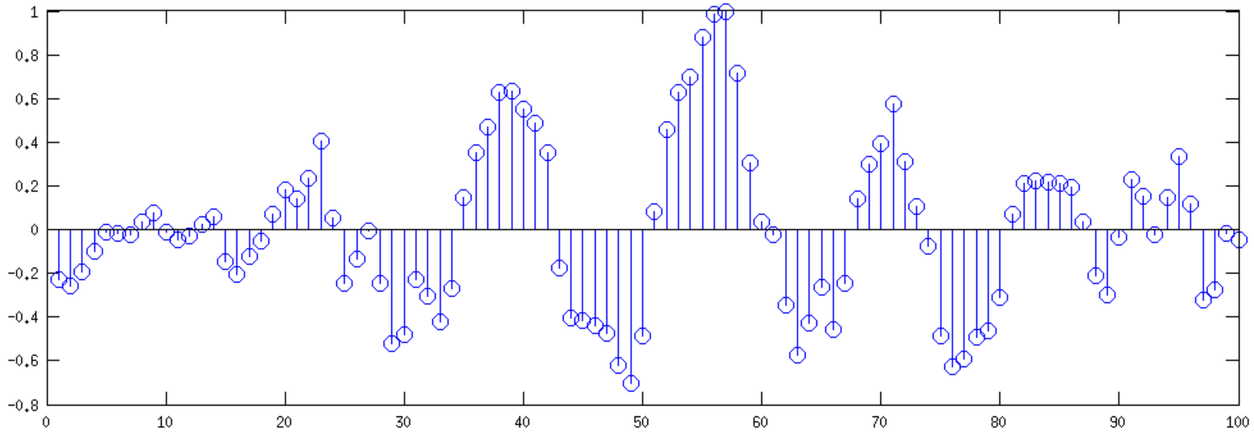
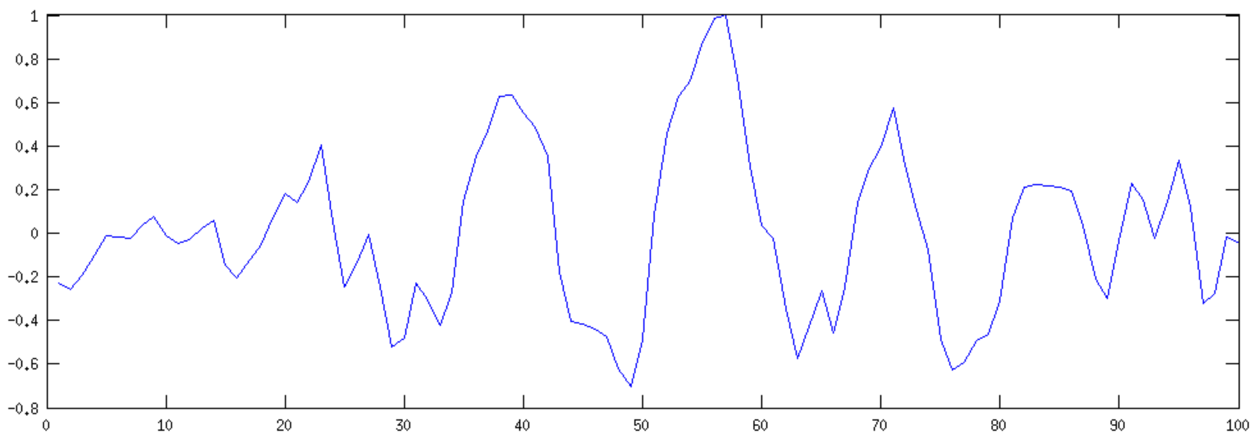
# Image processing

Honza Černocký, ÚPGM

# 1D signal

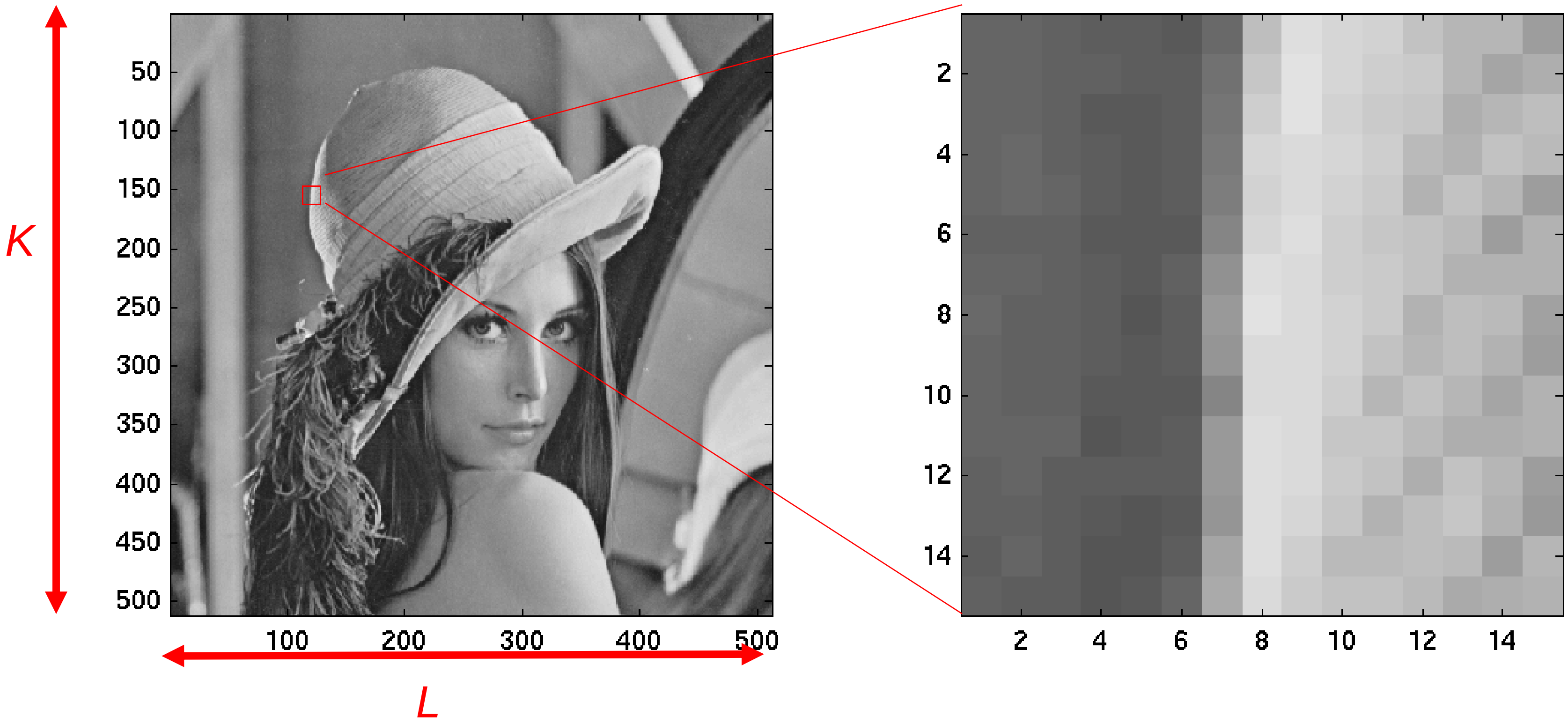

1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
10
20
30
40
50
60
70
80
90
100


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
10
20
30
40
50
60
70
80
90
100

# Images

- 2D – grayscale photo
- Multiple 2D – color photo
- 3D – medical imaging, vector graphics, point-clouds (depth information, Kinect)
- Video – time is another dimension

# Image



K
L
50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
2
4
6
8
10
12
14
2
4
6
8
10
12
14

# Mathematically

$x[k, l]$

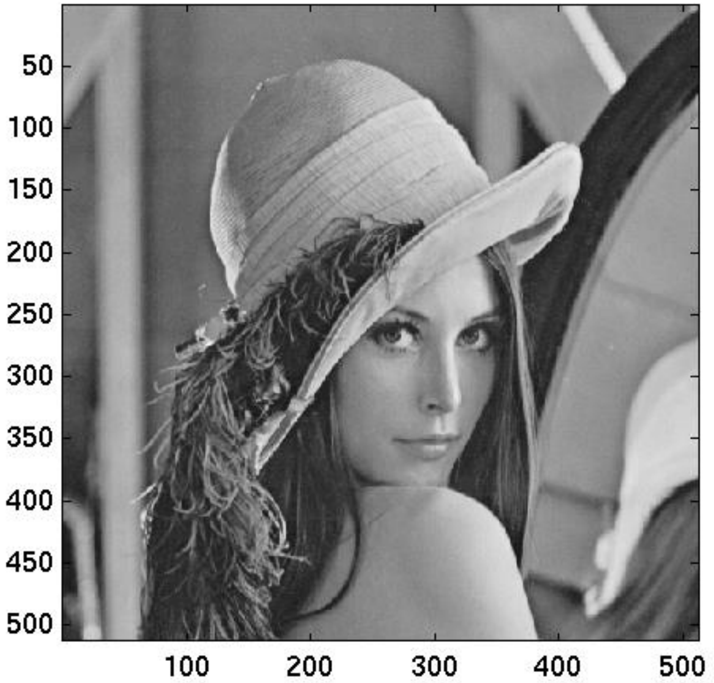
$$x[k,l] = \begin{bmatrix} x[0,0] & x[0,1] & \cdots & x[0,L-1] \\ x[1,0] & x[1,1] & \cdots & x[1,L-1] \\ \vdots & & & \vdots \\ x[K-1,0] & x[K-1,1] & \cdots & x[K-1,L-1] \end{bmatrix}$$

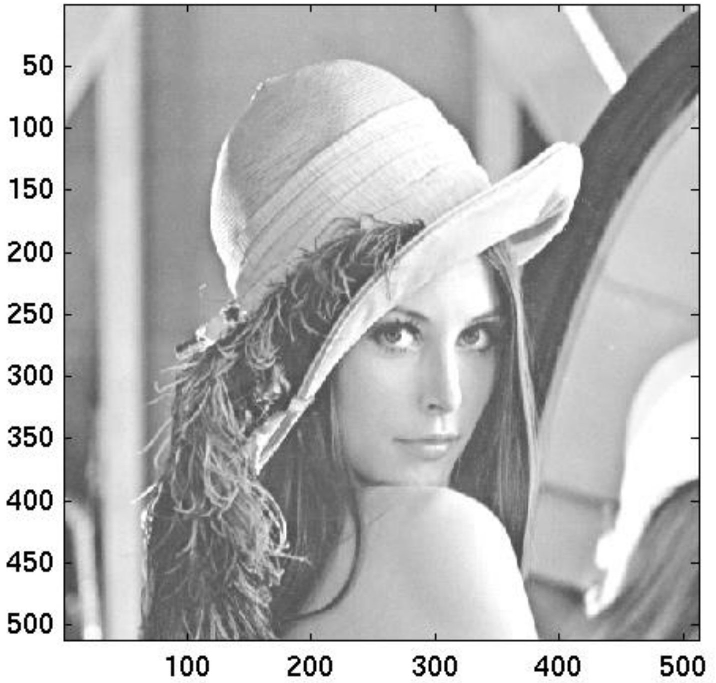
# Representation of pixels

- For storage – quantization
- 8 (normal) / 16 (HDR) bits
- 0…255 etc
- For computing [0…1]
- **floats**

# Operations on pixels – more brightness

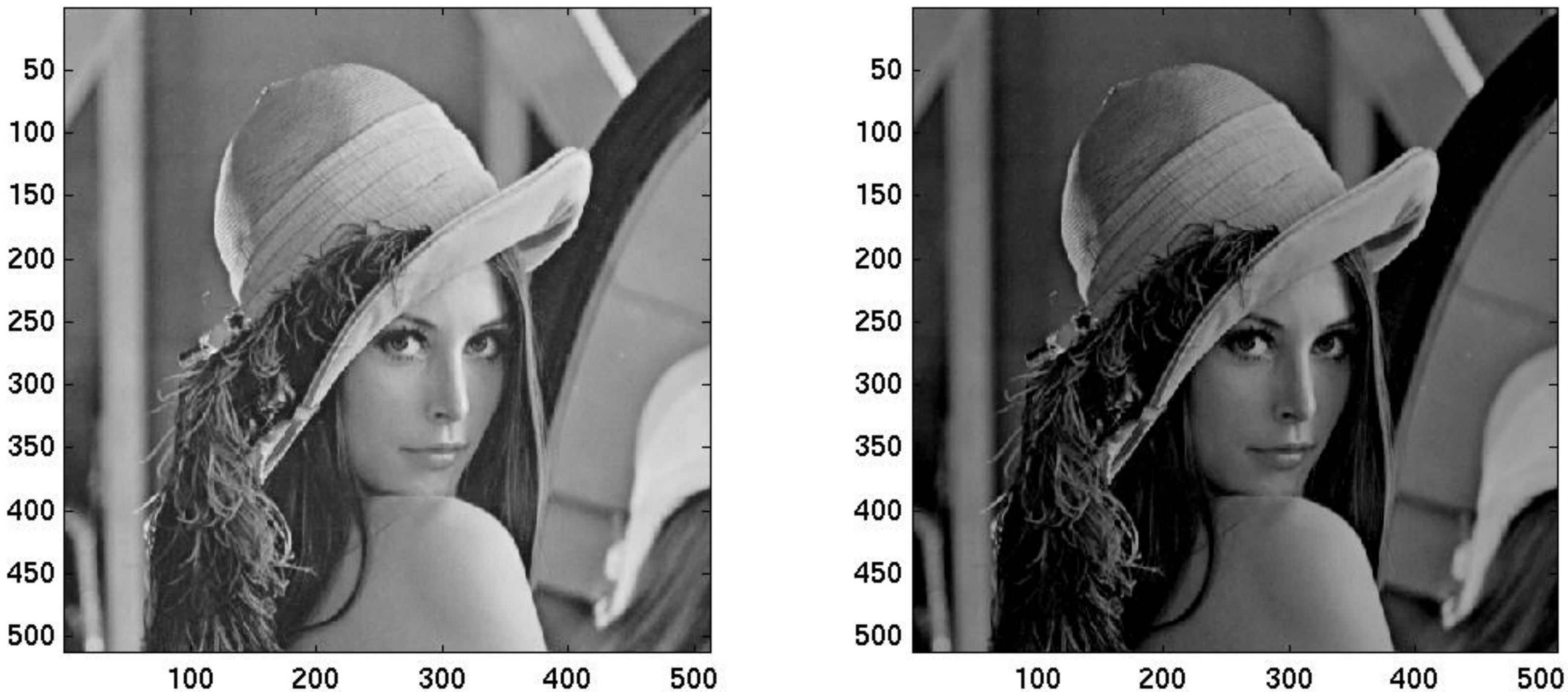
$$y[k,l] = x[k,l] + const.$$



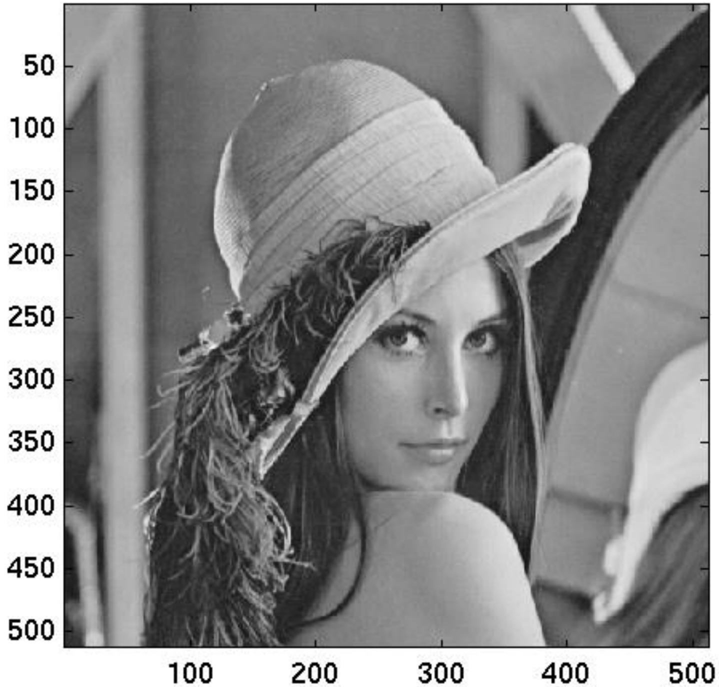
# Operations on pixels – less brightness

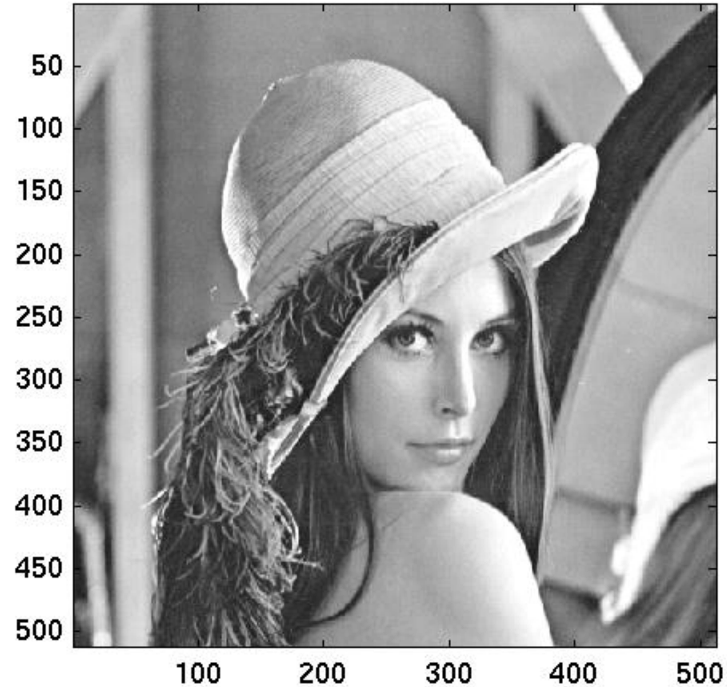
$$y[k,l] = x[k,l] + const.$$



# More contrast

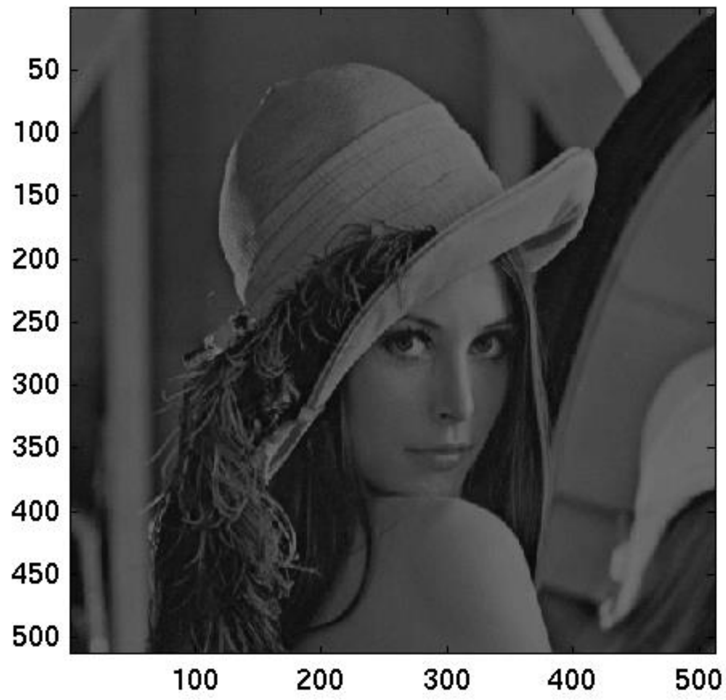
$$y[k,l] = x[k,l] \times const.$$



# Less contrast

$$y[k,l] = x[k,l] \times const.$$

# What about bad values ?

$x[k,l] < 0$ or $x[k,l] > 1$

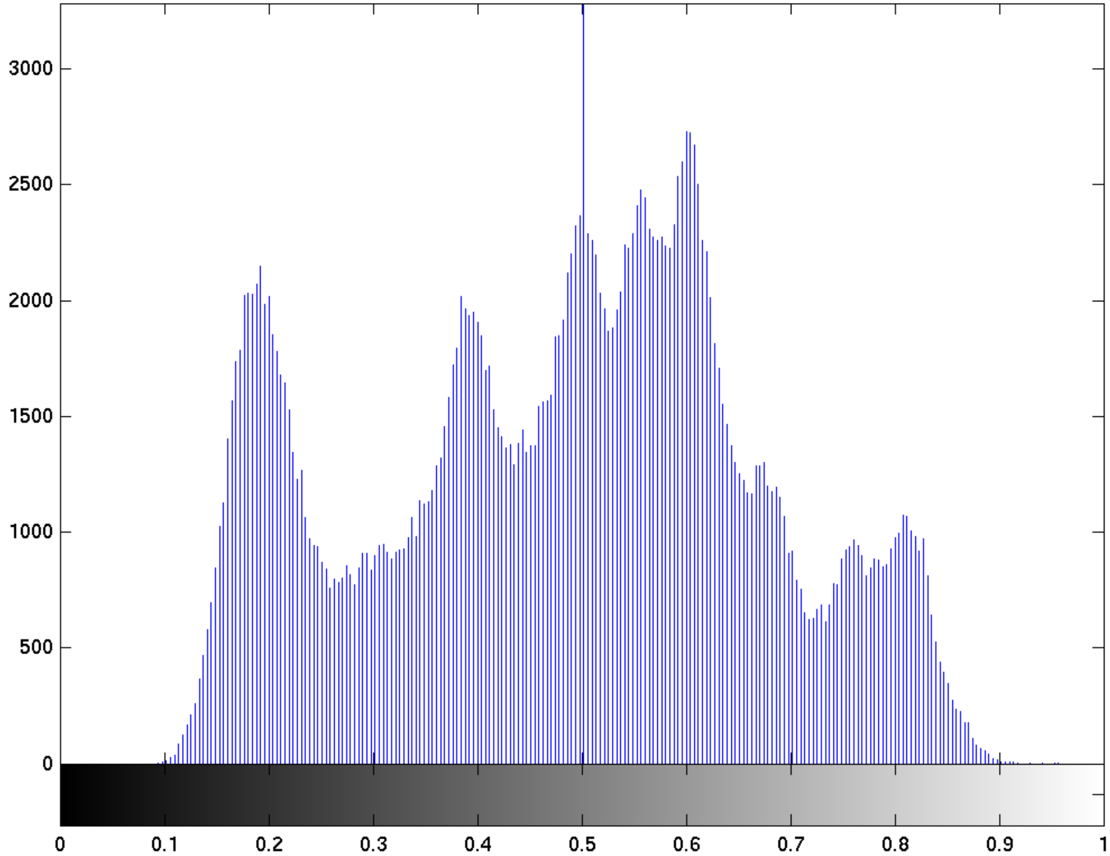
- Let them be …
- Clip them:

$$y[k,l] = 0, \quad \text{if} \;\; y[k,l] < 0$$

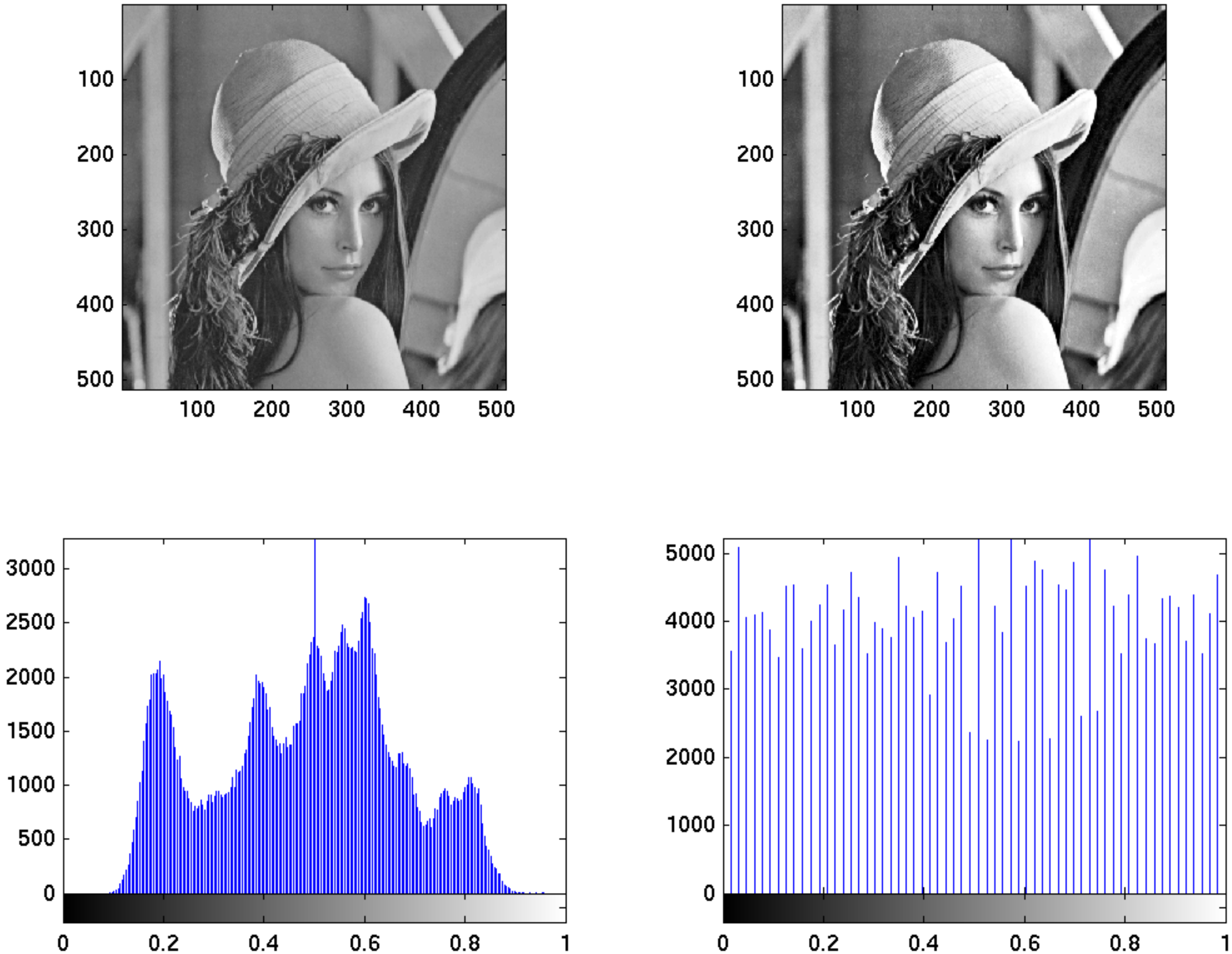
$$y[k,l] = 1, \quad \text{if} \;\; y[k,l] > 1$$

