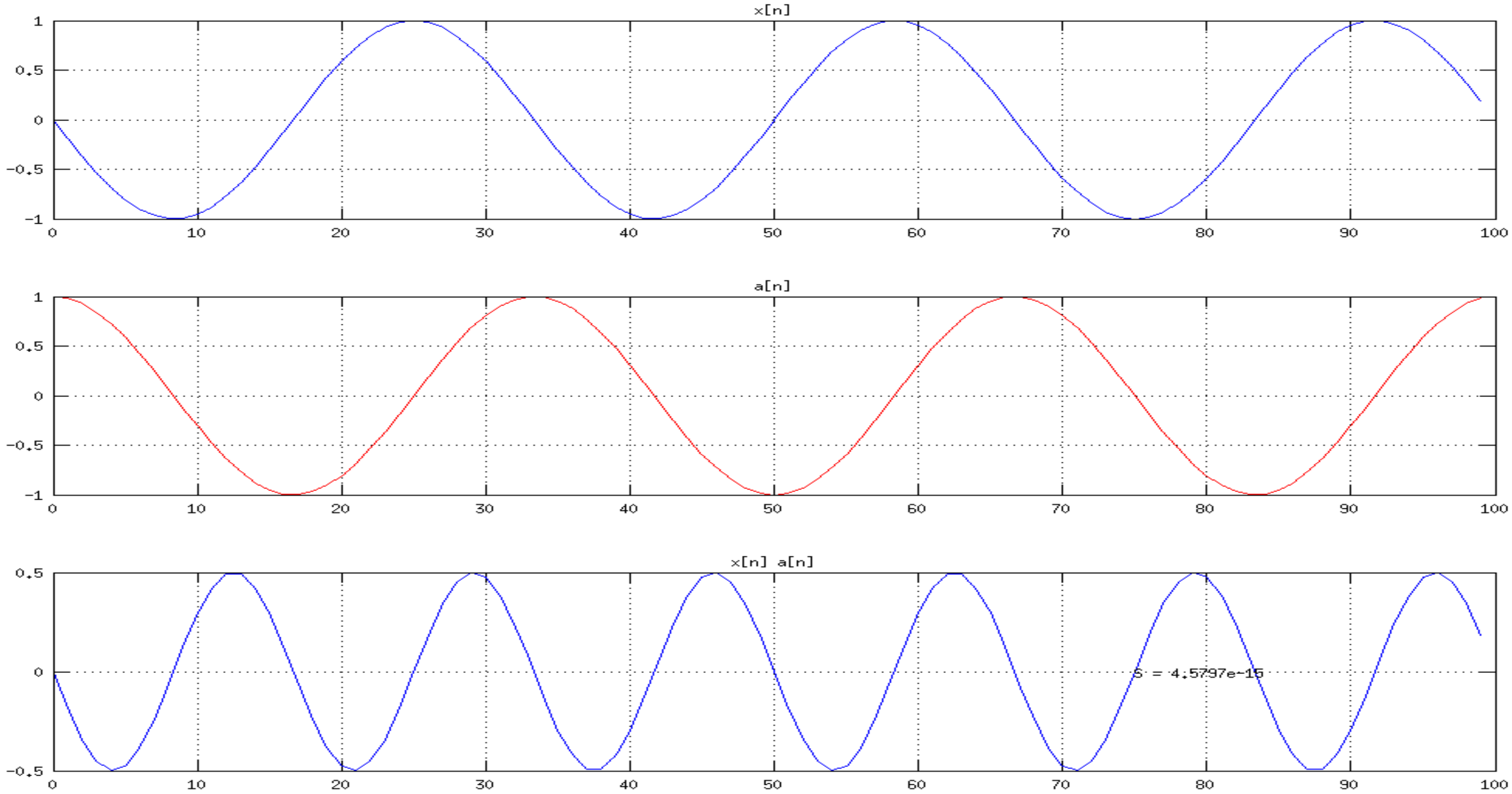
What’s the problem ??





x[n]
1
0.5
0
-0.5
-1
0
10
20
30
40
50
60
70
80
90
100
a[n]
x[n] a[n]
Σ = -50

Phase is the problem !



- How come it's zero when $\sin(x) = \cos(x - \frac{\pi}{2})$

We'll need also the sines …

- **DEMO 4 in Matlab**
- Get coefficient *a* by a projection to cos
- Get coefficient *b* by a projection to sin
- How will sqrt(a^2 + b^2) look like ?

