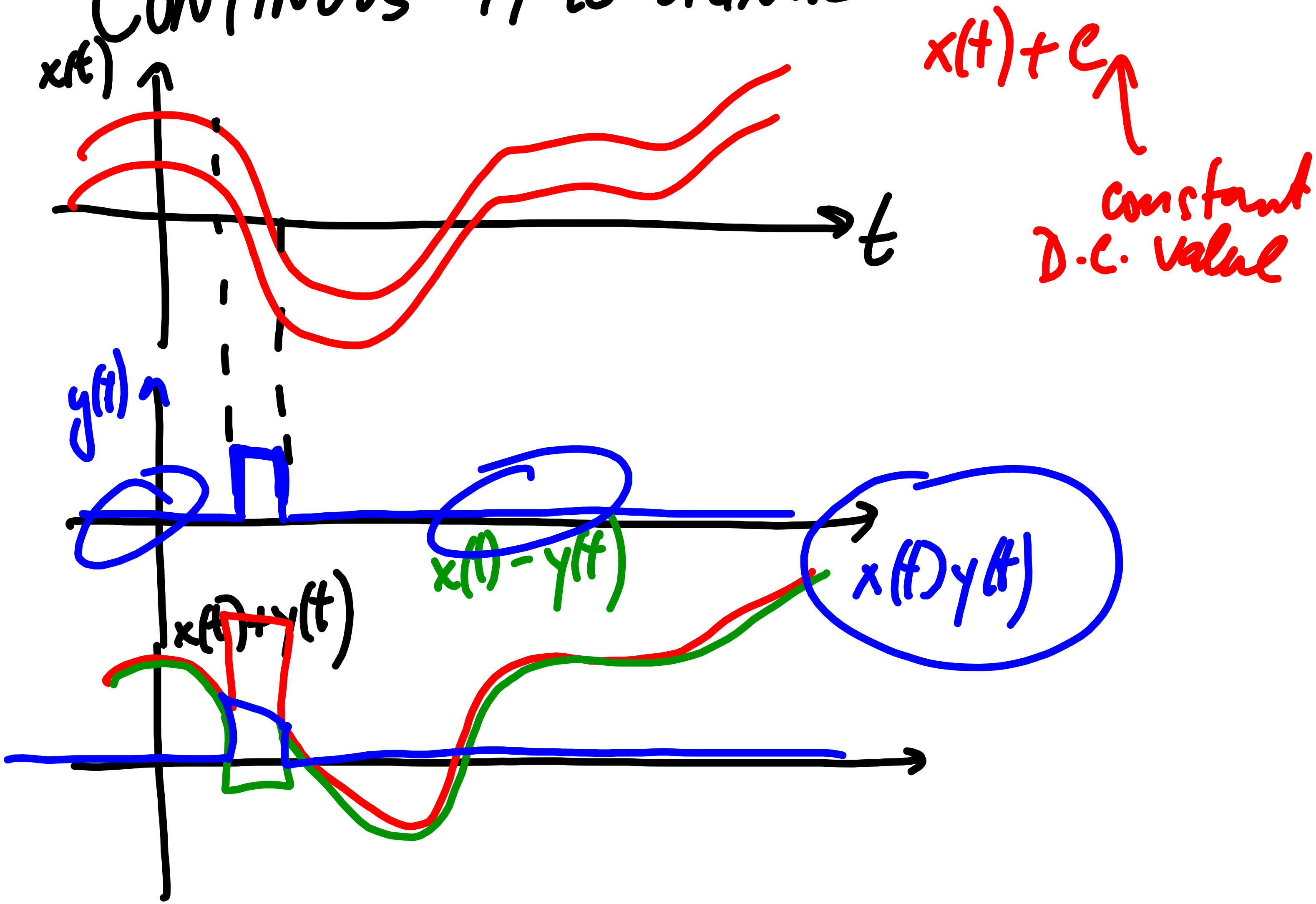