# Use statistics of values

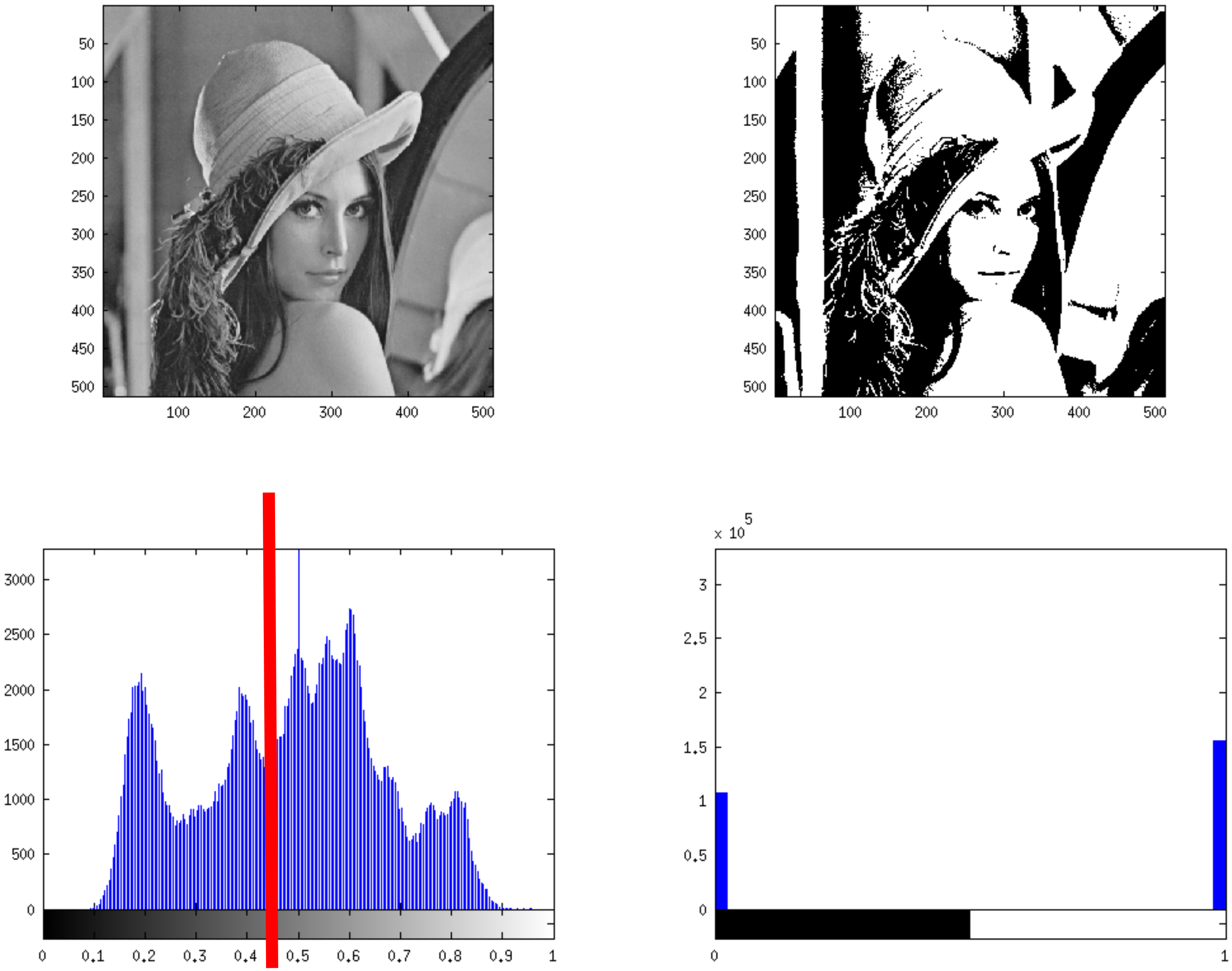
- Histogram



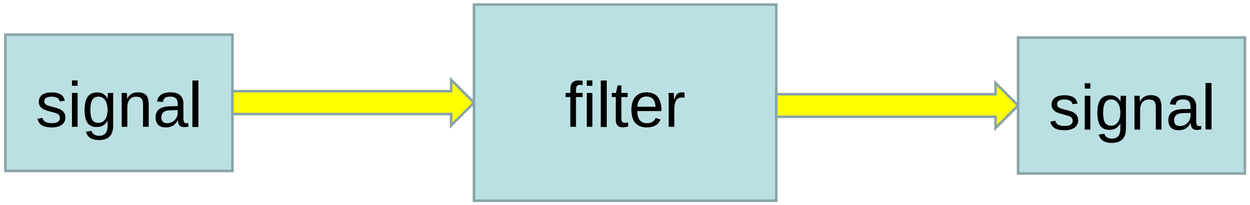
# Histogram equalization



# Thresholding



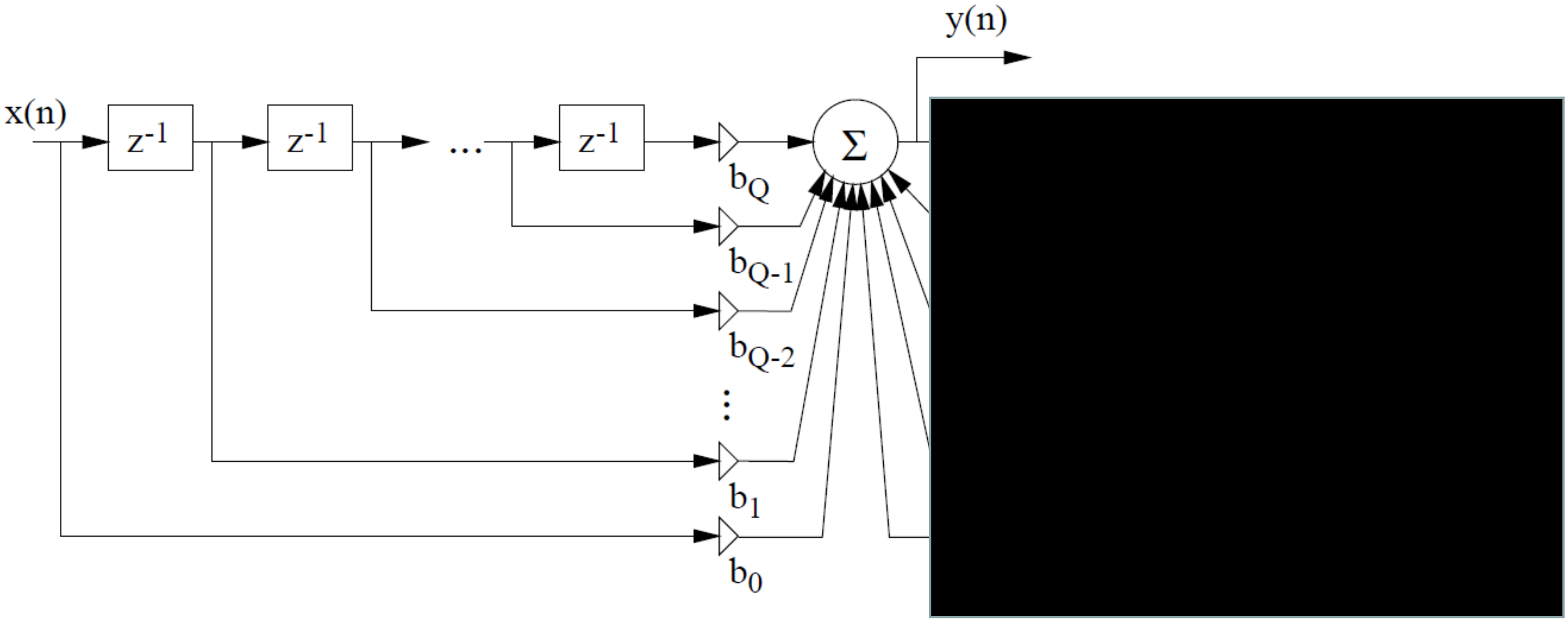
# Filtering




Task ?

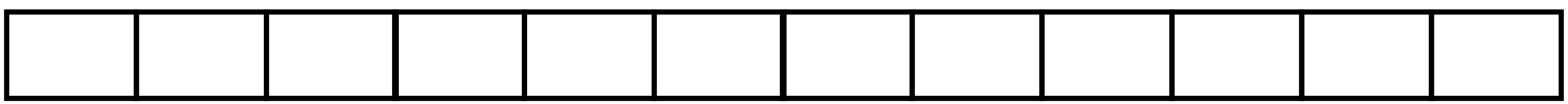
Obtain a **new signal** with desired properties.

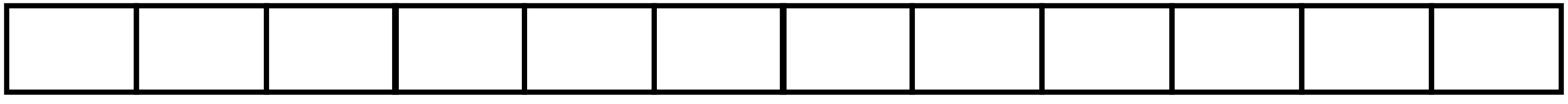
# Filtering – reminder of 1D



# Filtering – reminder of 1D

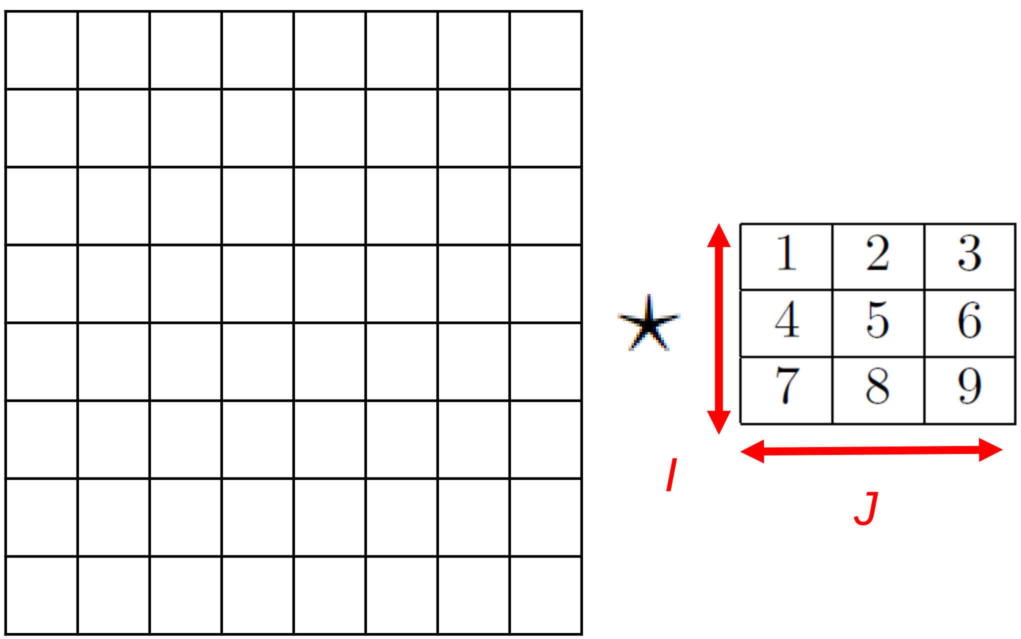
$$y[n] = x[n] \star h[n] = \sum_{k=0}^{Q} h[k]x[n-k]$$



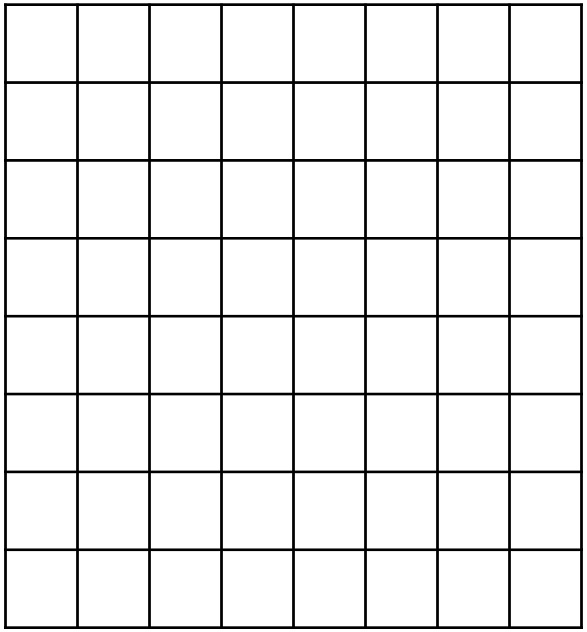


# 2D filter

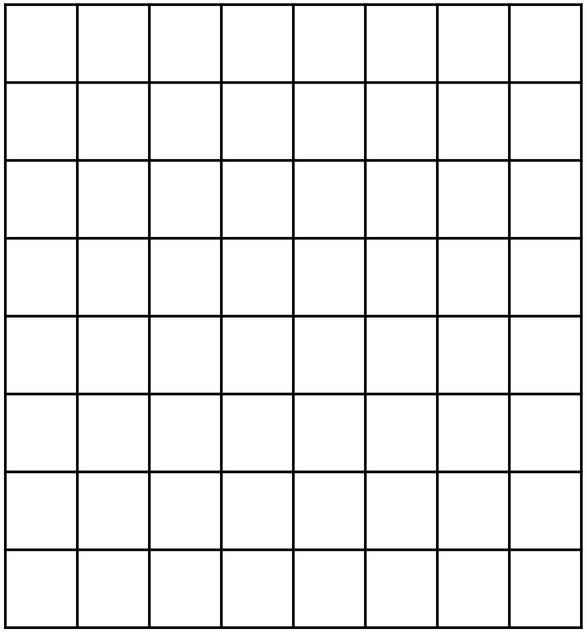
$$y[k,l] = x[k,l] \star h[k,l] = \sum_{m=-\frac{I-1}{2}}^{\frac{I-1}{2}} \sum_{n=-\frac{J-1}{2}}^{\frac{J-1}{2}} h[m,n]x[k-m,l-n]$$



# Filtering …



| | | |
|---|---|---|
| 9 | 8 | 7 |
| 6 | 5 | 4 |
| 3 | 2 | 1 |



Edge ? Insert zeros for example …

# Coefficients *h[k,l]*

**Terminology**

- Coefficients
- Mask
- Convolution kernel …

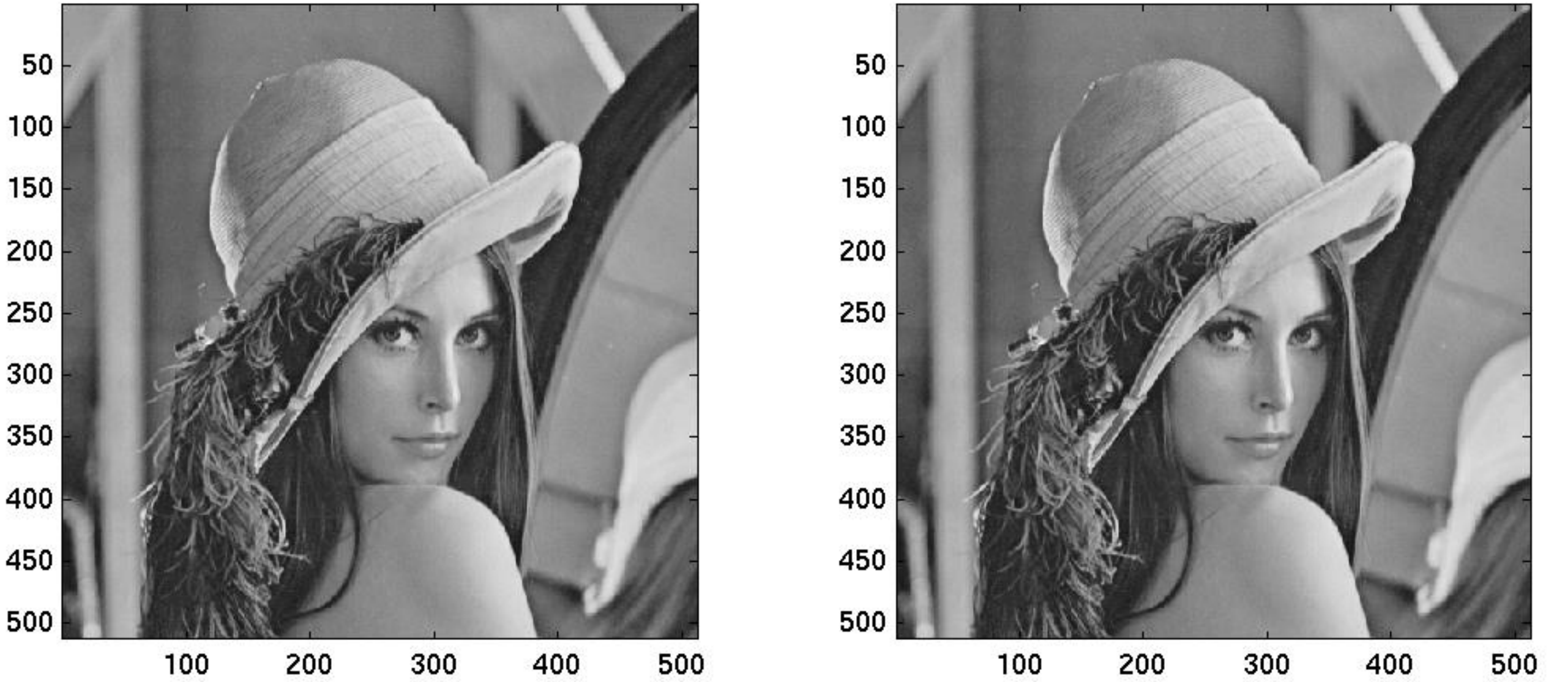
**We want them**

- to have some sense …
- not to change the dynamics of the signal

$$\sum_k \sum_l |h[k,l]| = 1$$

# Wire

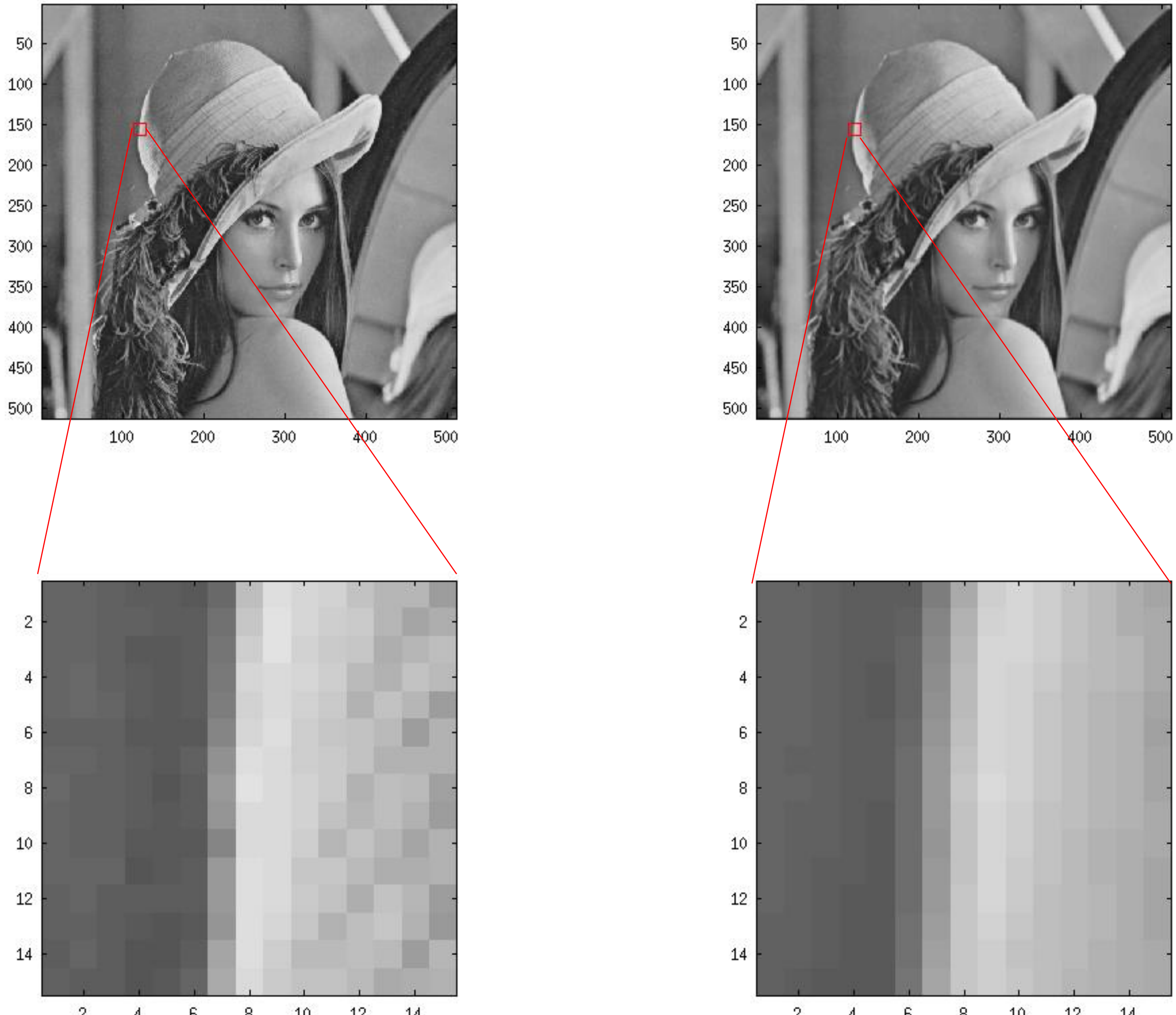
$$h[k,l] = [1]$$



50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
50
100
150
200
250
300
350
400
450
500
100
200
300
400
500