Analysis with whole groups of cosines and sines

$$a_0[n] = \cos(2\pi \frac{0}{N} n) \quad b_0[n] = \sin(2\pi \frac{0}{N} n)$$

$$a_1[n] = \cos(2\pi \frac{1}{N} n) \quad b_1[n] = \sin(2\pi \frac{1}{N} n)$$

$$a_2[n] = \cos(2\pi \frac{2}{N} n) \quad b_2[n] = \sin(2\pi \frac{2}{N} n)$$

$$\ldots \qquad \ldots$$

$$a_{\frac{N}{2}}[n] = \cos(2\pi \frac{\frac{N}{2}}{N} n) \quad b_{\frac{N}{2}}[n] = \sin(2\pi \frac{\frac{N}{2}}{N} n)$$

- How will the analysis signals for limit values look like ?

Let's go

$$\mathbf{c} = \mathbf{A}\mathbf{x}, \qquad \mathbf{d} = \mathbf{B}\mathbf{x}$$

$$c_0 = \sum_{n=0}^{N-1} a_0[n]x[n] \qquad d_0 = \sum_{n=0}^{N-1} b_0[n]x[n]$$

$$c_1 = \sum_{n=0}^{N-1} a_1[n]x[n] \qquad d_1 = \sum_{n=0}^{N-1} b_1[n]x[n]$$

$$c_2 = \sum_{n=0}^{N-1} a_2[n]x[n] \qquad d_2 = \sum_{n=0}^{N-1} b_2[n]x[n]$$

$$\ldots \qquad \ldots$$

$$c_{\frac{N}{2}} = \sum_{n=0}^{N-1} a_{\frac{N}{2}}[n]x[n] \qquad d_{\frac{N}{2}} = \sum_{n=0}^{N-1} b_{\frac{N}{2}}[n]x[n]$$

How did it work

- **DEMO 5 in Matlab**
  - Visualizationí
  - Re-synthesis

$$xs[n] = c_0 + c_1 \cos(2\pi \frac{1}{N} n) + c_2 \cos(2\pi \frac{2}{N} n) + \ldots + c_{\frac{N}{2}} \cos(2\pi \frac{\frac{N}{2}}{N} n)$$

$$+ d_1 \sin(2\pi \frac{1}{N} n) + d_2 \sin(2\pi \frac{2}{N} n) + \ldots + d_{\frac{N}{2}} \sin(2\pi \frac{\frac{N}{2}}{N} n)$$

- Nice ☺

cos and sin in one function – complex exponentials

$$X_k = c_k - j d_k$$

- The meaning of $|X_k|$
- *…* and arg(X_k) ?
- What is k ?

This veeeery complicated maths

$$
\begin{aligned}
X_k &= c_k - j d_k \\
&= \sum_{n=0}^{N-1} x[n] \cos(2\pi \frac{k}{N} n) - j \sum_{n=0}^{N-1} x[n] \sin(2\pi \frac{k}{N} n) \\
&= \sum_{n=0}^{N-1} x[n] \left[\cos(2\pi \frac{k}{N} n) - j \sin(2\pi \frac{k}{N} n) \right] \\
&= \sum_{n=0}^{N-1} x[n] e^{-j 2\pi \frac{k}{N} n}
\end{aligned}
$$

Discrete Fourier transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j 2\pi \frac{k}{N} n}, \quad k = 0 \ldots N-1$$

- What is what ?
  - *x[n]* and *n*
  - *X[k]* and *k*
  - *k/N* and multiplication by 2π ...

DFT in matrix way

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

How does complex exponential look like ?

- **DEMO 6 in Matlab**
- Physical model
- Make one yourself !

Use of DFT

- Select *N* samples out of your signal (good if it is a power of 2)
- Call it (`fft`, ne `dft` …)
- Limit samples to *0…N/2*
- Visualize it

A nice frequency axis

- **DEMO 7 in Matlab**
- Frequency axis
  - Indices *0 … N-1*
  - Normlized frequencies *0/N … (N-1)/N*
  - Real frequencies *0* … almost F_s
  - And attention, most often, we want to see just *N/2+1* samples

Inverse DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j2\pi \frac{k}{N} n}$$

- **Pokračování DEMO 7**

SUMMARY

- We analyze by multiplying and adding 7
- Difficult signals are analyzed by harmonically related functions
  - Cosines – not enough
  - Cosines and sines
  - But even better complex exponentials => DFT
- The results are there for *N* discrete frequencies from 0 till almost F_s
  - Of these, only N/2+1 are worth showing
  - But with a nice frequency axis !

TO BE DONE

- How is it with the phases ?
- Why this minus sign ? $X_k = c_k - jd_k$
- How is it possible, that the inverse DFT processes complex coefficients, the functions are complex too, and still it produces a real signal ?
- What to do if we need more points than *N* (making the plot more beautiful ?)
- Answers
  - Continuation of ISS
  - Or self-thinking supported by literature and online sources.

The END