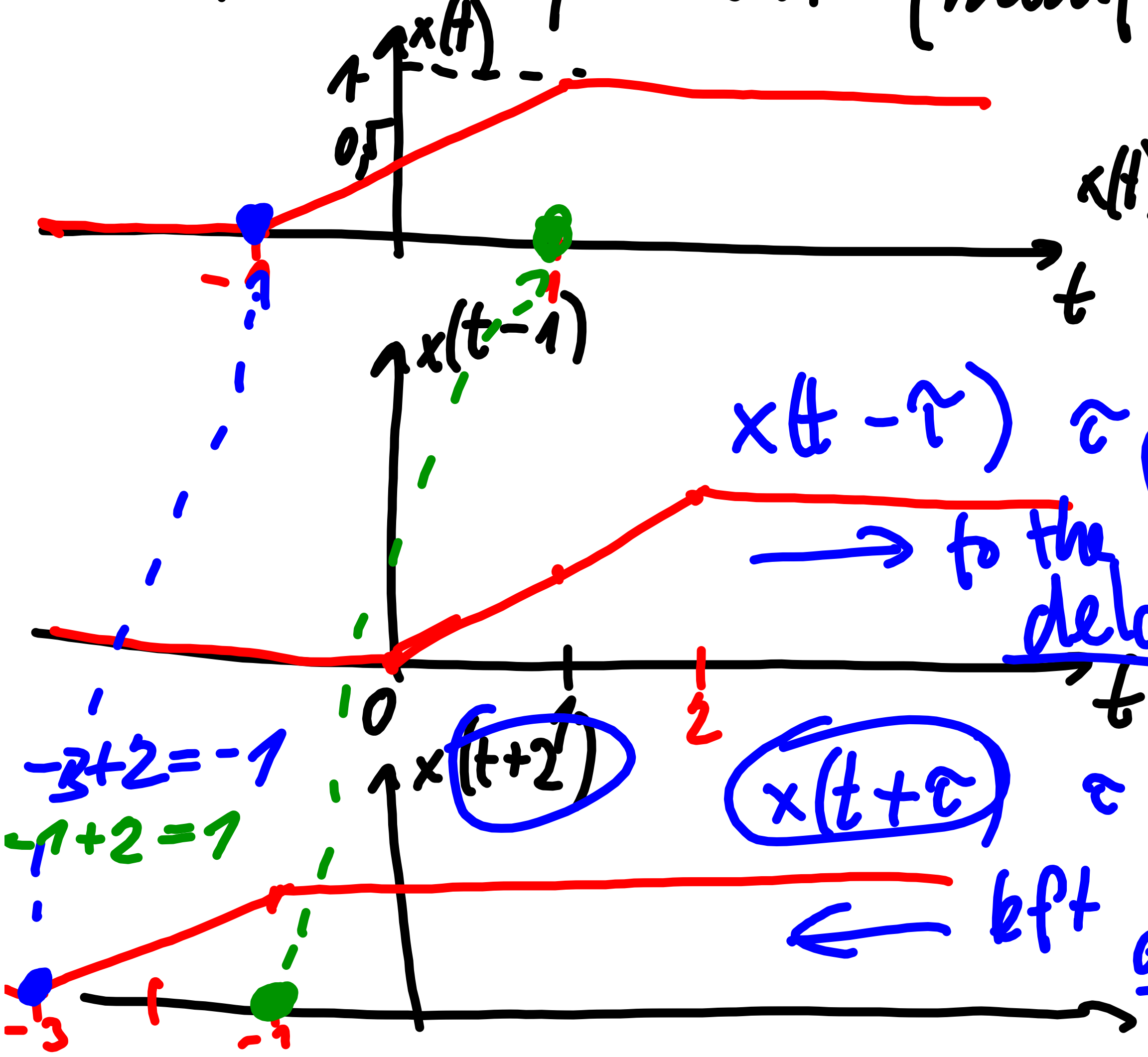
CONTINUOUS-TIME SIGNALS