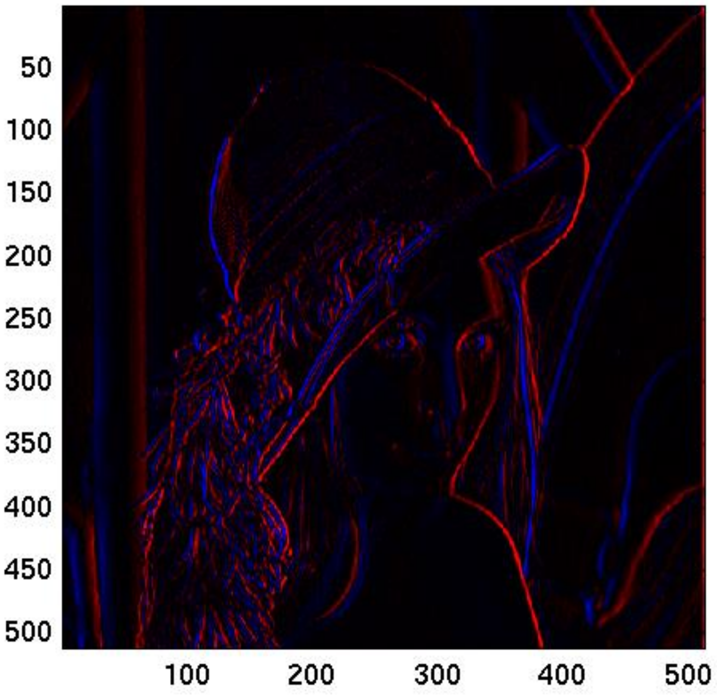
# Smoothing

$$h[k,l] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



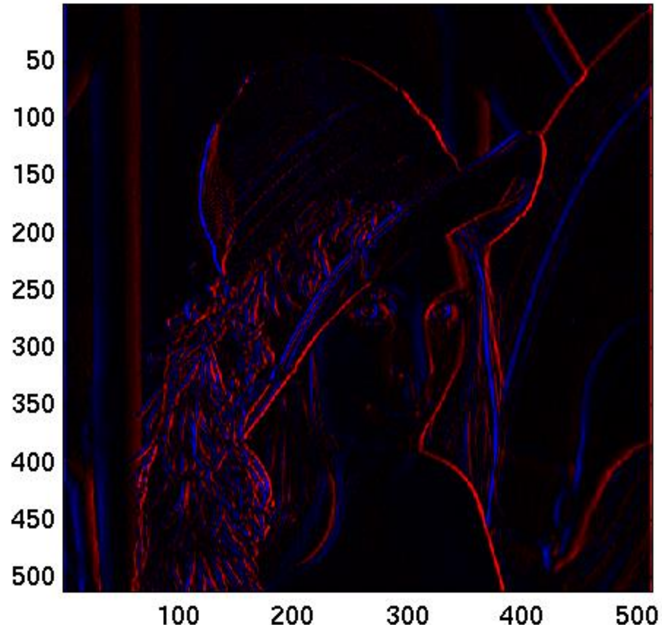
# Vertical edge detector

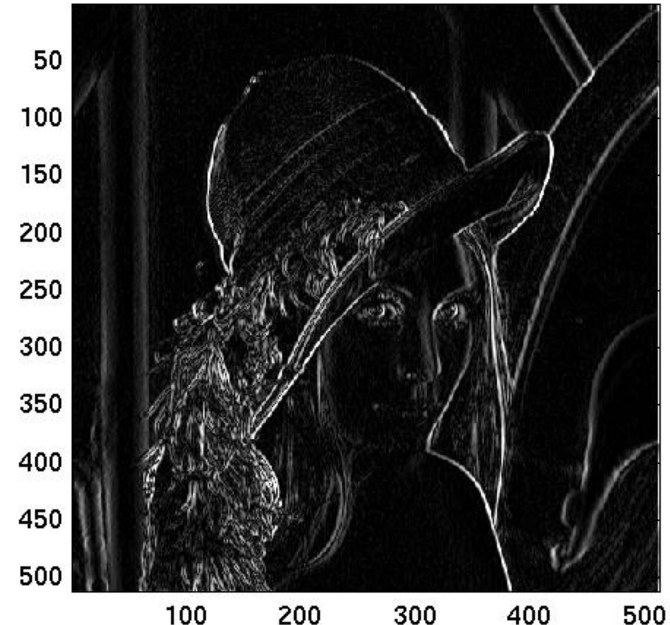
$$h_v[k,l] = \frac{1}{8}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

# What to do with negative samples?

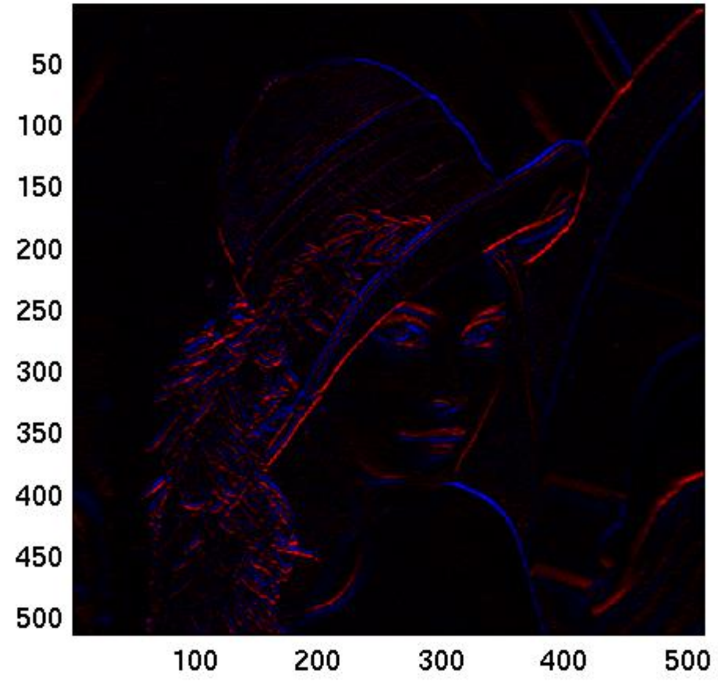
- Let them as they are for further computation…
- Or convert to [0…1] for visualization – for example take absolute value



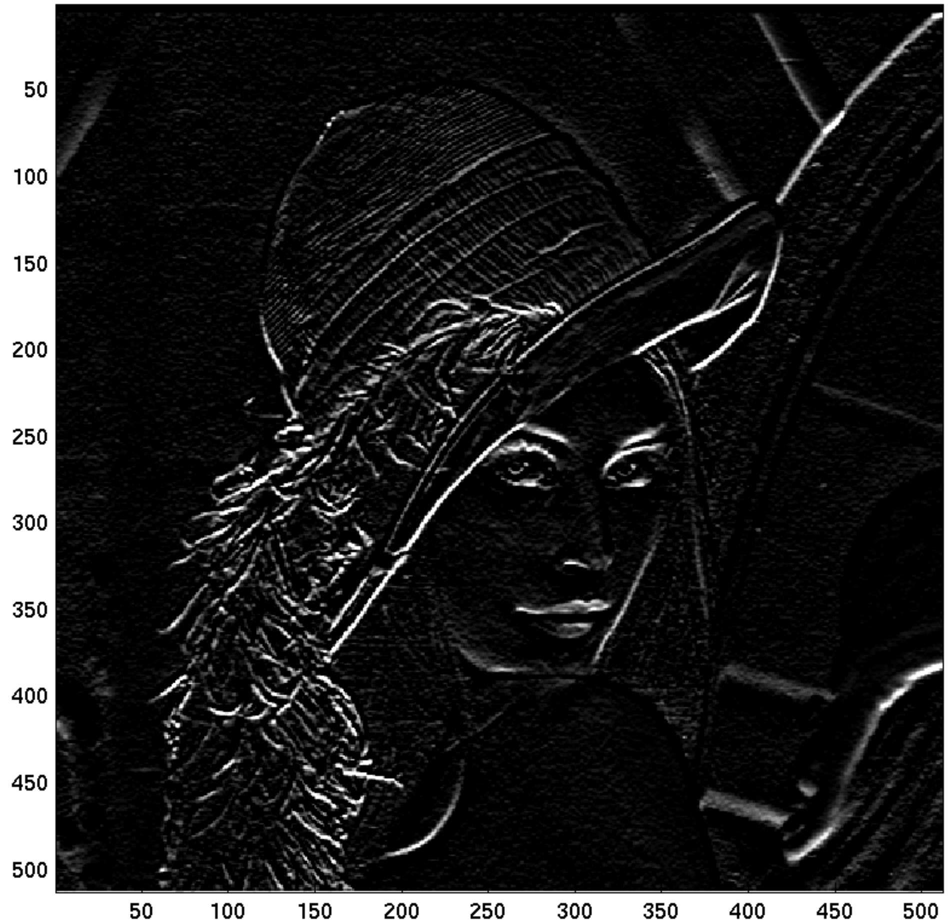
# Horizontal edge detector

$$h_h[k,l] = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

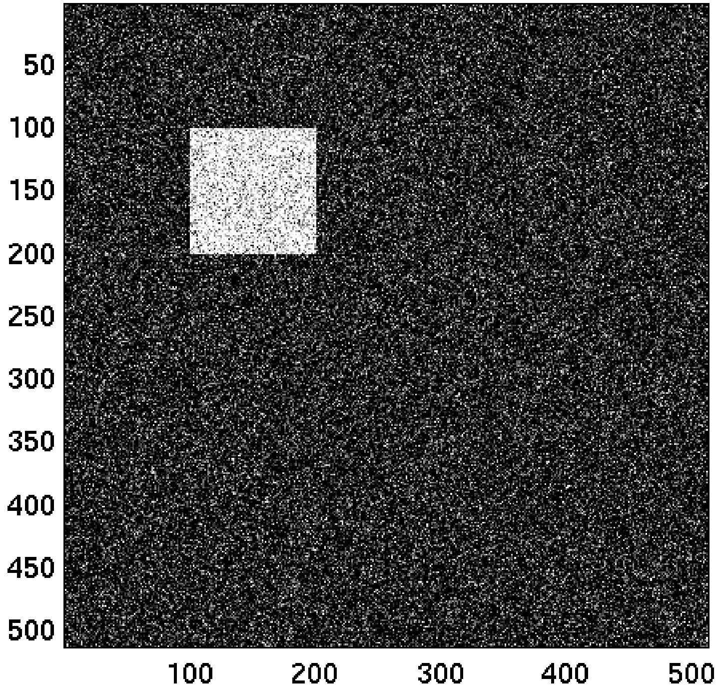
# Both together …

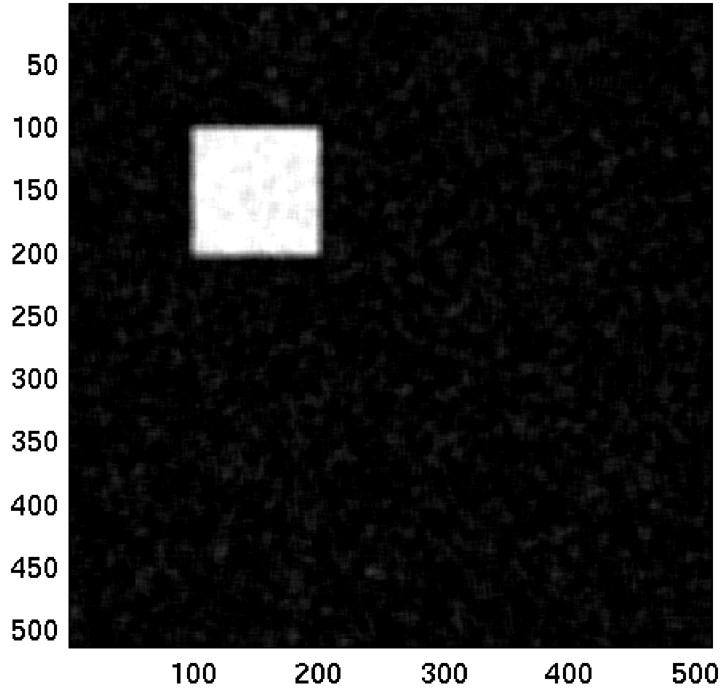
$$y[k,l] = |y_v[k,l]| + |y_h[k,l]|$$



# Denoising…

- Mask 9x9, values 1/81 … **zoom noise_low_pass.png**

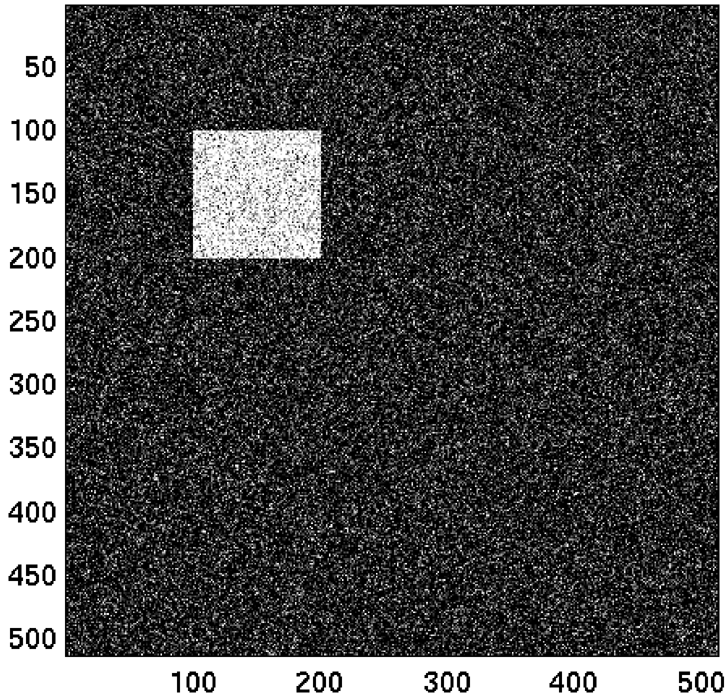


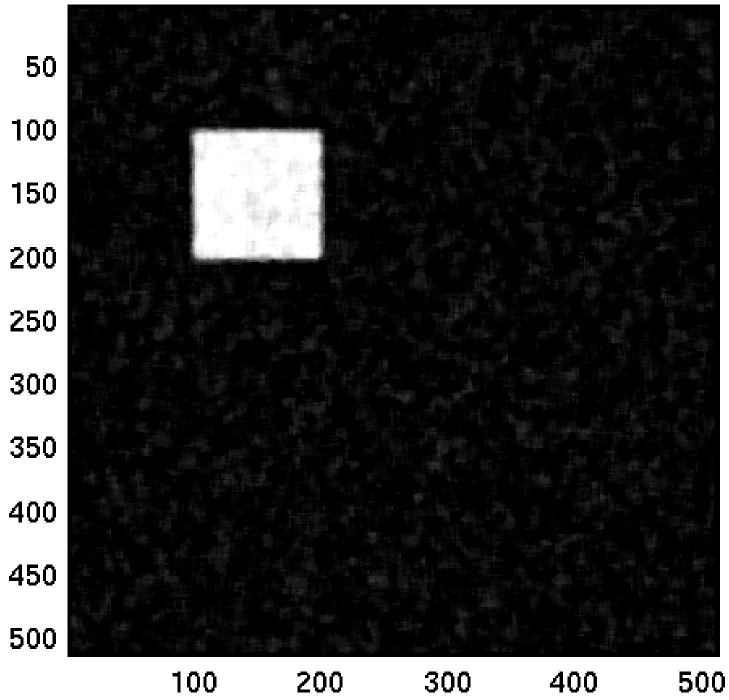



# Denoising II – median filter

$$y[k,l] = \text{median}_{k=-\frac{I-1}{2}\dots\frac{I-1}{2},\quad l=-\frac{J-1}{2}\dots\frac{J-1}{2}} x[k,l]$$

- **zoom noise_median.png**



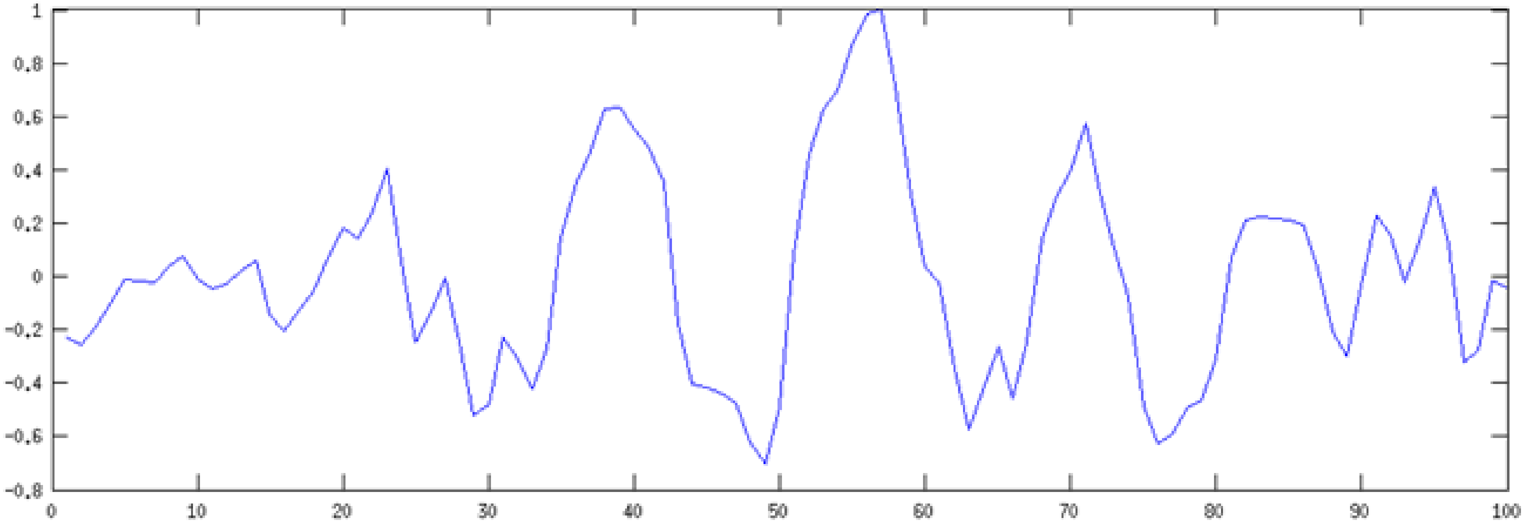
# Spectral analysis

- Task:

  **Determine, what is in the signal on different frequencies**

- Why ?
  - Visualization,
  - Feature extraction (think of Facebook)
  - Filtering (conversion to spectral domain and multiplication therein can be more efficient)
  - Coding (think of JPEG)

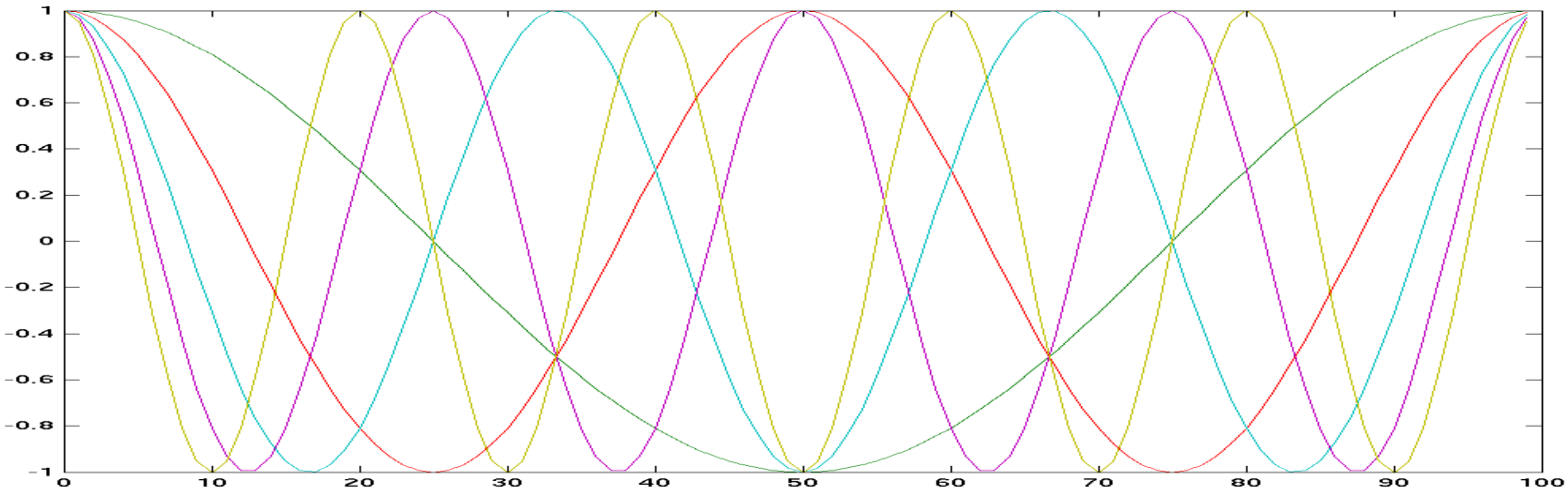
# Reminder – 1D




- What is in ?

$$c = \sum_{n=0}^{N-1} x[n]a[n]$$

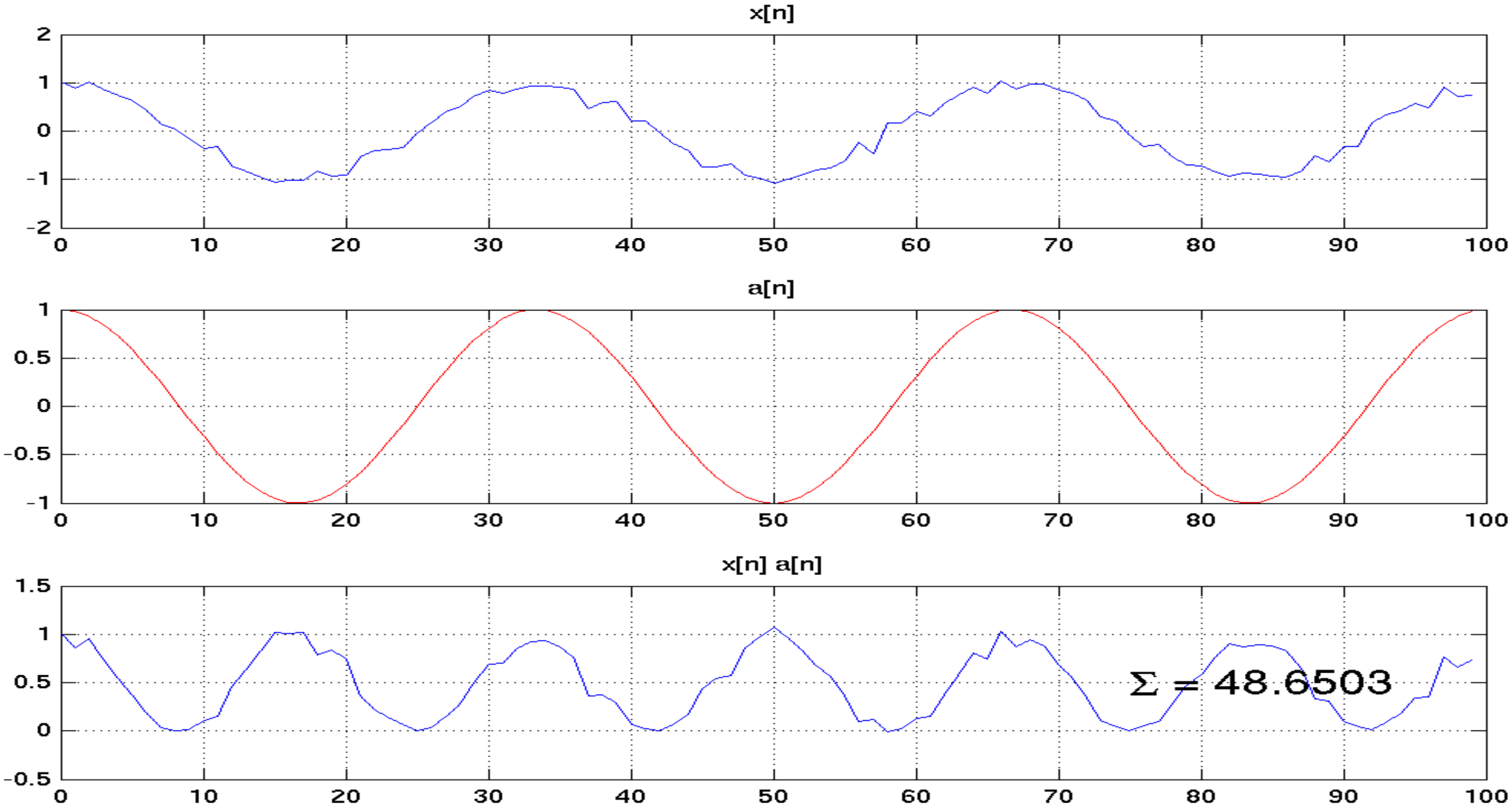
# Reminder – 1D cosine bases




$$\cos(2\pi \frac{k}{N} n)$$

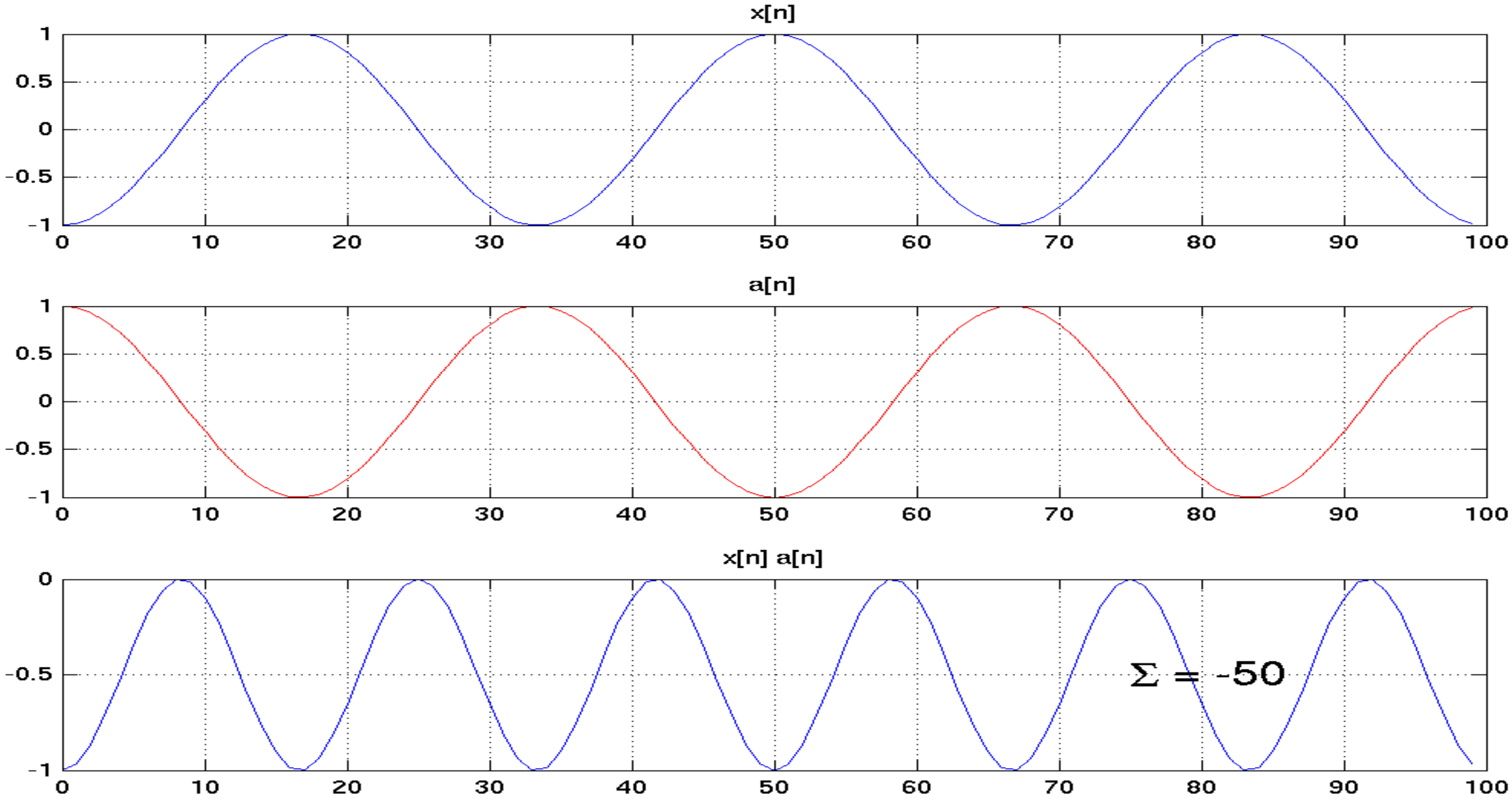
Problem with the phase – the signal “wave” begins not necessarily at zero …

# OK ...

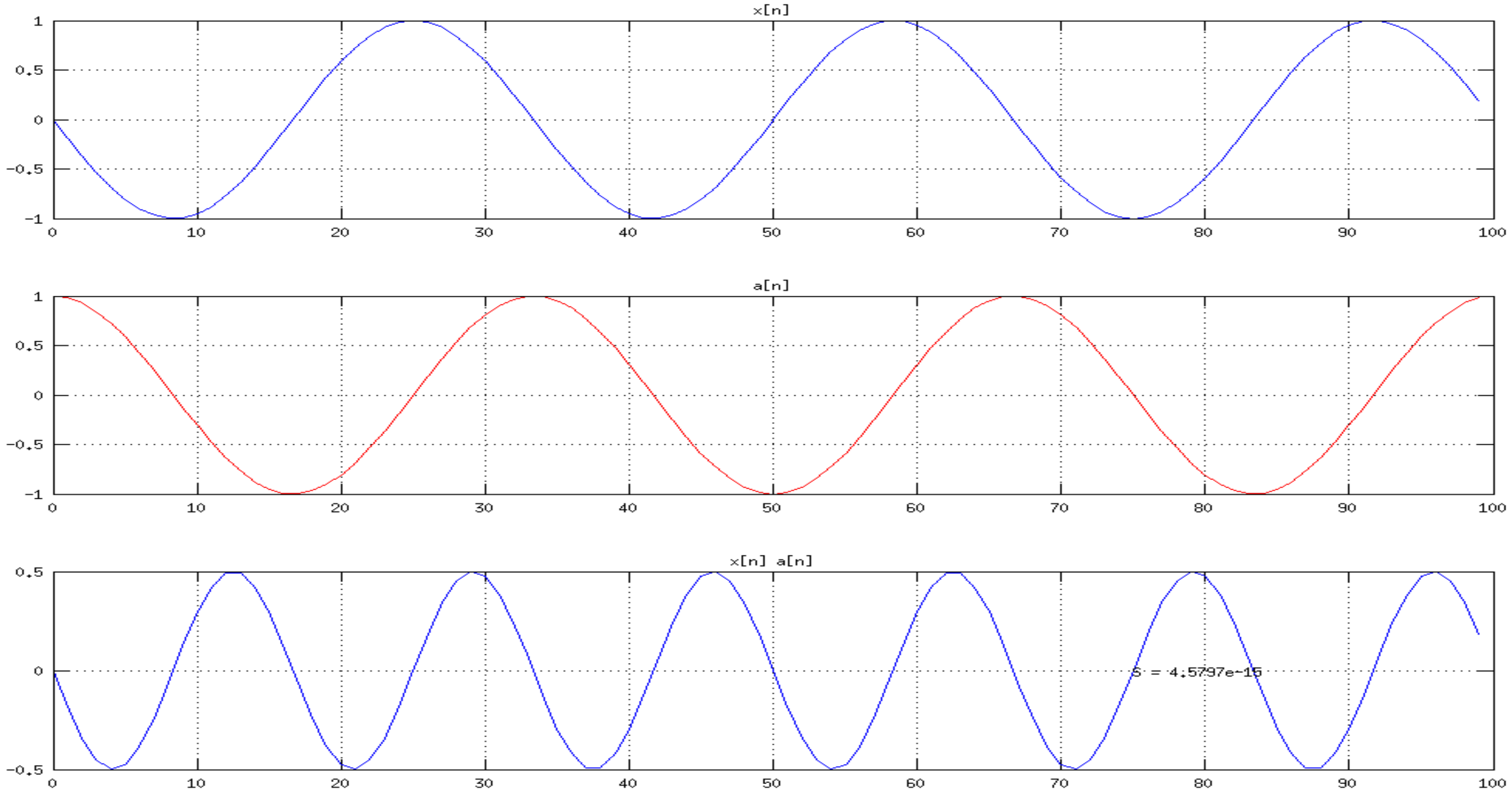


x[n]
2
1
0
-1
-2
0
10
20
30
40
50
60
70
80
90
100
a[n]
1
0.5
0
-0.5
-1
0
10
20
30
40
50
60
70
80
90
100
x[n] a[n]
1.5
1
0.5
0
-0.5
0
10
20
30
40
50
60
70
80
90
100
Σ = 48.6503

# Also OK …

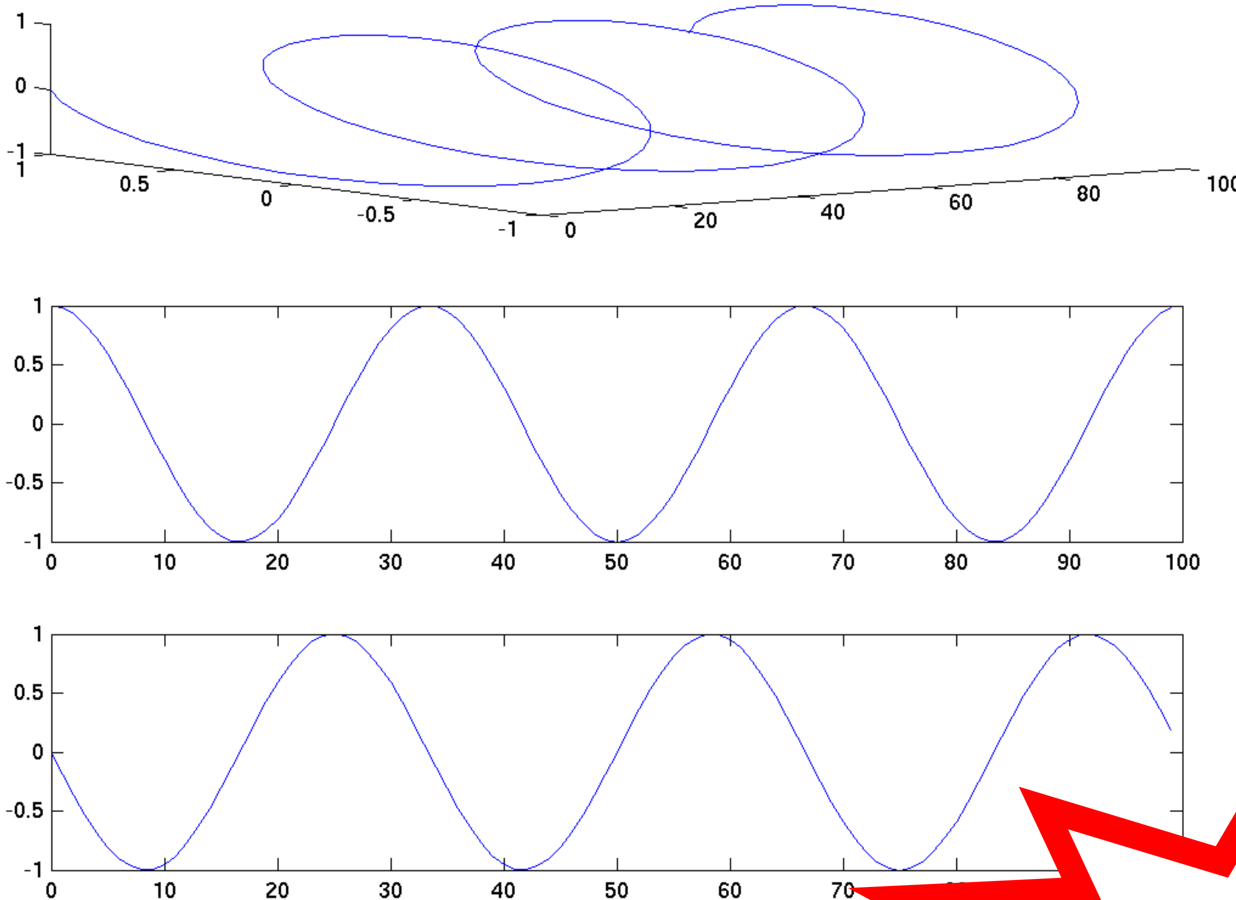


# Bad bad bad ☹



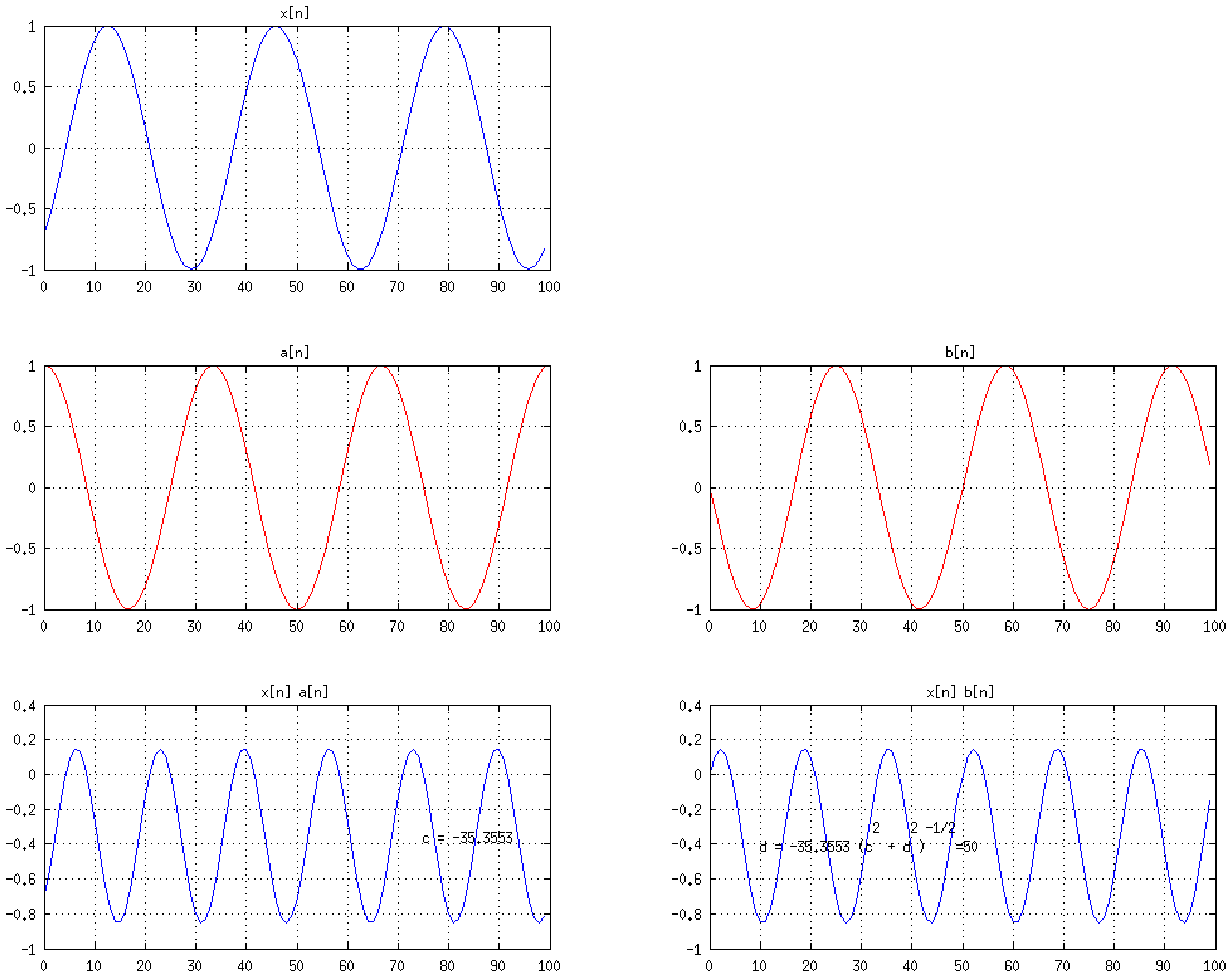

- How come zero when $\sin(x) = \cos(x - \frac{\pi}{2})$ ?

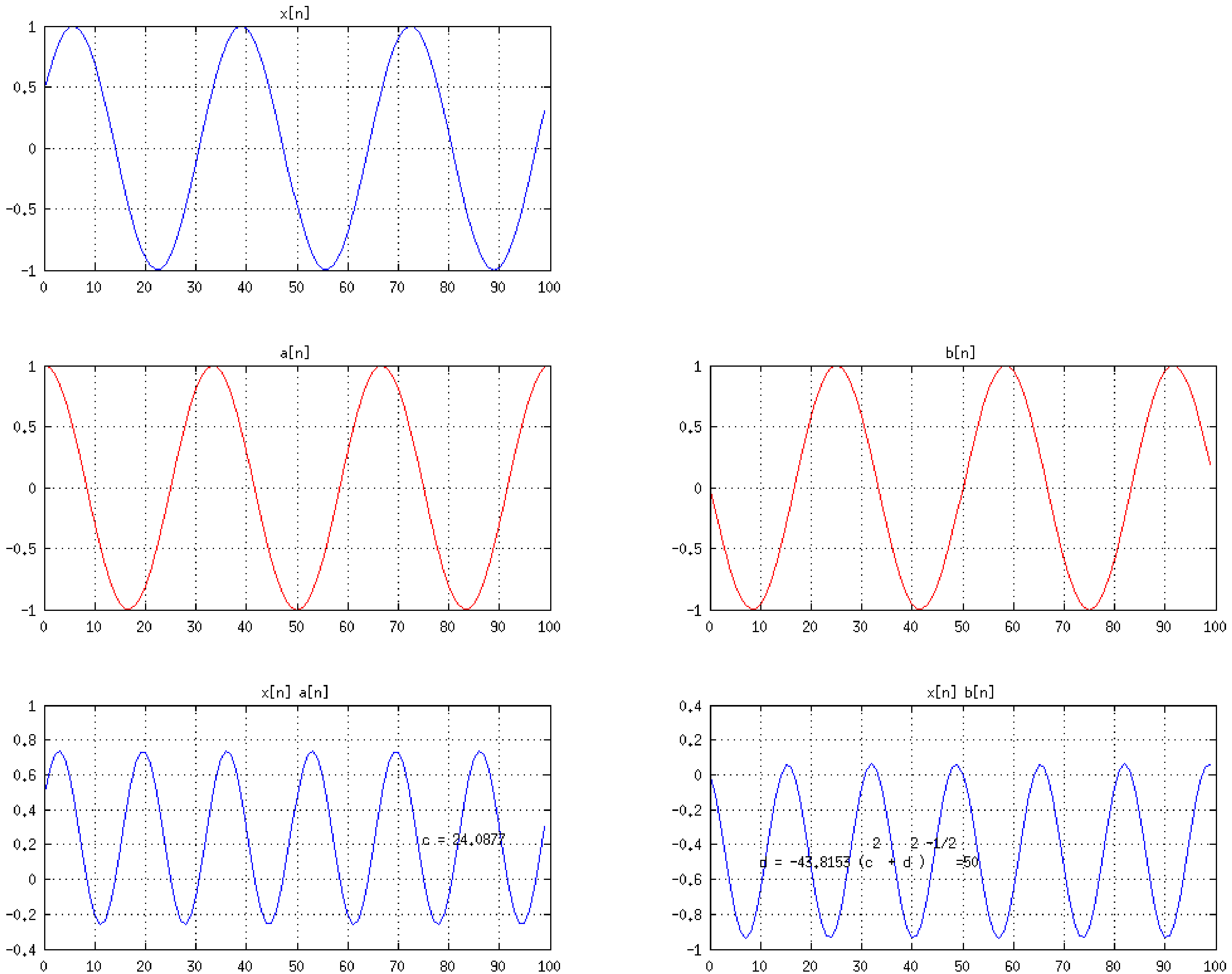
# Complex exponentials




$$e^{jx} = \cos(x) + j\sin(x)$$

$$e^{-j2\pi\frac{k}{N}n} = \cos(2\pi\frac{k}{N}n) - j\sin(2\pi\frac{k}{N}n)$$


x[n]
a[n]
b[n]
x[n] a[n]
x[n] b[n]
c = -35.3553
d = -35.3553 (c^2 + d^2)^-1/2 =50


x[n]
a[n]
b[n]
x[n] a[n]
c = 24.0877
x[n] b[n]
d = -43.8153 (c^2 + d^2)^-1/2 = 50