Time modification (modif. of time axis)

$$x(t)=\begin{cases}0 & \text{for } t<-1\\ 0{,}5t+0{,}5 & \text{for } t\in[-1,1]\\ 1 & \text{for } t>1\end{cases}$$



$x(t-\tau)$ τ positive

→ to the right

delay

$x(t+\tau)$ τ positive

← left

advance

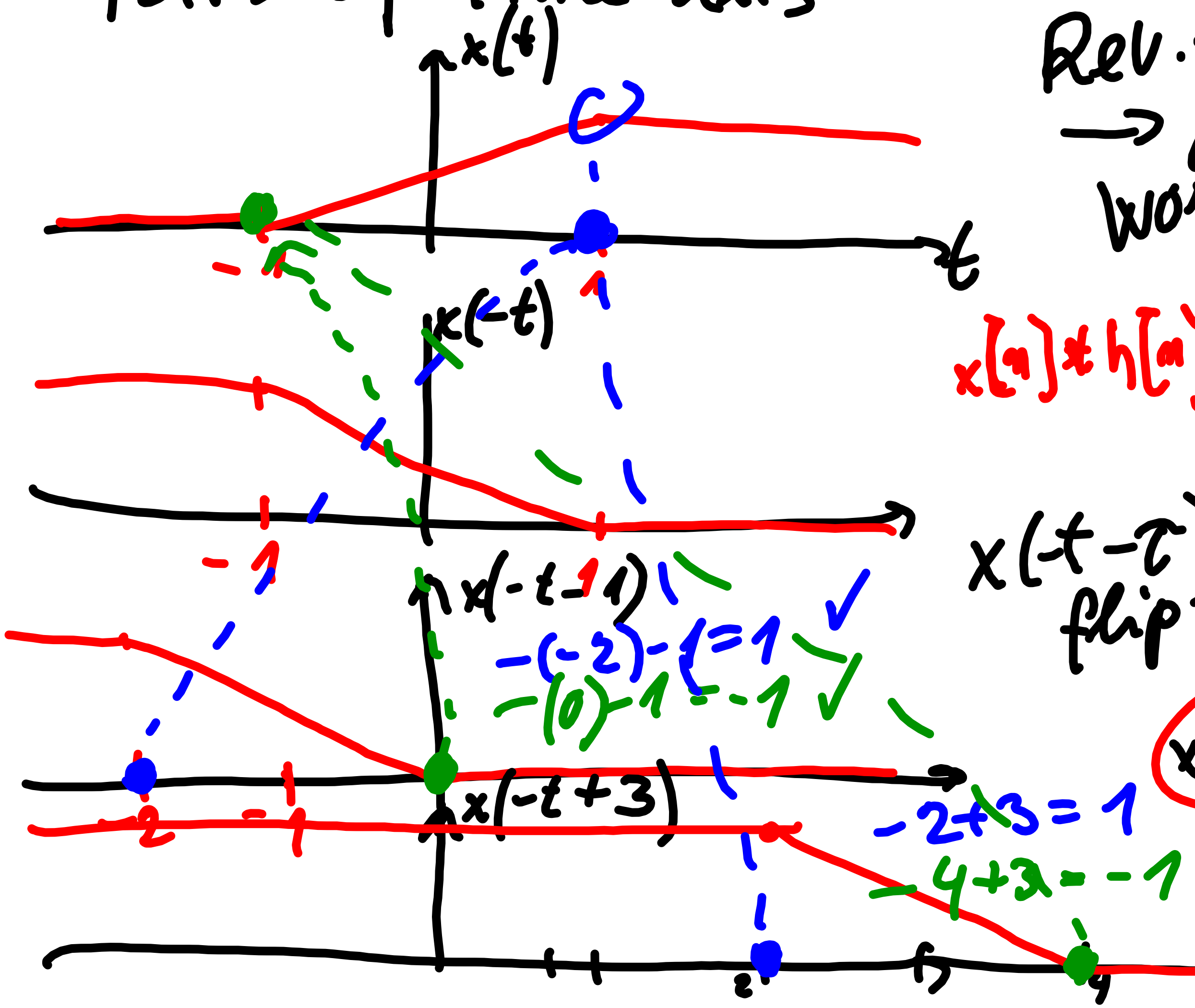
CHECK

1) select some time on the t axis

2) eval the time modif

3) check what is the orig.