# 1D ultimate result - DFT

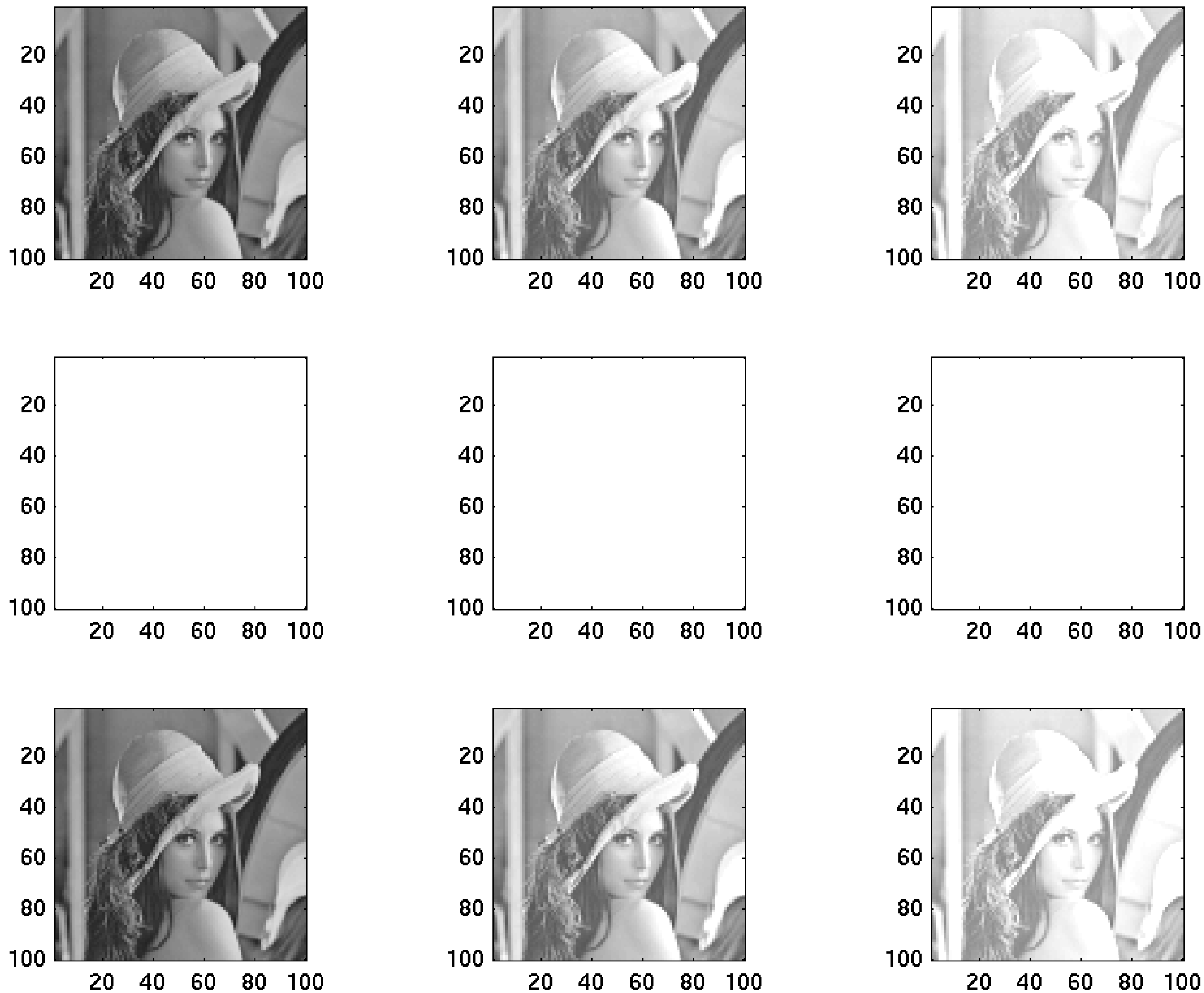
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}, \quad k = 0 \ldots N-1$$

# Now finally 2D

- Correlation = determination of similarity = projection to bases
- **We have seen this, it's boring, it's all the time the same … YES IT IS !**

$$c = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k,l] a[k,l]$$

# Analyzing signal = d.c. $a[k,l] = 1$

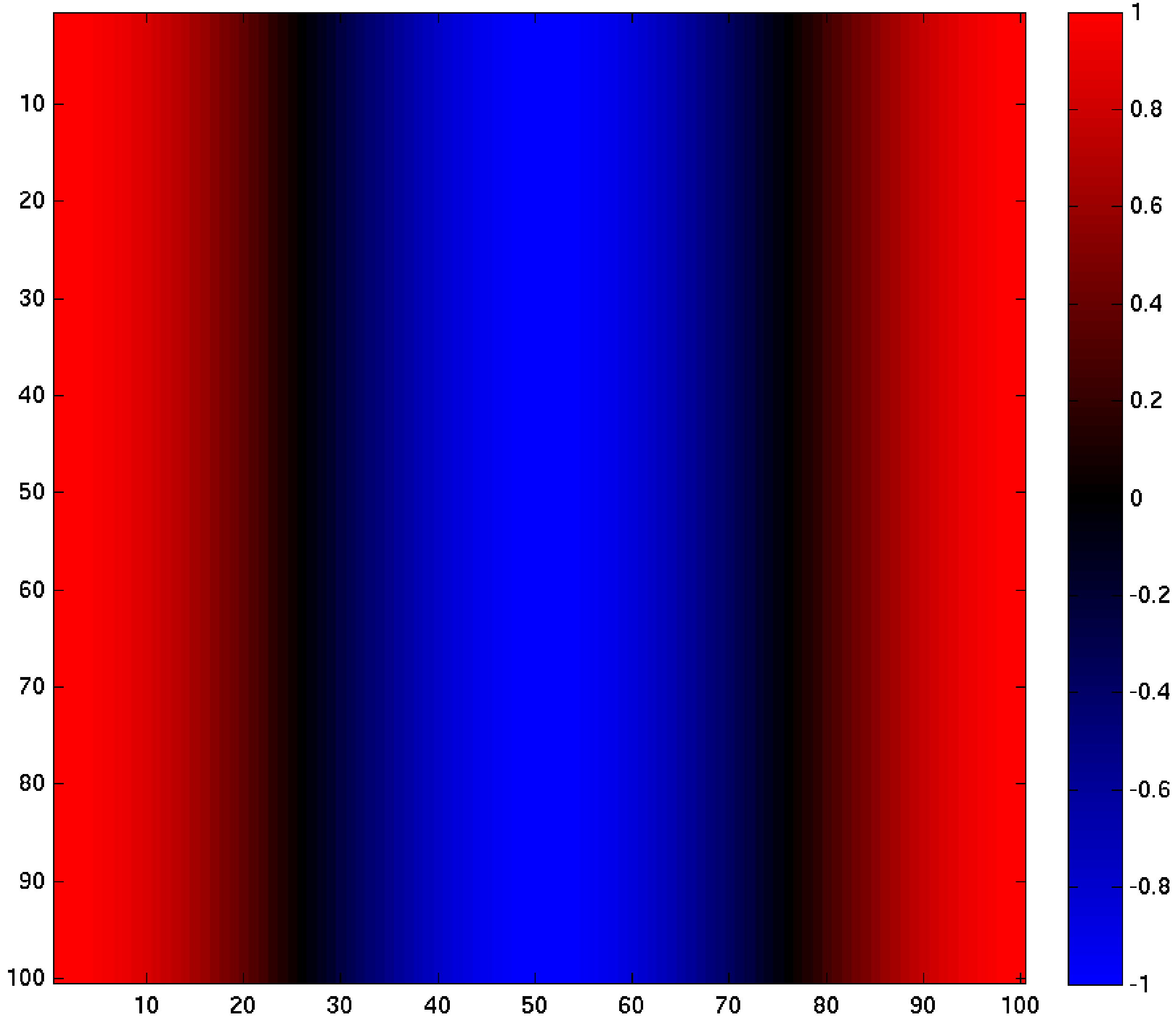



c=4.8287e+03 c=6.8287e+03 c=8.8287e+03

# Analyzzing signal – horizontal cosine

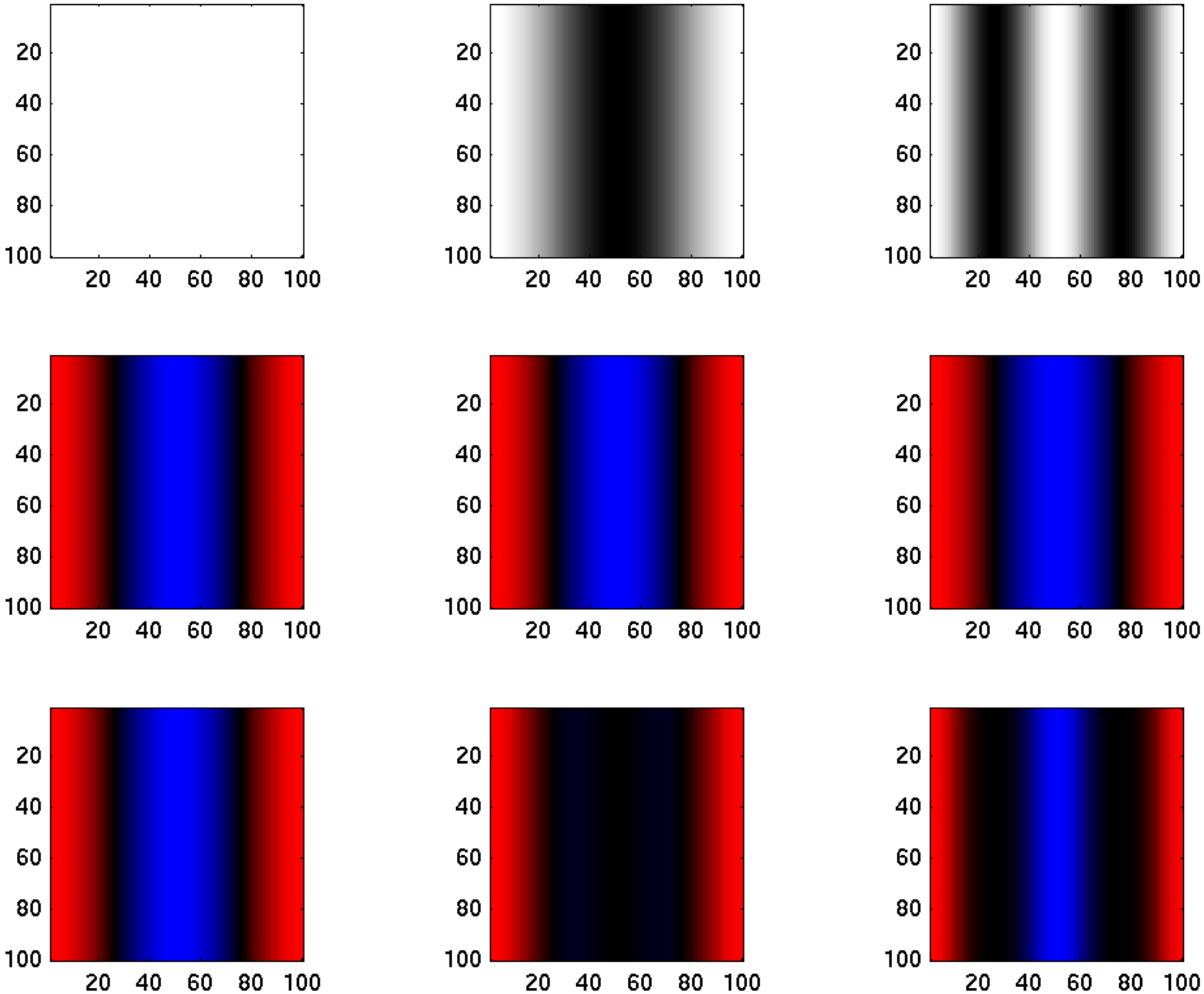
$$a[k, l] = \cos(2\pi \frac{1}{100} l)$$

- Changes only in one direction
- We’ll need to visualize negative values too.




Geeks: how to do this in Matlab ??

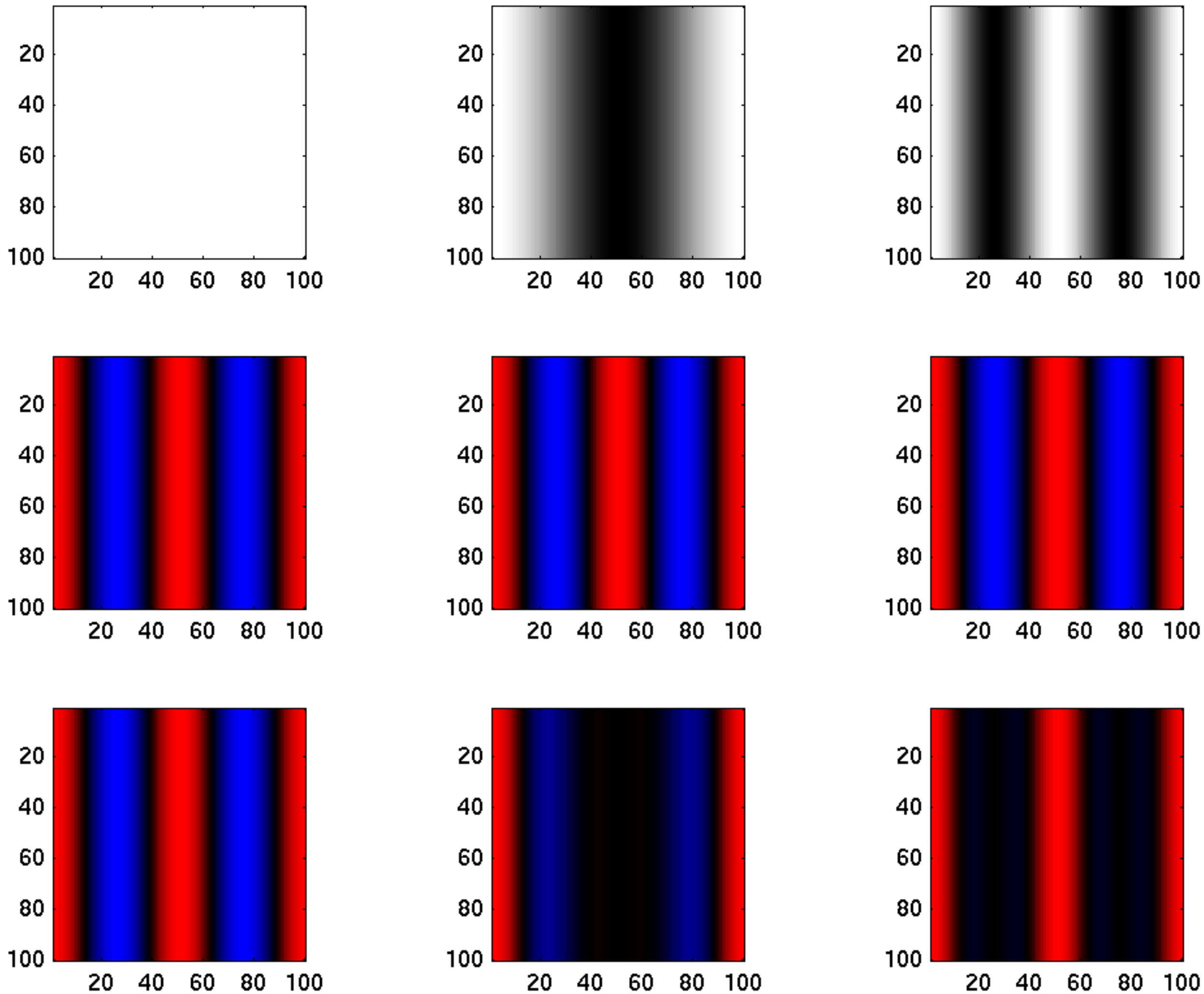
# Analysis for $a[k,l] = \cos(2\pi \frac{1}{100} l)$




c=4.5475e-13 c=2500 c=3.4106e-13

# 2 x faster horizontal cos

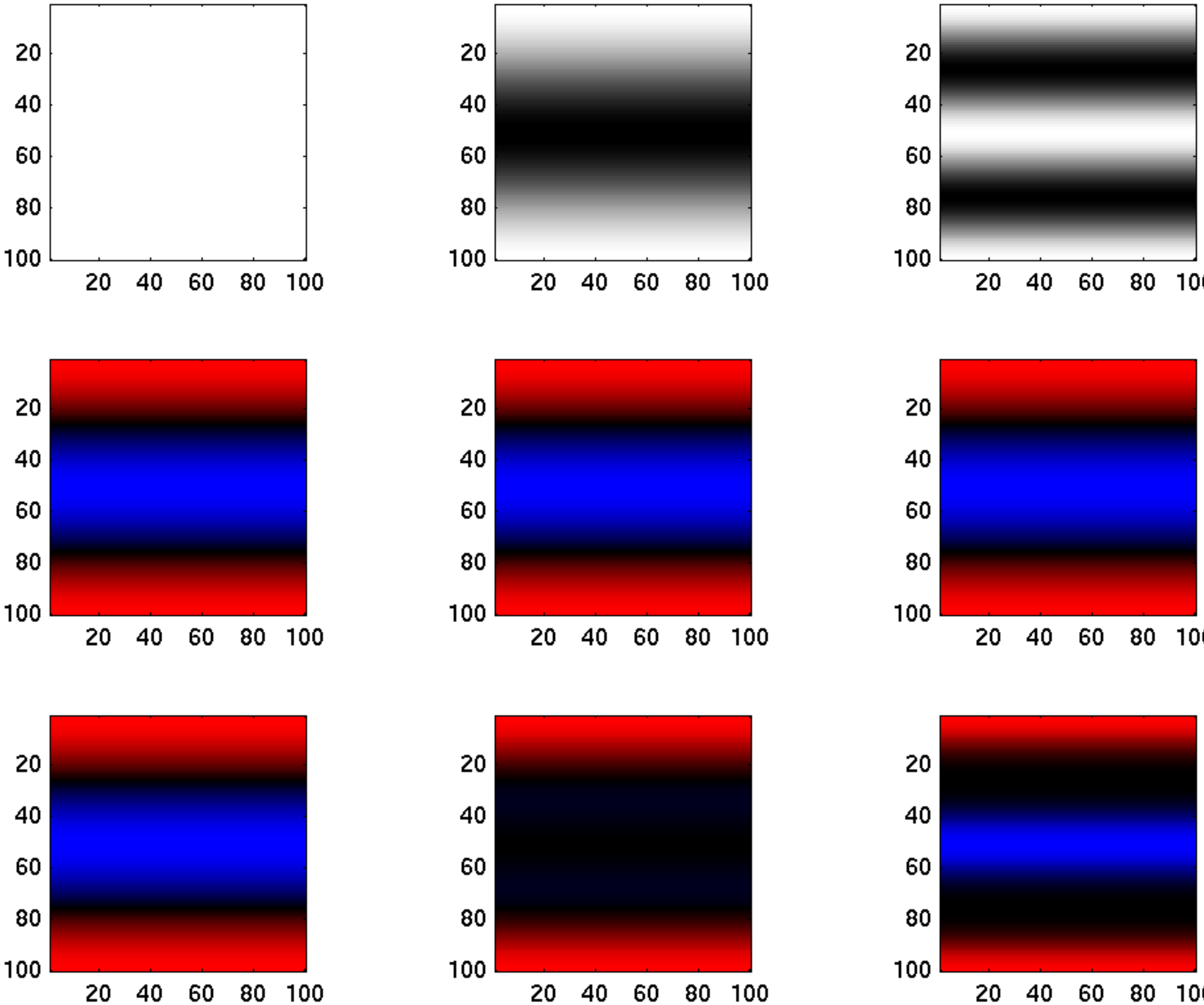
$$a[k,l] = \cos(2\pi \frac{2}{100} l)$$




c=2.9843e-13 c=1.5064e-12 c=2.5000e+03

# Vertical cos

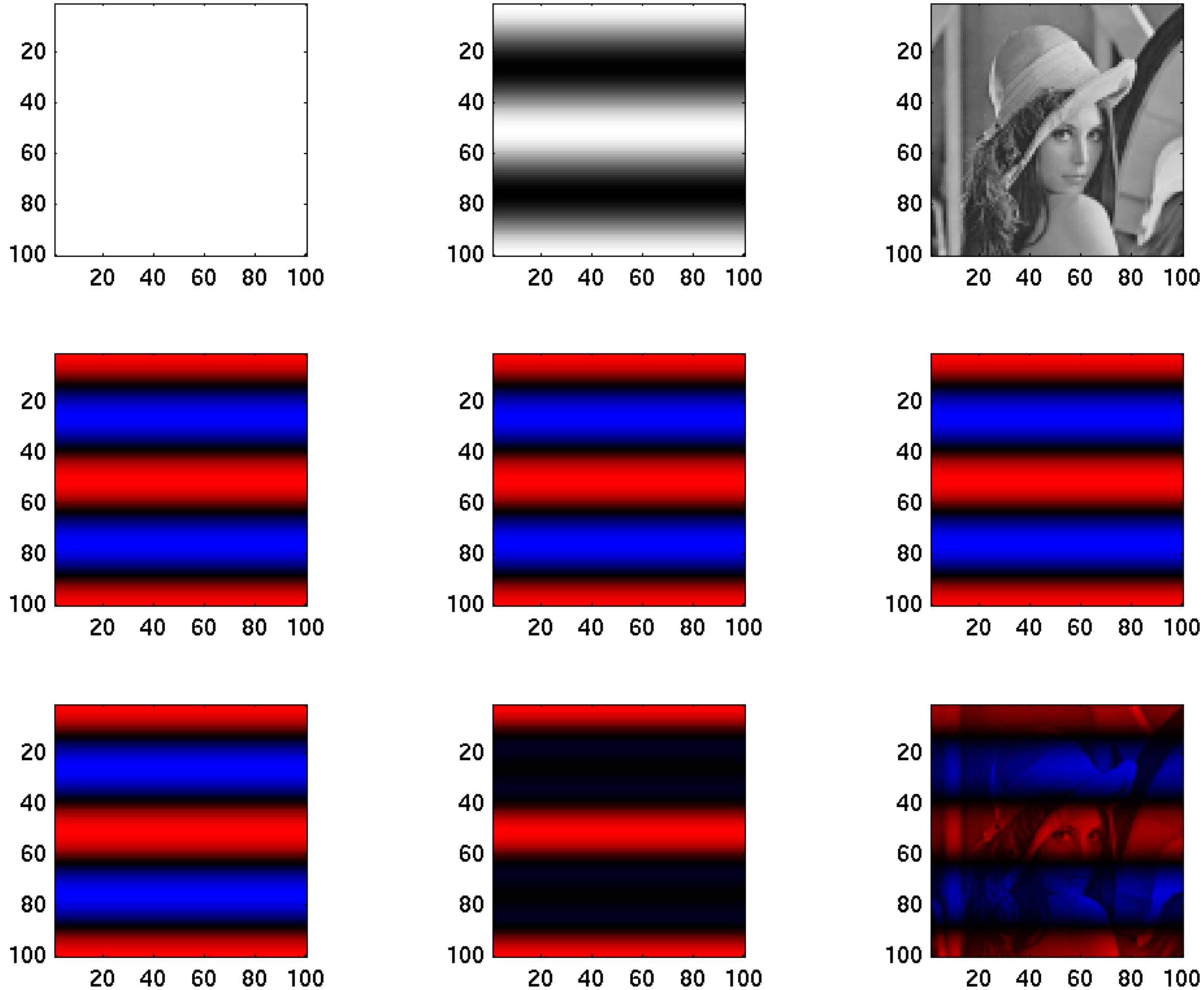
$$a[k,l] = \cos(2\pi \frac{1}{100} k)$$




c= -7.7716e-14 c=2.5000e+03 c=6.9944e-13

# 2 x faster vertical cos

$$a[k,l] = \cos(2\pi \frac{2}{100} k)$$

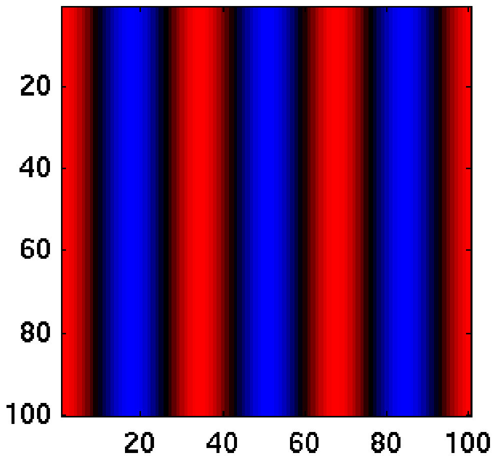


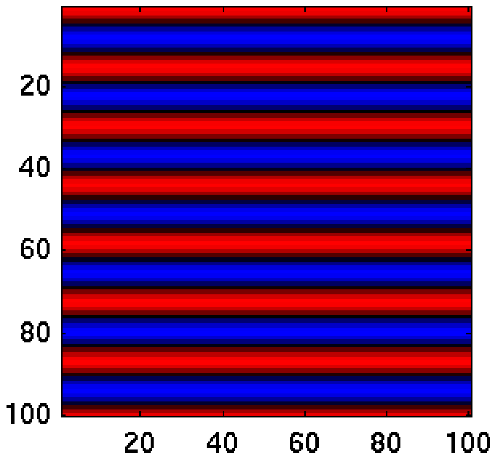

c= 4.9960e-13 c=2.5000e+03 c=-65.4694

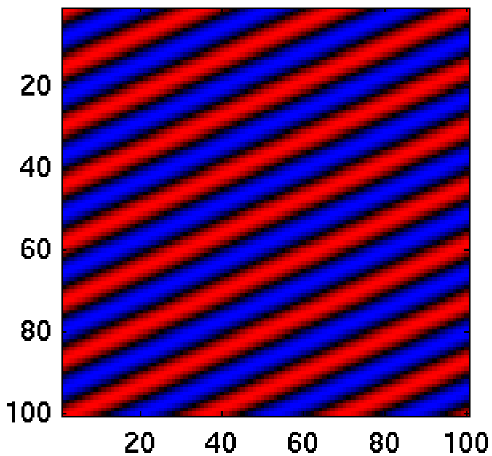
# Mix of both directions …

$$a[k,l] = \cos(2\pi \frac{3}{100} l) \qquad a[k,l] = \cos(2\pi \frac{7}{100} k)$$

$$a[k,l] = \cos\left[2\pi(\frac{7}{100}k + \frac{3}{100}l)\right]$$



# Generalization

$$X[m,n] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k,l] \cos\left[2\pi(\frac{m}{K}k + \frac{n}{L}l)\right]$$

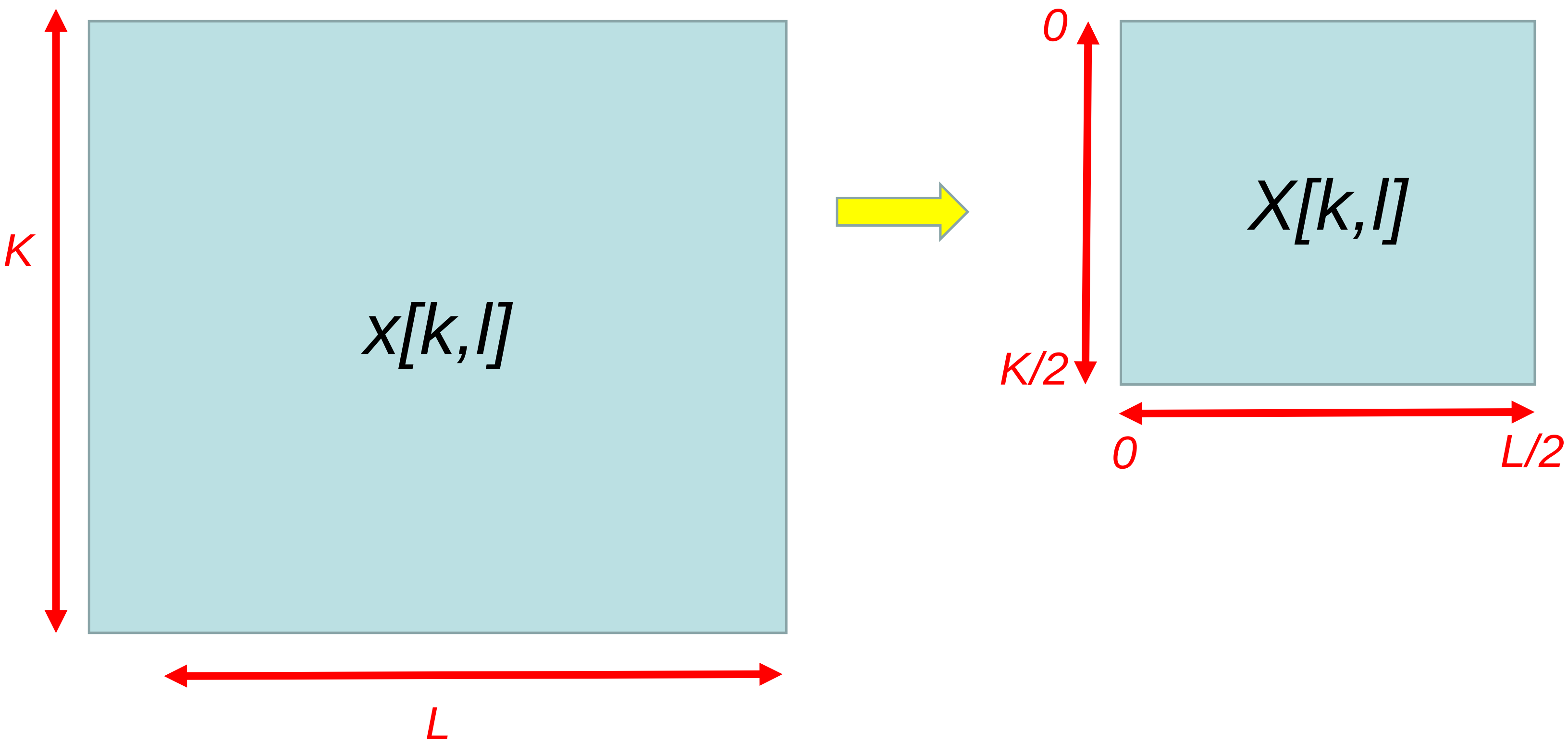
- *m/K* – horizontal frequency
- *n/L* – vertical frequency

# Reasonable ranges of frequencies ?

- *m/K* and *n/L*

  0 … ½ => OK

  > ½ => hm…



# More on frequencies …

- 1D

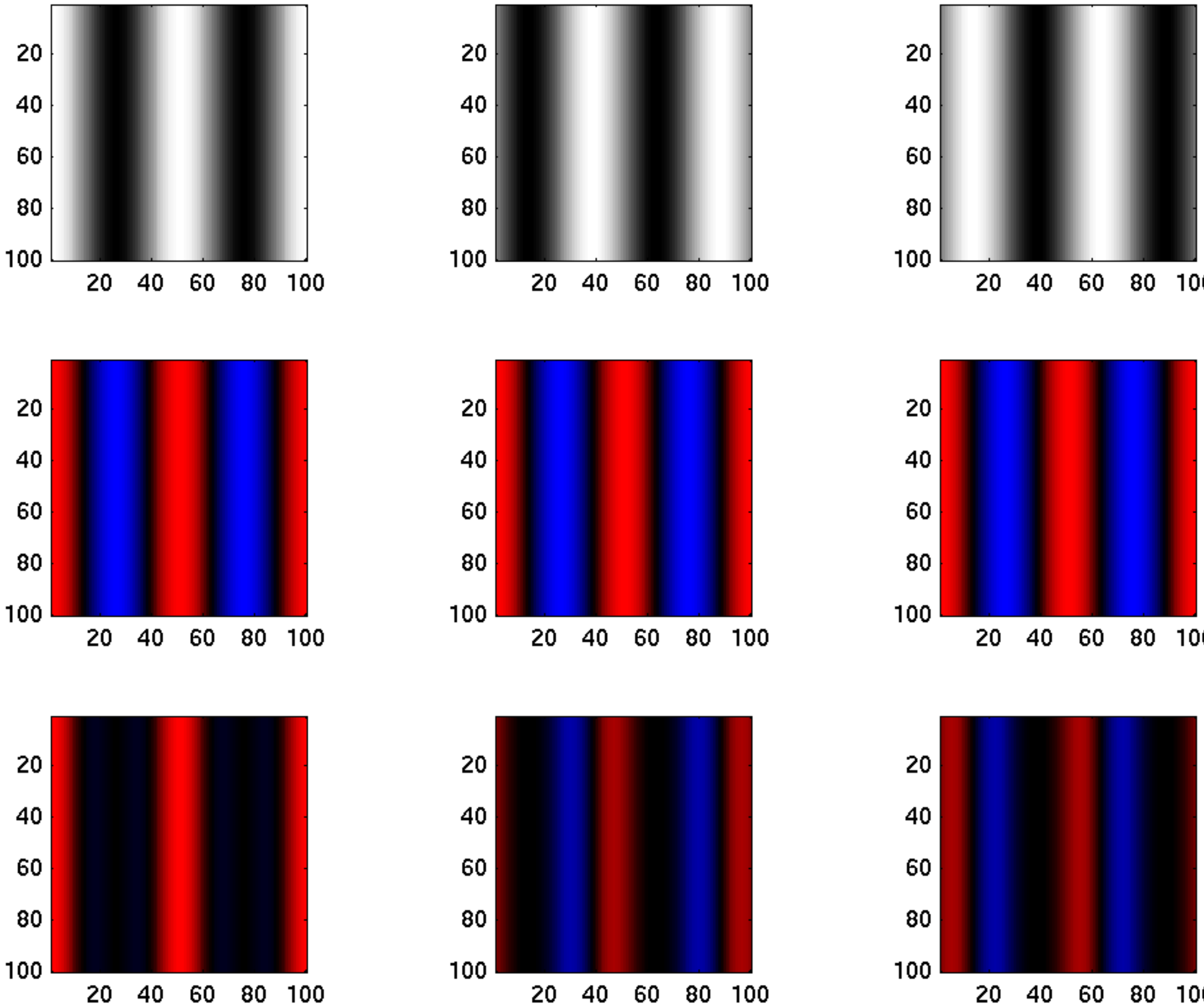
$$Hz = \frac{1}{s} \qquad F_s = \frac{\#samples}{s} \qquad f_{norm} = \frac{f_{skut}}{F_s} \qquad f_{skut} = \frac{k}{N} F_s$$

- 2D

$$dpi = \frac{1}{inch} \qquad F_s = \frac{\#pixels}{inch} \qquad f_{norm} = \frac{f_{skut}}{F_s}$$

$$f_{skut,vert} = \frac{m}{K} F_s, \quad f_{skut,horiz} = \frac{n}{L} F_s$$

# Phase – problem again ☹




c=2.5000e+03 c=9.9476e-13 c=-5.6843e-14

# Solution – complex exponentials (again)

$$X[m,n] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k,l] e^{-j\left[2\pi\left(\frac{m}{K}k + \frac{n}{L}l\right)\right]}$$

Reminder what is what :

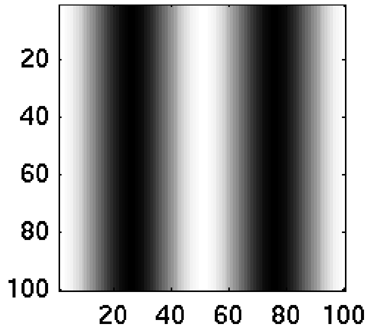
- *k,l* indices of pixels (input)
- *m,n* indices of frequencies (result)
- *m/K* normalized vertical frequency
- *n/L* normalized horizontal frequency

# Imagining it I.

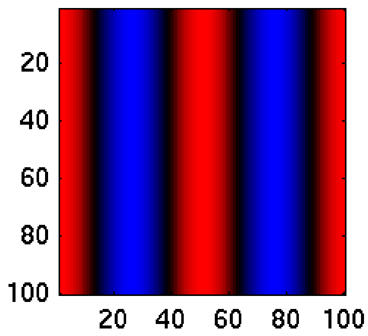
$$X[0,1] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k,l] e^{-j\left[2\pi\left(\frac{0}{K}k + \frac{1}{L}l\right)\right]} = \ldots$$

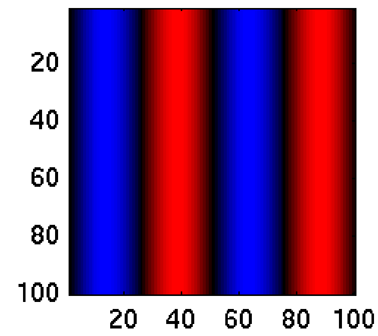
# Imagining it II.

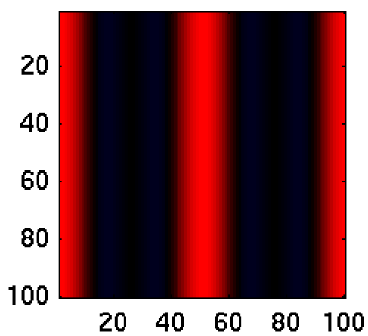
$$X[3,0] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k,l] e^{-j\left[2\pi\left(\frac{3}{K}k + \frac{0}{L}l\right)\right]} = \dots$$

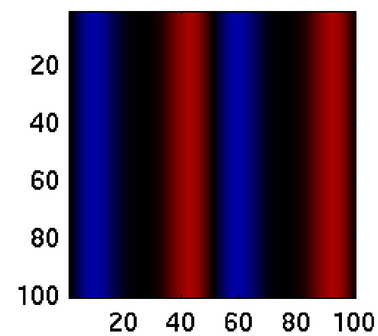



# Correlation with *x[n]*



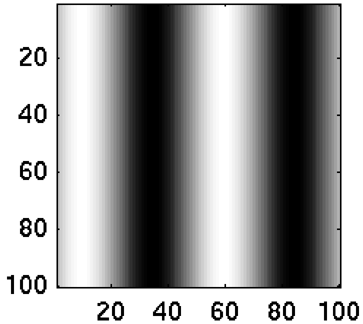



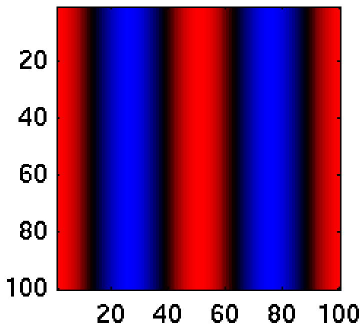

Re(c) = 2500

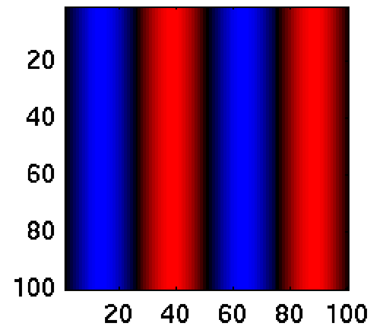
Im(c) = 0

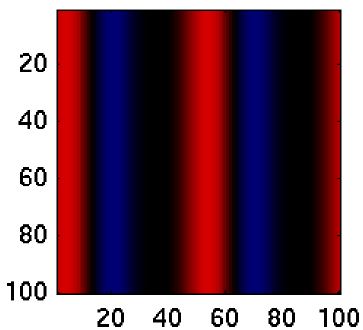
**|c| = 2500**

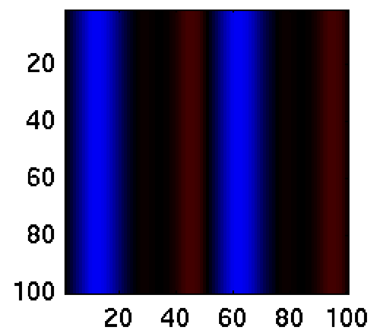
# Correlation with *x[n]*



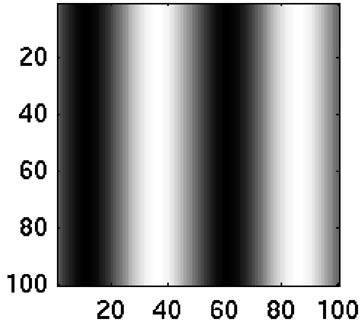



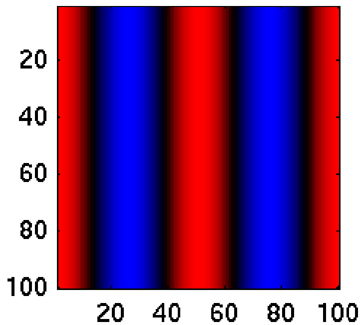

Re(c) = 1250

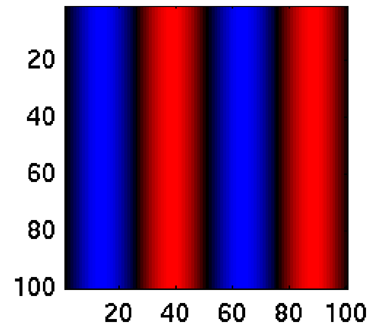
Im(c) = -2165

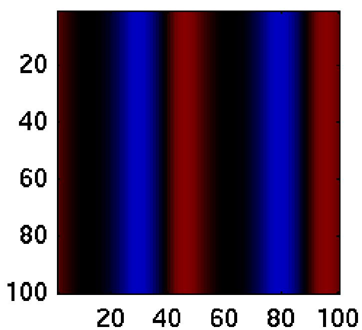
**|c| = 2500**

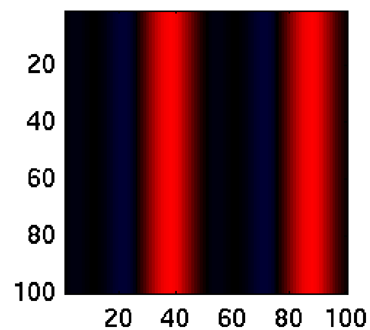
# Correlation with *x[n]*



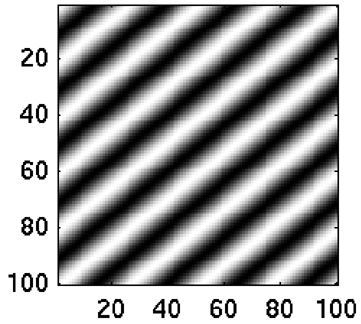



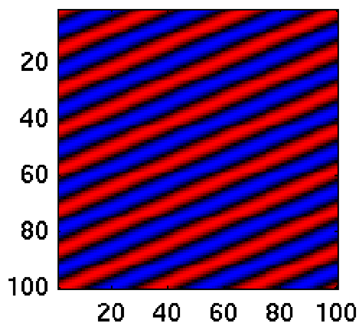

Re(c) = -772

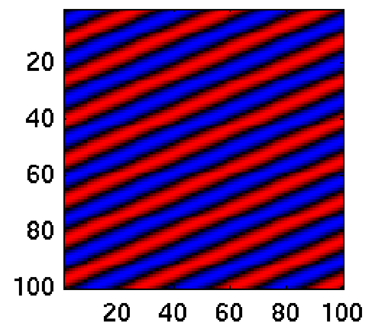
Im(c) = 2378

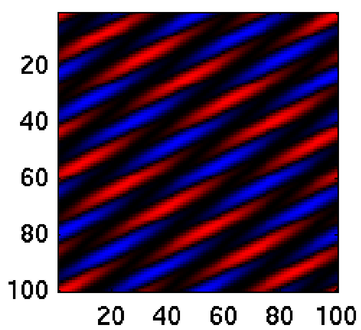
**|c| = 2500**

# Change in both directions…*X[7,3]*

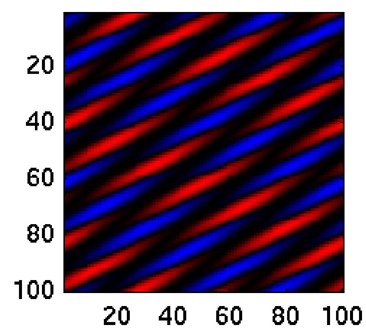


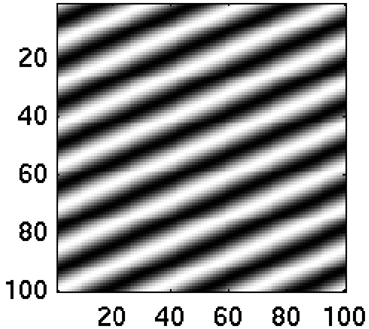

Re(c) = 0

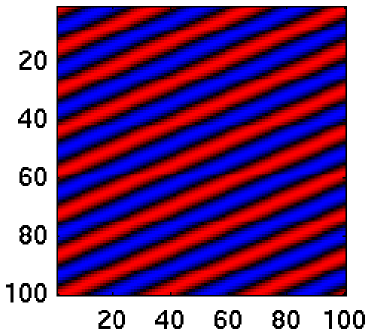


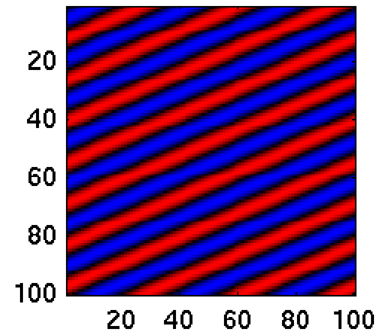

Im(c) = 0

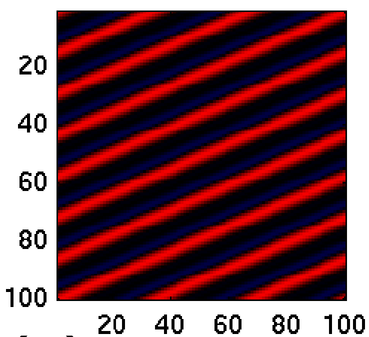
**|c| = 0 … not similar**

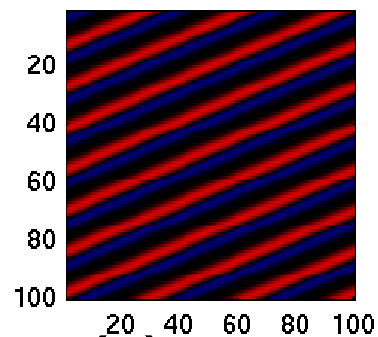
# Change in both directions…*X[7,3]*




Re(c) = 2165

Im(c) = 1250

**|c| = 2500 … similar**

# 2D DFT using 2 x 1D DFT

$$X[m,n] = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} x[k,l] e^{-j2\pi\left(\frac{mk}{K}+\frac{nl}{L}\right)} =$$