FLIPS of time axis



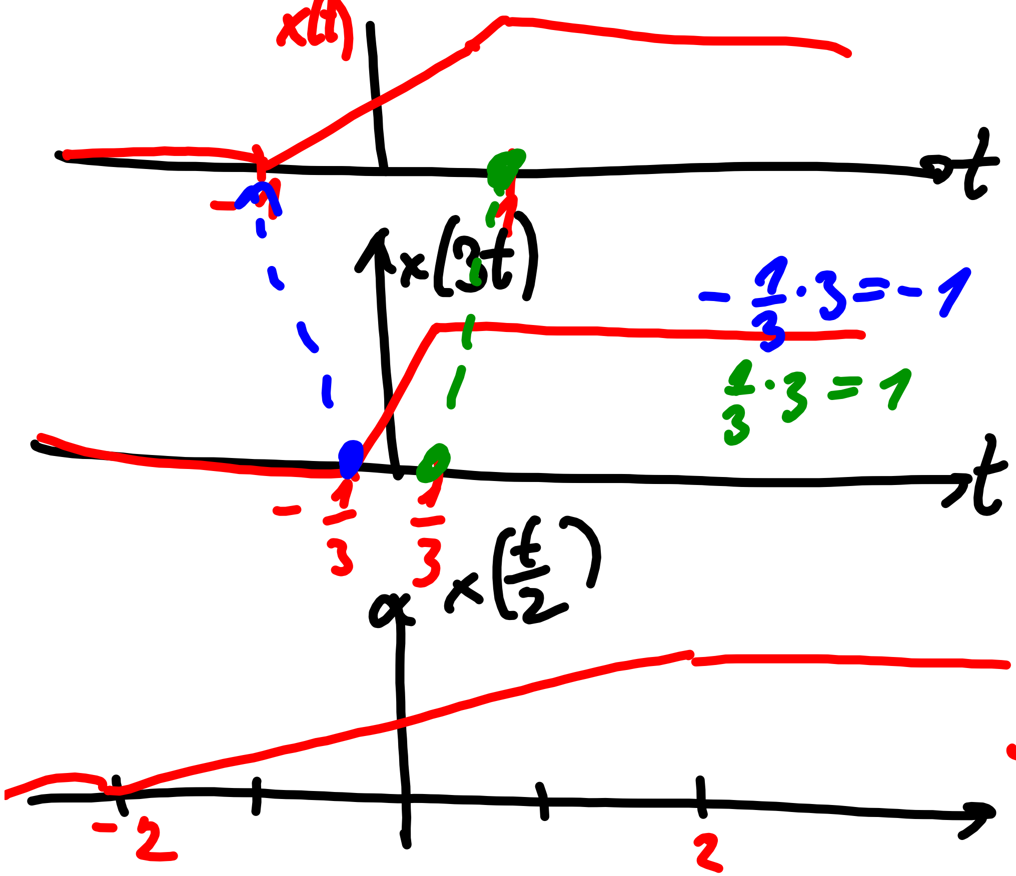
Rev. time axis → all shifts work inversely!