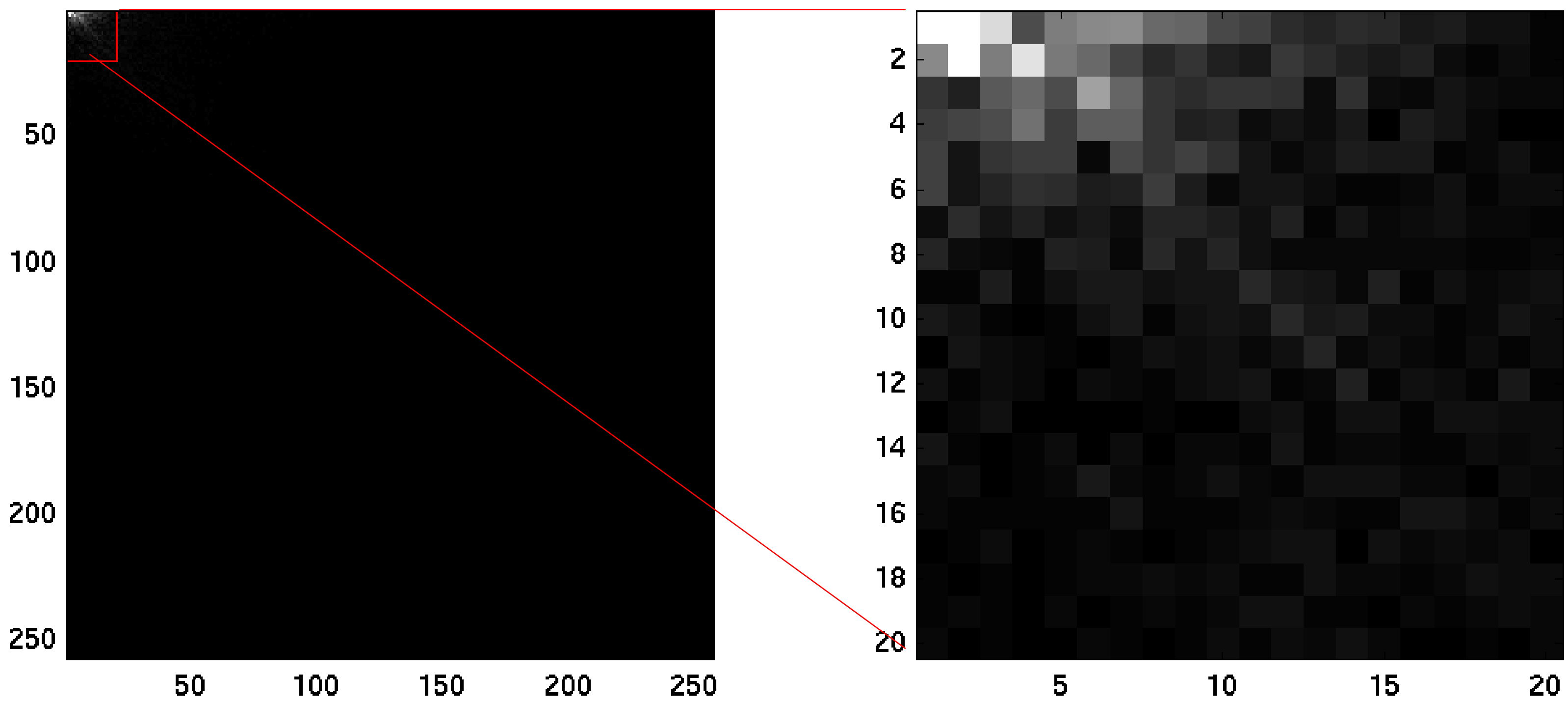
$$= \sum_{k=0}^{K-1} e^{-j2\pi\frac{mk}{M}} \sum_{l=0}^{L-1} x[k,l] e^{-j2\pi\frac{nl}{L}}, \quad \ldots \quad \text{or vice versa}$$

- so that

$$2DDFT\{x[k,l]\} =$$

$$= 1DDFT_{columns}\{1DDFT_{rows} x[k,l]\} \quad \ldots \quad \text{or vice versa}$$

# 2D DFT for a real signal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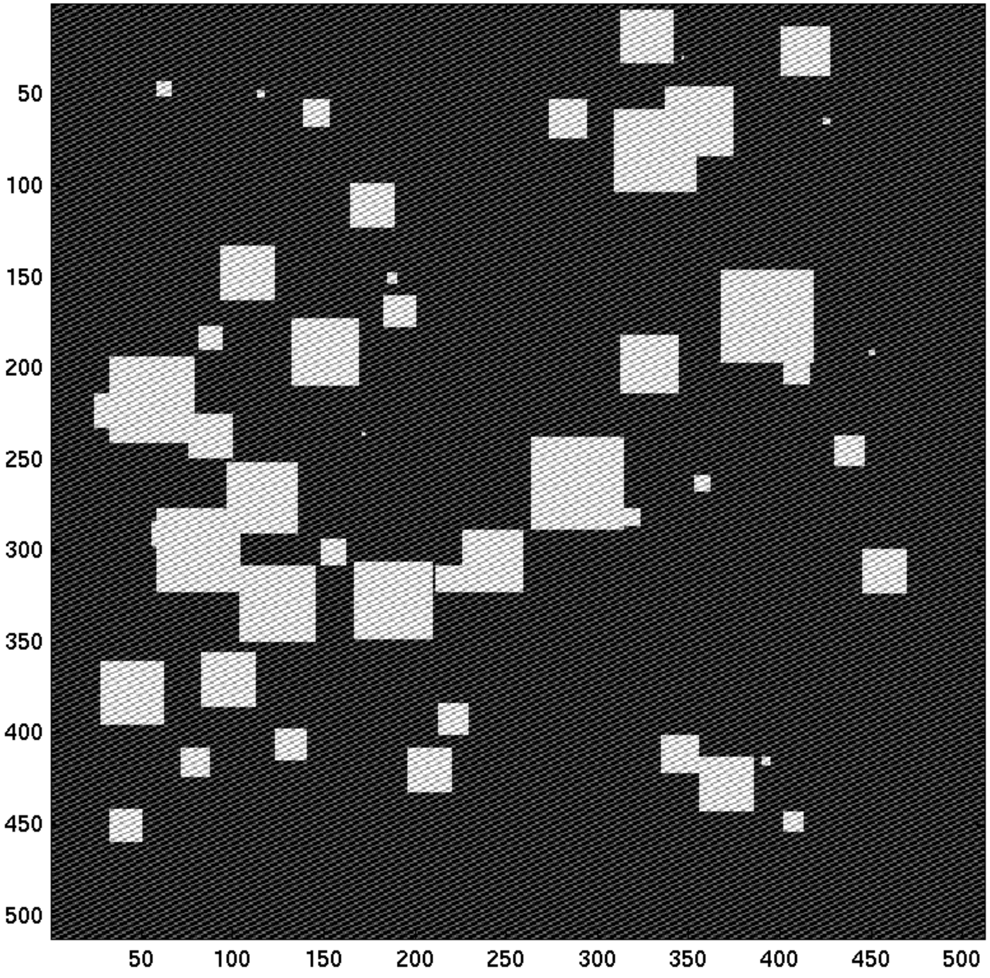
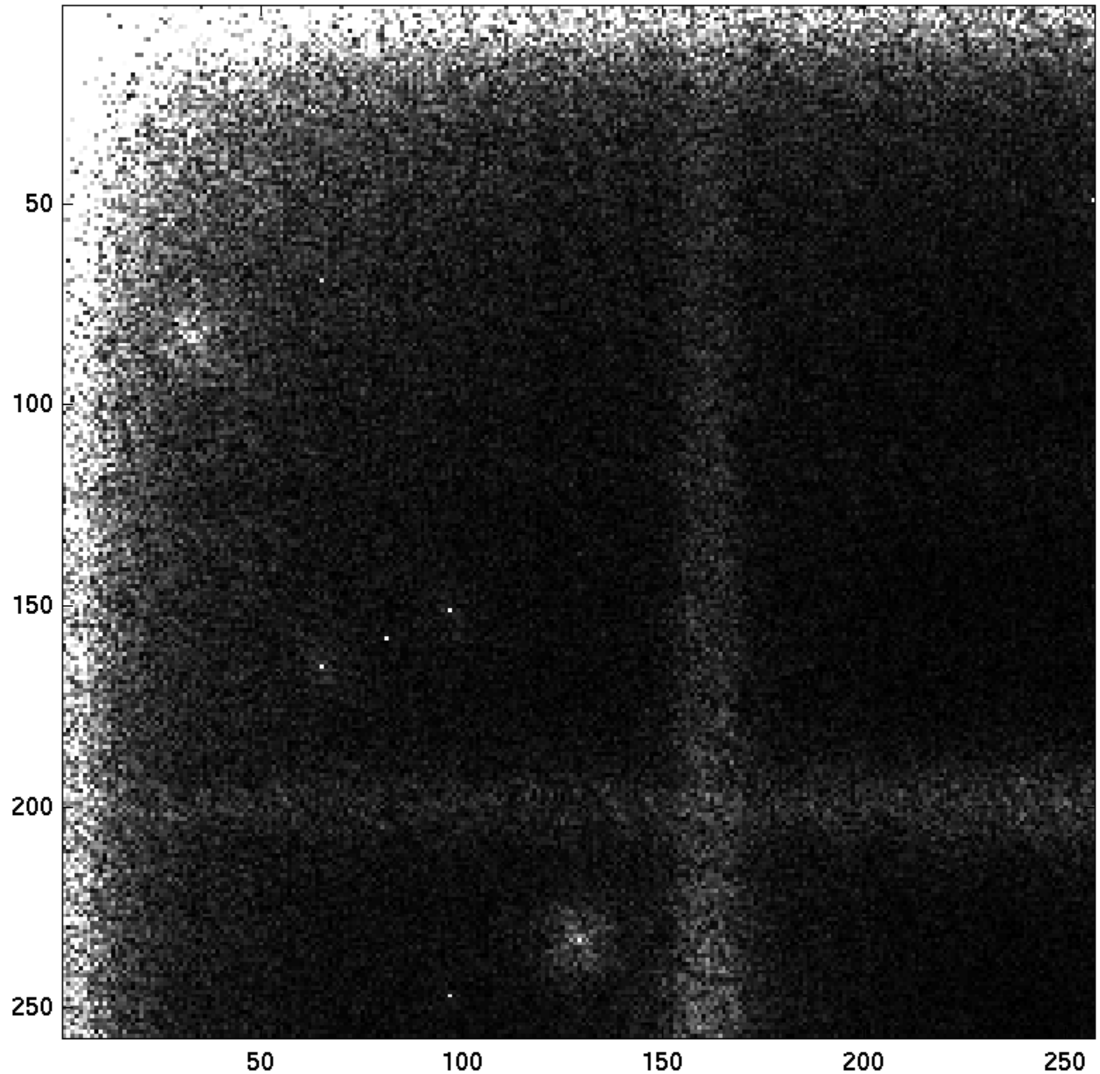
# Something with higher frequencies

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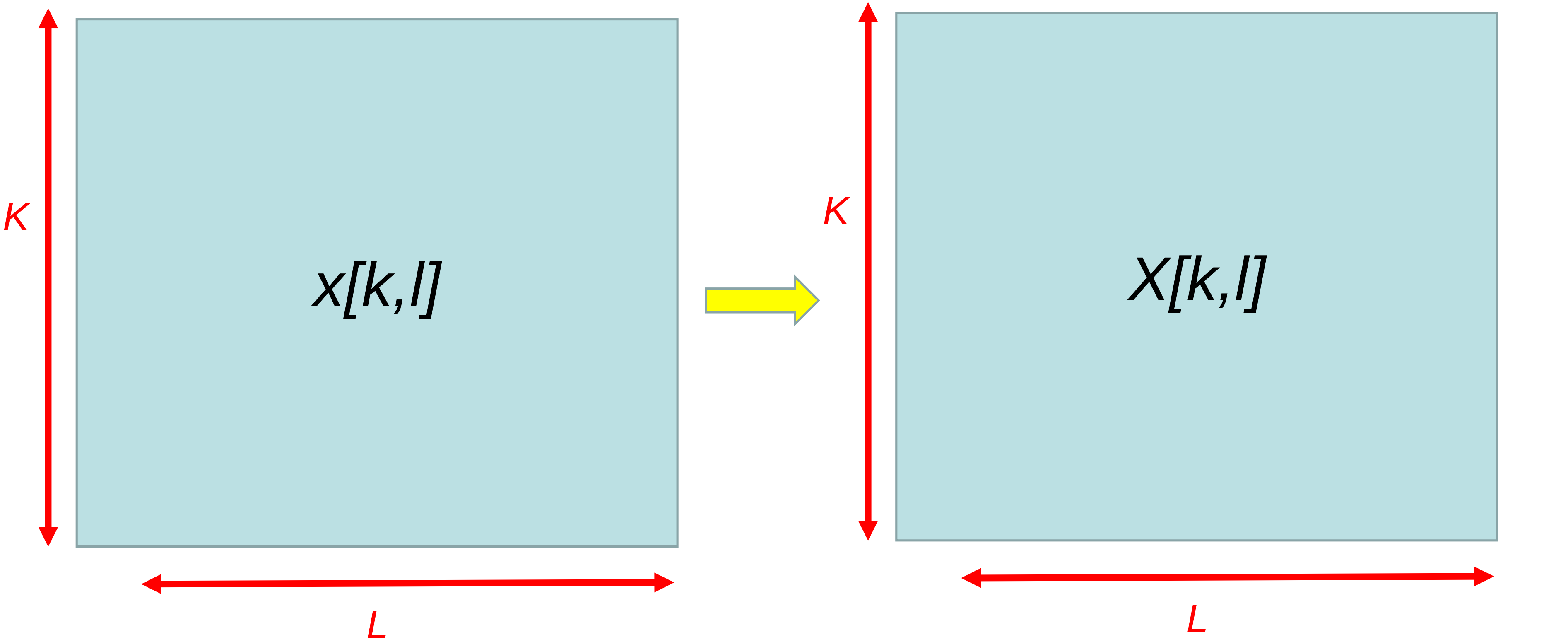
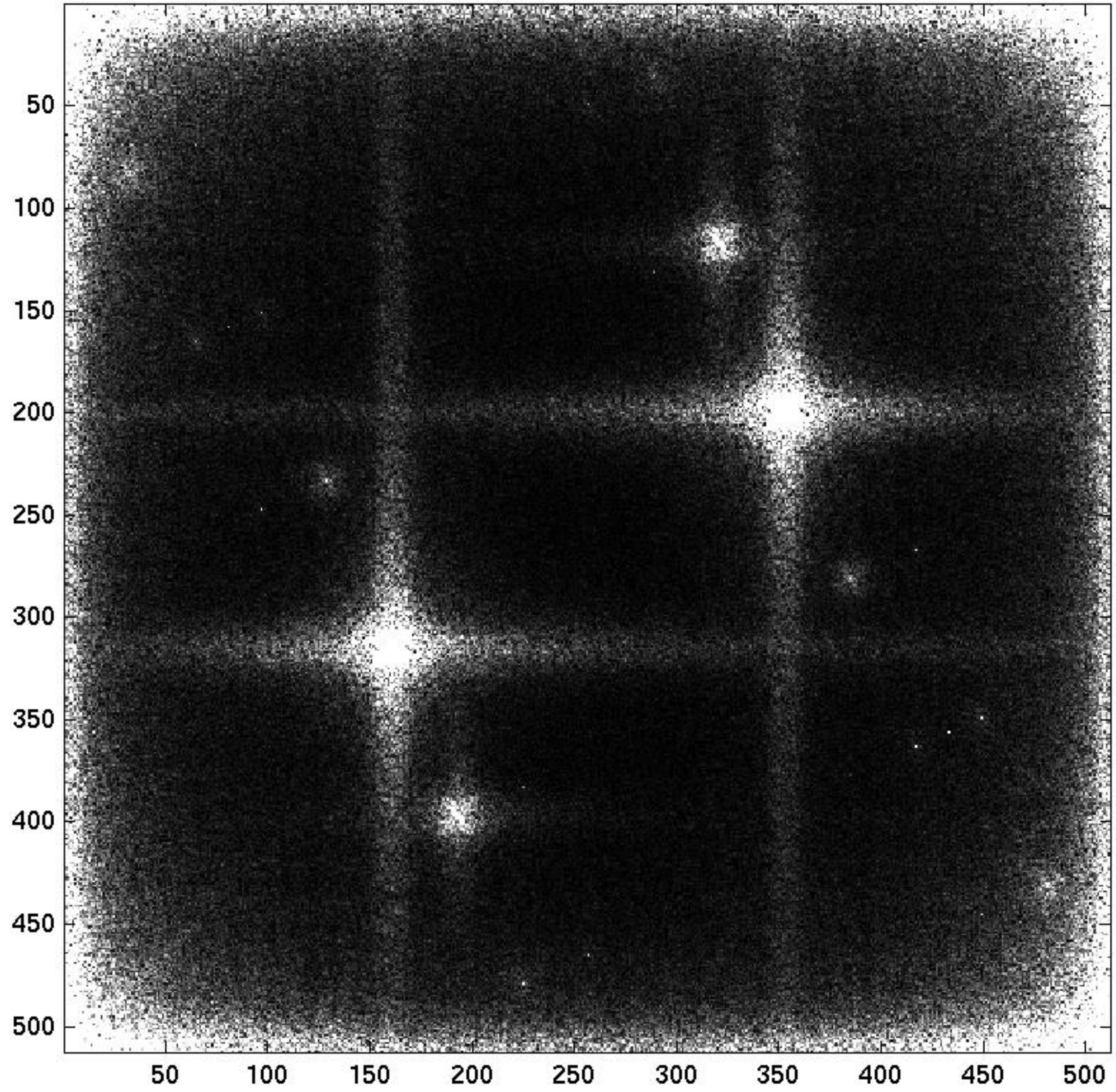
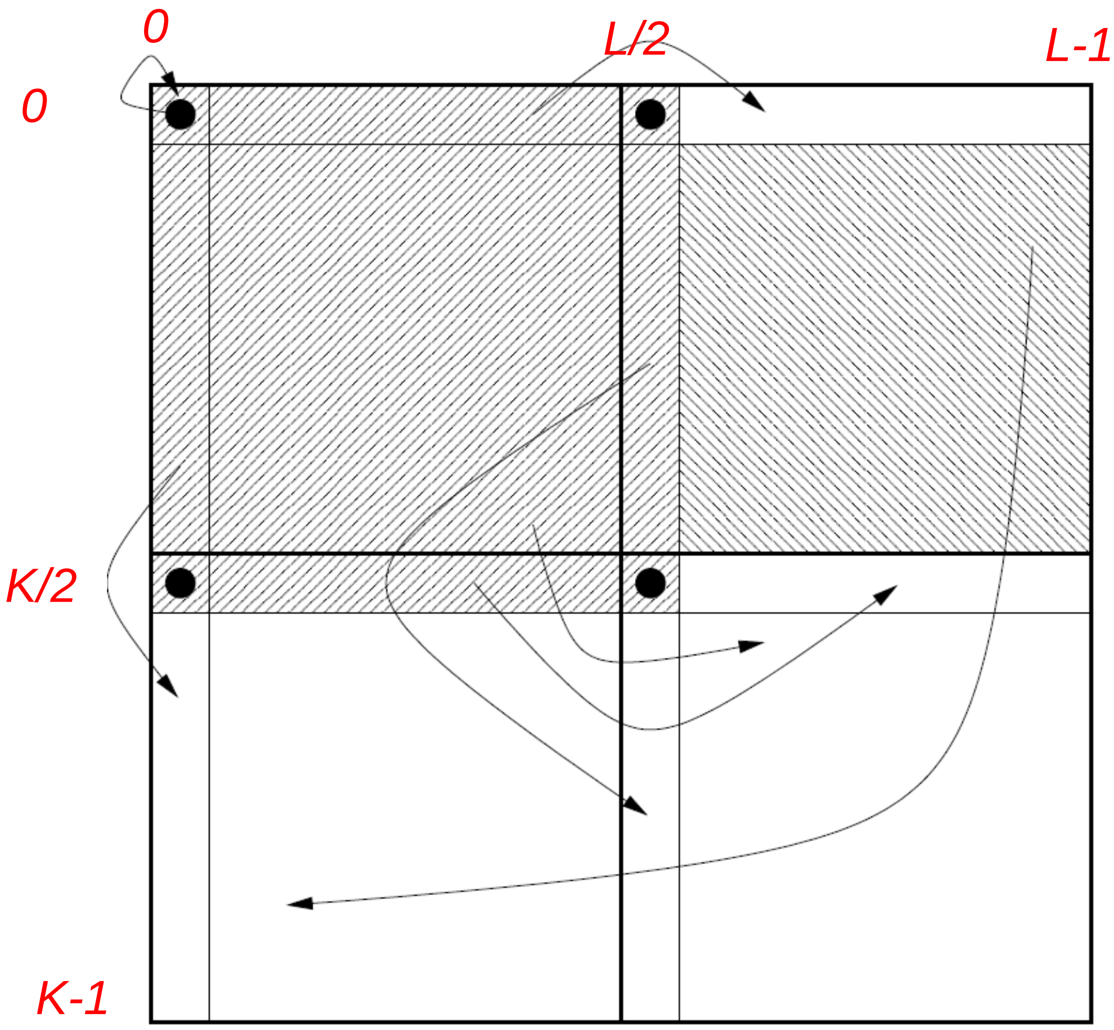
# DFT produces *K x L* points !



# The whole result for *k=0…K-1*, *l=0…L-1*



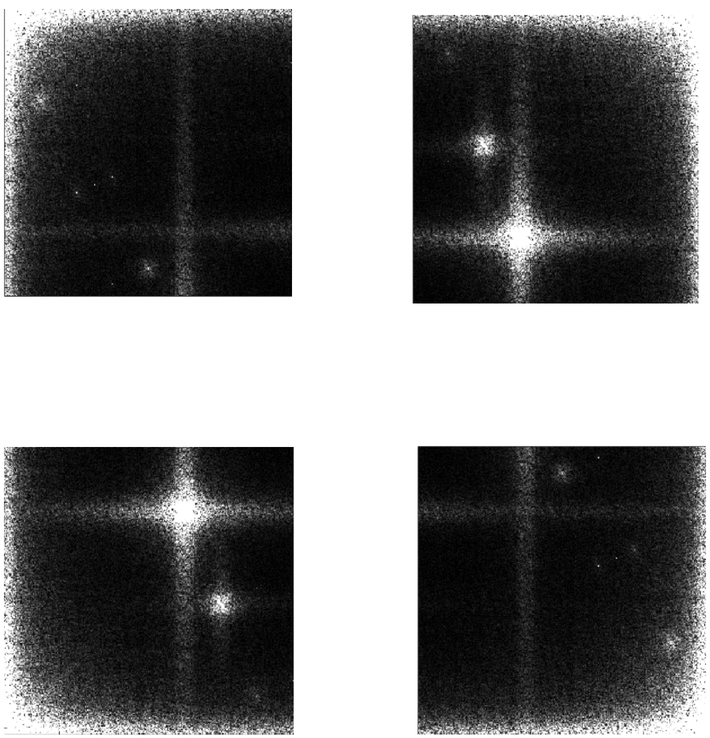
# Symmetry ? $X[m,n] = X^{\star}[K-m, L-n]$

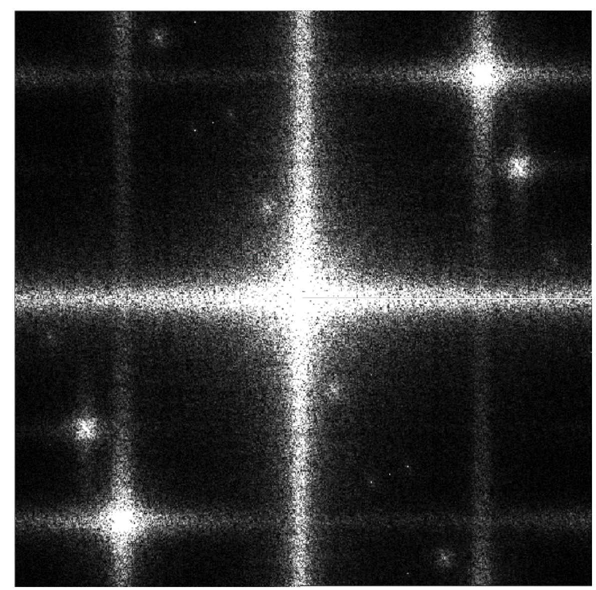



Source: http://www-personal.umich.edu/~hyunjinp/notes/n-dft.pdf

# Re-shuffling…

- Low frequencies to the center

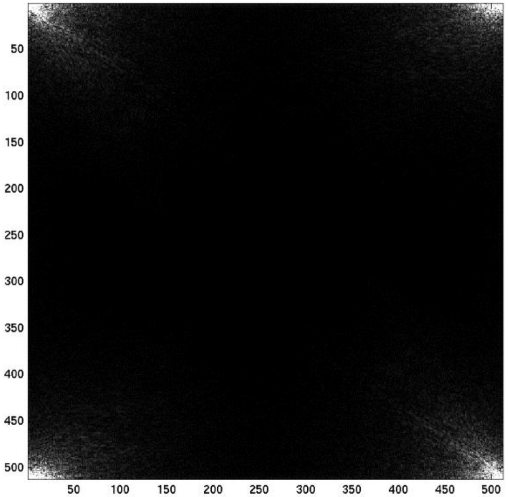




# Inverse 2D DFT

$$x[k,l] = \frac{1}{KL} \sum_{m=0}^{K-1} \sum_{n=0}^{L-1} X[m,n] e^{+j2\pi\left(\frac{mk}{K} + \frac{nl}{L}\right)}$$

# Playing with frequencies

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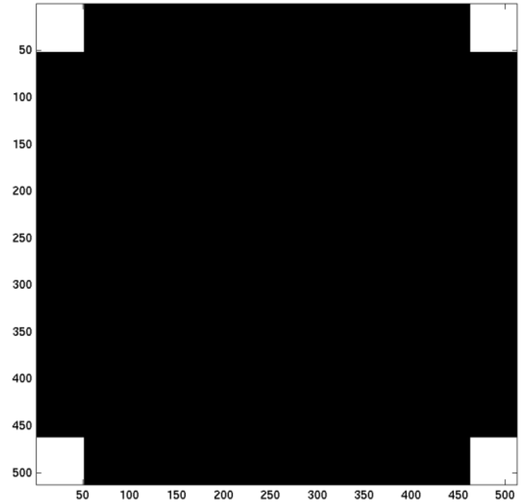


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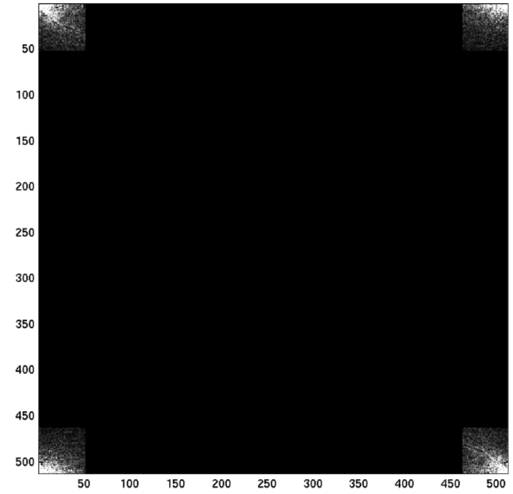

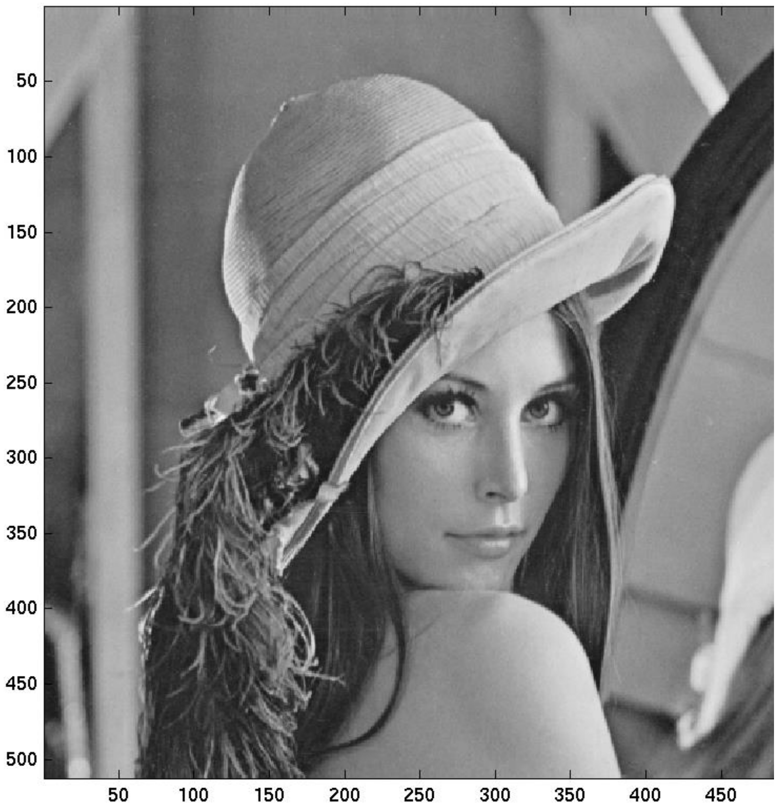
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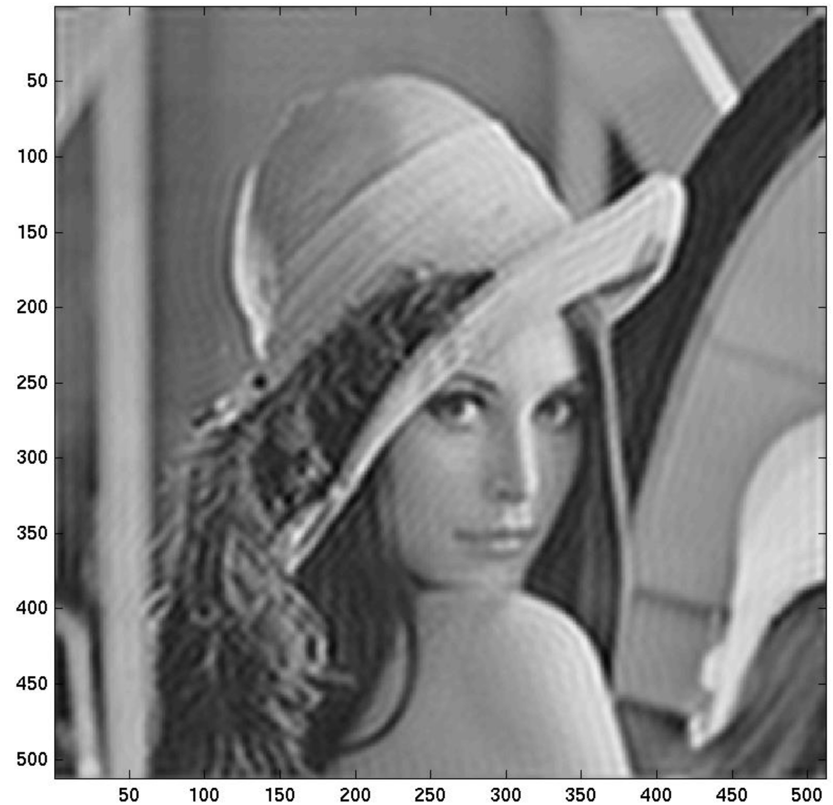
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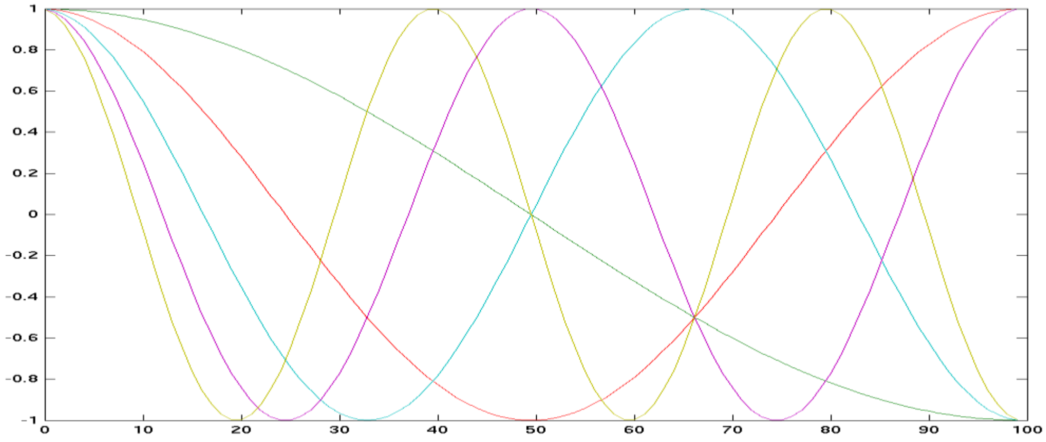
# DCT

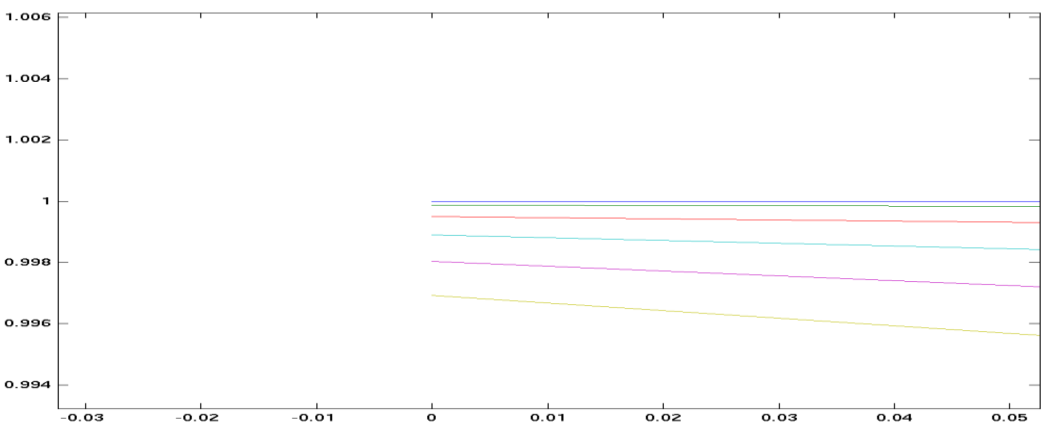
Why ?

- People don't like complex exponentials
- People don't like symmetries and "useless" values …
- For image *K x L,* we want *K x L* real values.

# 1D DCT

$$X[k] = \sum_{n=0}^{N-1} x[n] \cos\left[\frac{\pi}{n}(n+\frac{1}{2})k\right]$$




+ normalization of coefficients…

$$X[0] \ldots \times \sqrt{\frac{1}{N}}$$

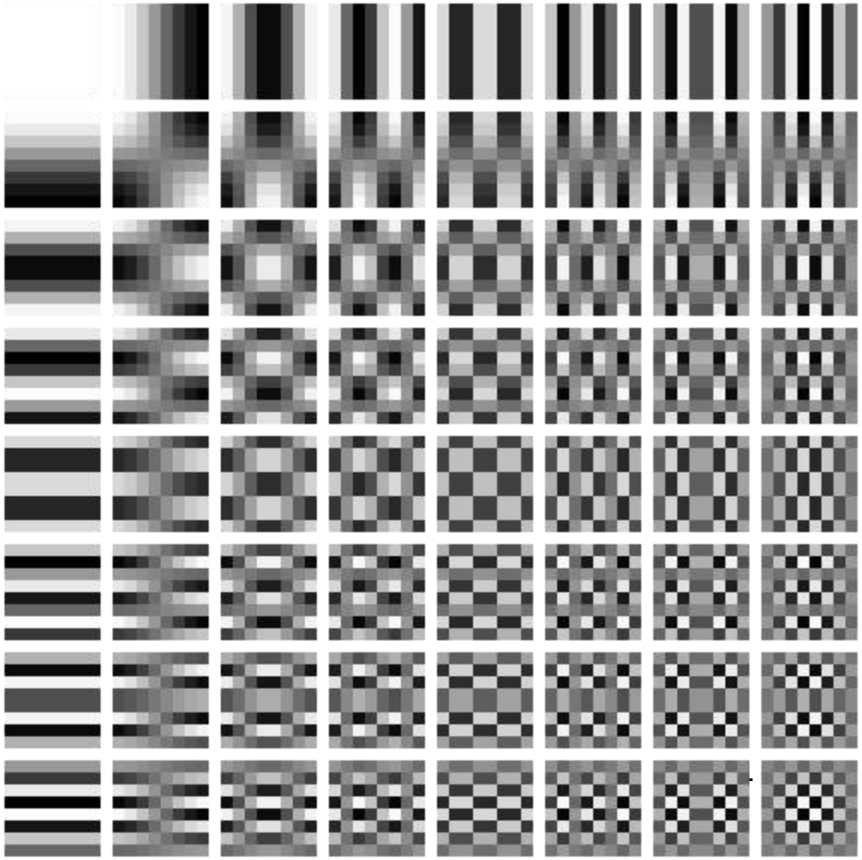
$$X[1 \ldots N-1] \ldots \times \sqrt{\frac{2}{N}}$$

# 2D DCT bases

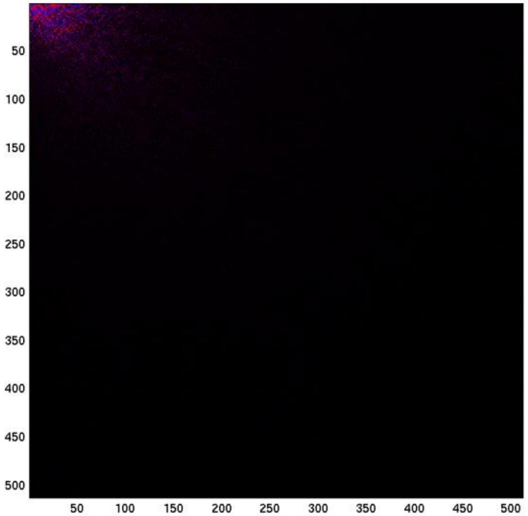
$$X_{k_1,k_2} = \sum_{n_1=0}^{N_1-1} \left( \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_2}\left(n_2+\frac{1}{2}\right)k_2\right]\right) \cos\left[\frac{\pi}{N_1}\left(n_1+\frac{1}{2}\right)k_1\right]$$

$$= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_1}\left(n_1+\frac{1}{2}\right)k_1\right] \cos\left[\frac{\pi}{N_2}\left(n_2+\frac{1}{2}\right)k_2\right].$$

Source:
https://en.wikipedia.org/wiki/Discrete_cosine_transform#Multidimensional_DCTs



# Filtering in DCT

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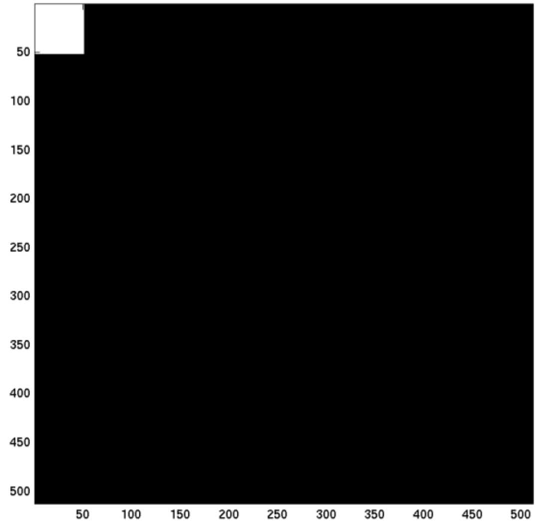


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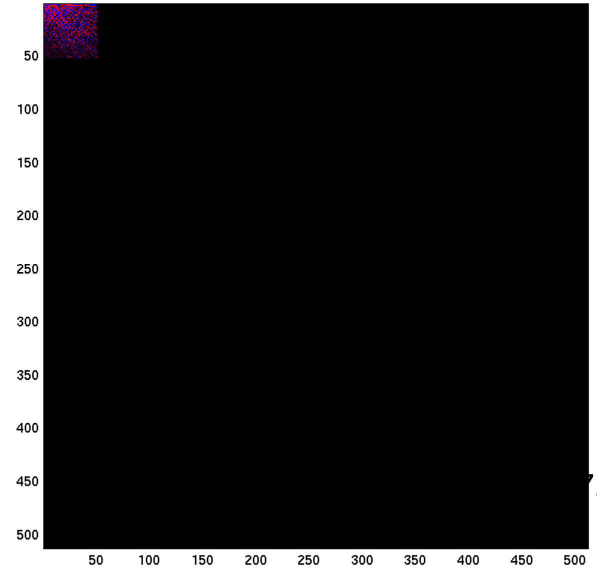


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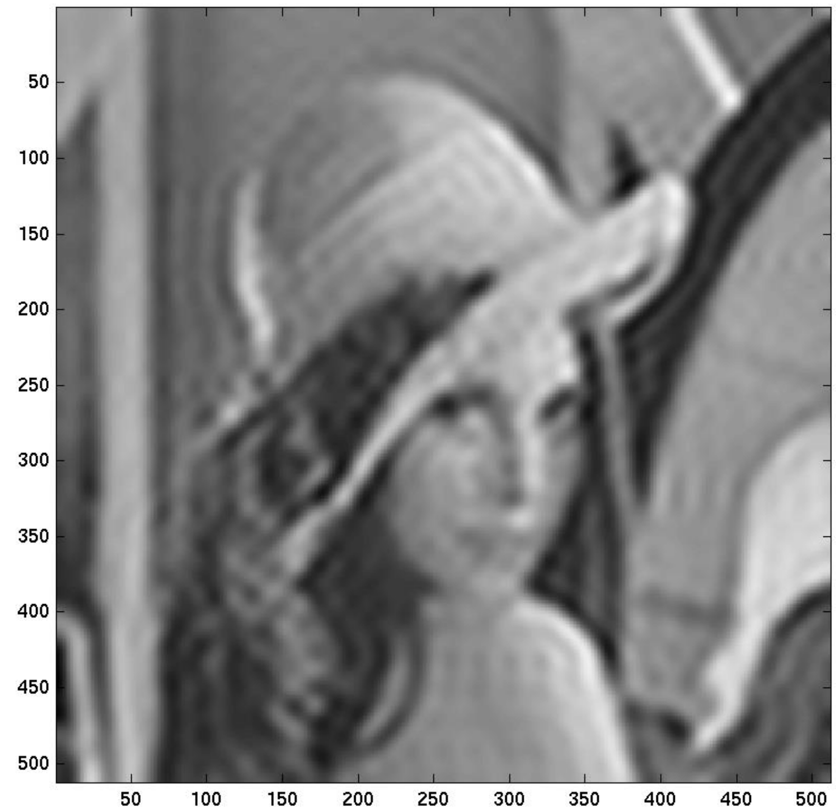
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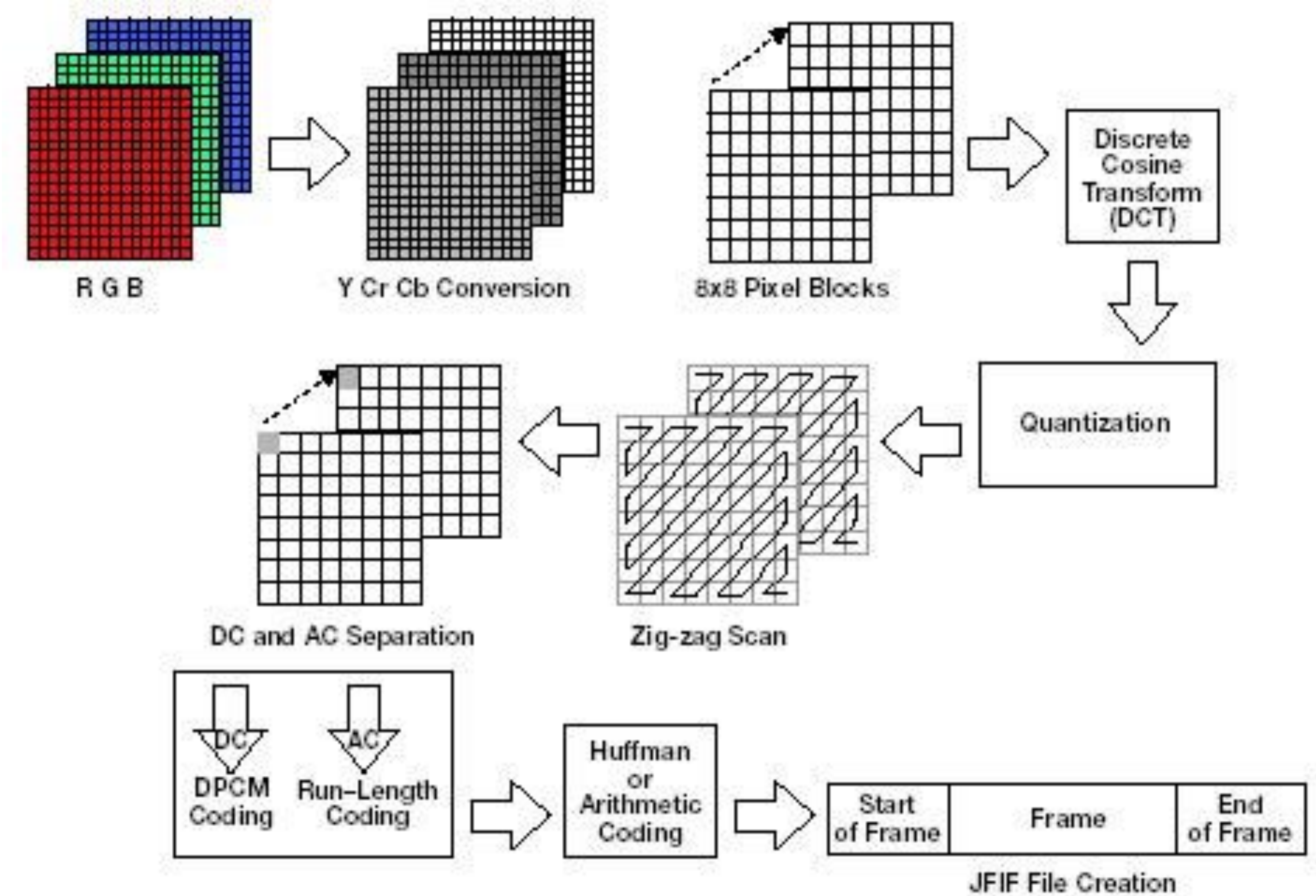
# JPEG




- Source: http://www.eetimes.com/document.asp?doc_id=1225736
- More in computer graphics courses and lab exercise.

# SUMMARY

- Image is a 2D signal
- Filtering
  - Convolution, analogy with 1D FIR filters
  - No feedback (IIR) in images
- Sweeping mask over the image, multiplying everything that is underneath with its coefficients, adding.

# SUMMARY II.

- Frequency analysis 2D
  - Similar to 1D – projection to bases
  - Image frequencies also have some sense.
  - Cos bases are a good exercise, but not enough.
- 2D DFT
  - Projection/similarity/correlation with complex exponentials
  - Coefficients are complex – magnitude and angle.
  - Lots of symmetries in the resulting DFT matrix
  - Can use spectrum for filtering

# SUMMARY III.

- DCT
  - 2x „slower“ bases than DFT
  - Slightly More complicated definition
  - Produces real coefficients, low frequencies (only!) at the beginning.
  - Use in JPEG

# TO BE DONE

- Determining frequency response of a 2D filter
- How is it exactly with the symmetries of 2D DFT ?
- Why are 1D and 2D DCT exactly defined and why this half-sample shift ?
- Colors (color models, etc).
- How does the face matching on Facebook work?
  - Hint: https://research.facebook.com/researchers/684639631606527/yann-lecun/

# The END