$$x[n] * h[n] = \sum_k x[k]\,h[n-k]$$

$x(-t-\tau)$ flip and advance, positive τ

$x(-t+\tau)$ flip and delay

Contraction and dilatation of time



time speeds up
contraction
$\cos(3\omega_0 t)$

time slows down
dilatation

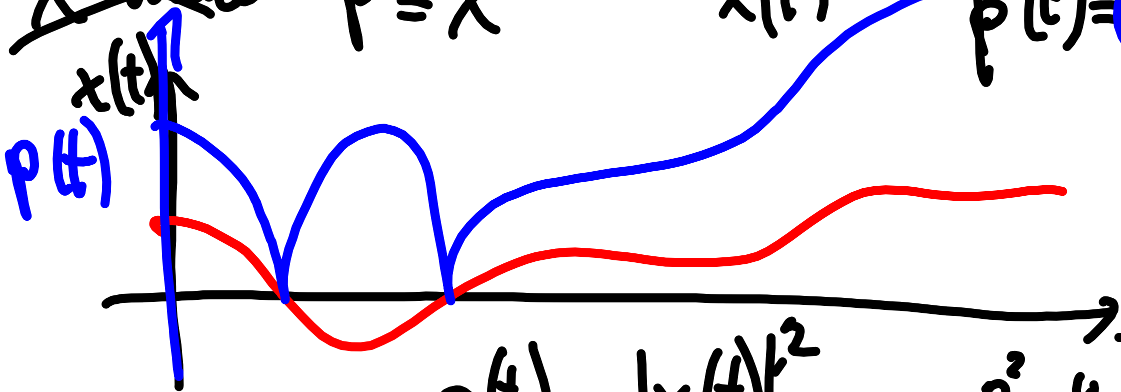
TDNN - time-~~delayed~~ dilated

ENERGY / POWER of signals

$$P = U \cdot I = \frac{U^2}{R} = I^2 R \qquad U = I \cdot R$$

instantaneous power

~~X value~~ $\quad P = X^2 \qquad x(t) \qquad p(t) = x^2(t)$



p(t) Real! Non-negative.

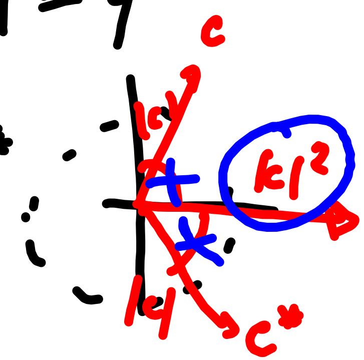
$$p(t) = |x(t)|^2 \qquad 2^2 = 4 \quad (-2)^2 = 4$$

necessary if the sig is complex!

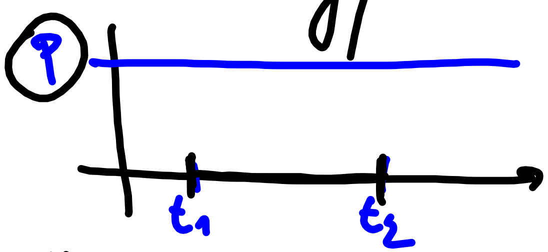
$2j \qquad \cancel{(2j)^2 = 4} \qquad |2j|^2 = 4$

```
p = np.pow(np.abs(x), 2.0);
p = np.conj(x) * x;
```

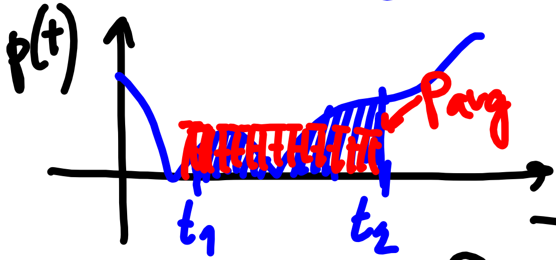
$$|c|^2 = c \cdot c^*$$



Energy



$$E_{t_1,t_2} = P\cdot(t_2-t_1)$$



$$E_{t_1,t_2} = \int_{t_1}^{t_2} p(t)\,dt$$

must be non-neg.

Average (mean) power

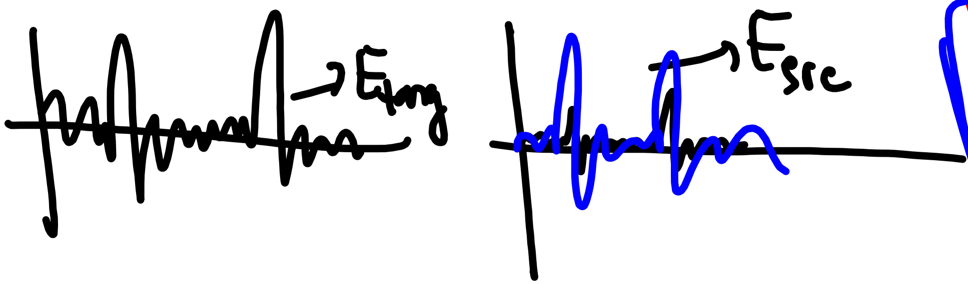
$$P_{avg} = \frac{E_{t_1,t_2}}{t_2-t_1}$$

Effective value = RMS root mean square

$$X_{ef} = \sqrt{P_{avg}} = \sqrt{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} x^2(t)\,dt}$$

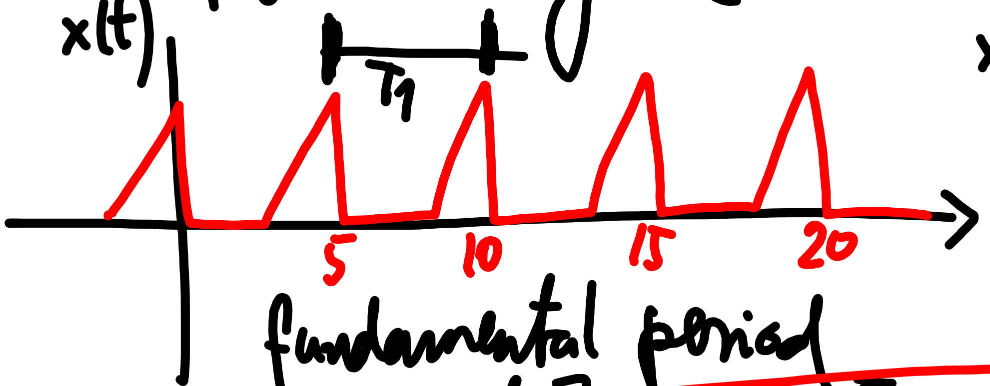
root mean square

constant signal that would give the same average power



$$\sqrt{\frac{E_{target}}{E_{src}}}$$

Periodic signal



$x(t) = x(t-5)$
$= x(t-10)$
$= x(t-15)$
$\vdots$

fundamental period

T_1 [seconds] $T_1 = 5\,s$ → unique

not true for discrete signals !!!

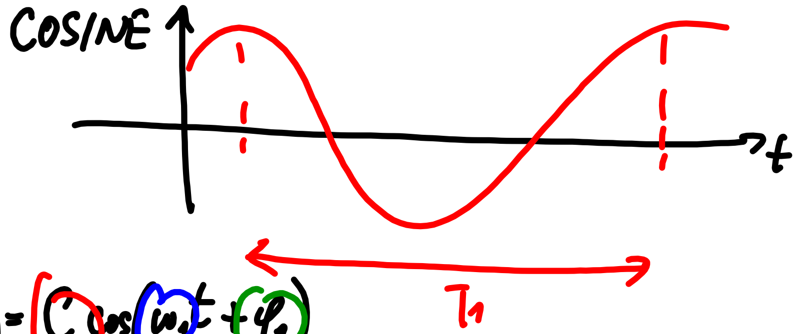
$f_1 = \frac{1}{T_1}$ [Hz] fundamental freq.

$\omega_1 = 2\pi f_1$ [rad/s] fund. angular frequency

Energy/power over 1 period

$E_{T_1} = \int_{T_1} x^2(t)\,dt$ $\qquad P_{avg} = \frac{1}{T_1}\int_{T_1} x^2(t)\,dt$

$\int_{0}^{T_1}$ $\quad \int_{-T_1/2}^{T_1/2}$



$$x(t) = C_1\cos(\omega_1 t + \varphi_1)$$

[any unit] rad/s initial phase [rad] $2\pi \approx$ the whole period.

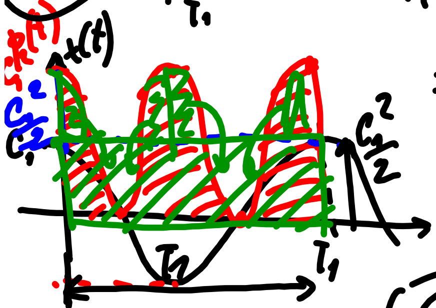
how big? where? how shifted?

Average power of a cosine $C_1\cos(\omega_1 t)$

$$P_{avg} = \frac{1}{T_1}\int_{T_1} x^2(t)\,dt = \frac{1}{T_1}\int_0^{T_1}\left[C_1\cos(\omega_1 t)\right]^2 dt =$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{T_1}\int_0^{T_1} C_1^2\left[\frac{1+\cos 2\omega_1 t}{2}\right] dt =$$

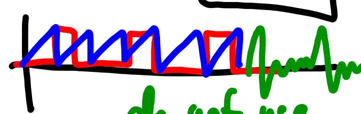


$$= \frac{1}{T_1}\int_0^{T_1}\left[\frac{C_1^2}{2} + \frac{C_1^2}{2}\cos 2\omega_1 t\right] dt =$$

$$= \frac{1}{\cancel{T_1}}\,\frac{C_1^2}{2}\cdot\cancel{T_1} = \frac{C_1^2}{2}$$

$$X_{ef} = \sqrt{\frac{C_1^2}{2}} = \boxed{\frac{C_1}{\sqrt{2}}} \leftarrow \text{cos only!}$$

B:25

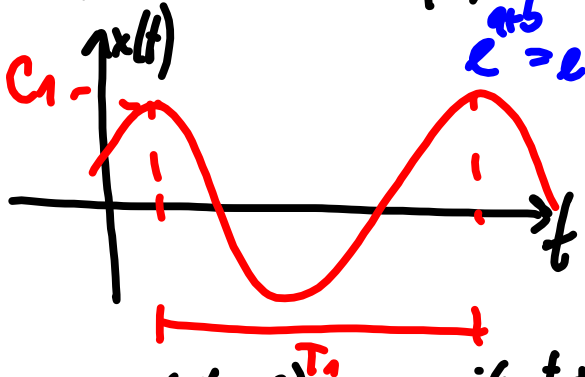


please do not use $\frac{C_1}{\sqrt{2}}$

Breaking cos into 2 complex exponentials.

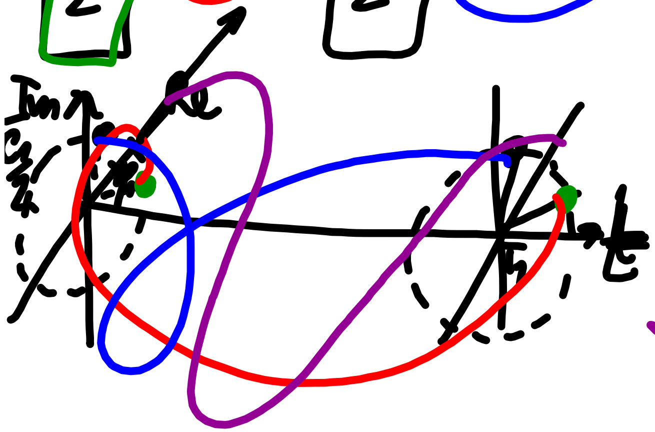
$$x(t) = C_1 \cos(\omega_1 t + \varphi_1)$$

$$e^{a+b} = e^a \cdot e^b$$



$$x(t) = \frac{C_1}{2} e^{j(\omega_1 t+\varphi_1)} + \frac{C_1}{2} e^{-j(\omega_1 t+\varphi_1)} =$$

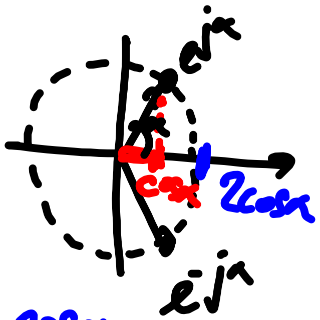
$$= \frac{C_1}{2} e^{j\varphi_1} e^{j\omega_1 t} + \frac{C_1}{2} e^{-j\varphi_1} e^{-j\omega_1 t}$$



→ www of ISS demo

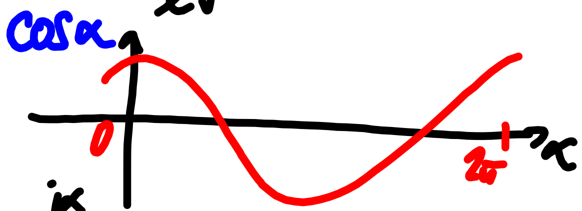
→ Python NB.

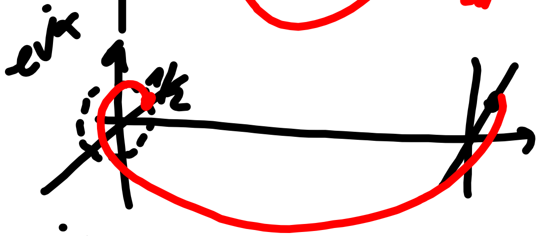
$$\cos x = \frac{e^{jx}+e^{-jx}}{2}$$

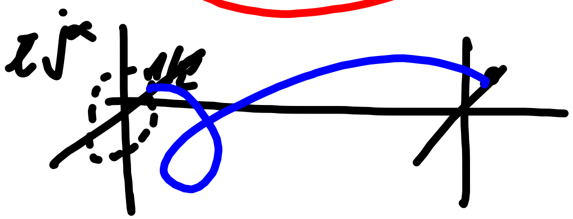


cos x

$$\cos x = \frac{e^{jx}+e^{-jx}}{2}$$

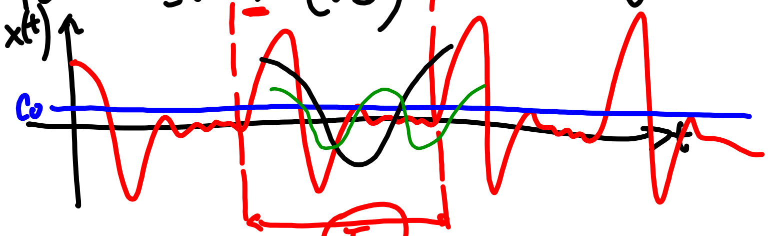






Frequency analysis of periodic signals.

Fourier Series (FS)



3 basic questions:

1. where/which frequencies?
2. how big?
3. (how shifted?)

T_1 ω_1 C_1 ...

$\omega_1 = \frac{2\pi}{T_1}$ fund. frequency

$$x(t) = c_0 + C_1\cos(\omega_1 t + \varphi_1) + C_2\cos(2\omega_1 t + \varphi_2) + \dots + C_k\cos(k\omega_1 t + \varphi_k) + \dots$$

constant D.C. component

synthesis Real form of FS.

COMPLEX form of FS.

$$x(t) = \dots + c_{-k}e^{-jk\omega_1 t} \dots + c_{-2}e^{-j2\omega_1 t} + c_{-1}e^{-j\omega_1 t} + c_0 + c_1 e^{j\omega_1 t} + c_2 e^{j2\omega_1 t} + \dots + c_k e^{jk\omega_1 t} + \dots$$

$\cos x = \frac{e^{jx} + e^{-jx}}{2}$ ...

I want: $C_k\cos(k\omega_1 t + \varphi_k) = c_{-k}e^{-jk\omega_1 t} + c_k e^{jk\omega_1 t}$

$$|c_{-k}| = |c_k| = \frac{C_k}{2} \qquad \arg c_{-k} = -\varphi_k \qquad \arg c_k = \varphi_k$$

compact writing:

Real: $x(t) = c_0 + \sum_{k=1}^{\infty} C_k\cos(k\omega_1 t + \varphi_k)$

Complex: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t}$

Synthesis formula

1. where?
2. how much?
3. how shifted

abs. value

arg

Computing coefficients of FS ← this is the ~~freq. analysis~~

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t}$$

coefficients

bases! $\underline{b_k(t)}$

$$c_k = \frac{1}{T_1} \int_{T_1} x(t)\, b_k^*(t)\, dt$$

① complex basis → multiply with complex-conjugate basis

② check that we have good bases!
→ orthogonal ✓
→ normal

$$\int_{T_1} e^{jk\omega_1 t} e^{-jl\omega_1 t} dt = \emptyset$$ unless $k = l$

we have orthogonal bases!

$$\int_{T_1} |\underbrace{e^{jk\omega_1 t}}_{1}| dt = \int_{T_1} 1\, dt = T_1$$ ooooooh bad! I need 1 !!!

$$c_k = \frac{1}{T_1} \int_{T_1} x(t)\, b_k^*(t)\, dt = \boxed{\frac{1}{T_1} \int_{T_1} x(t)\, e^{-jk\omega_1 t} dt}$$

analysis formula of FS.

Results of FS:

−10 - - -

−9 - - -

⋮

0 - - -

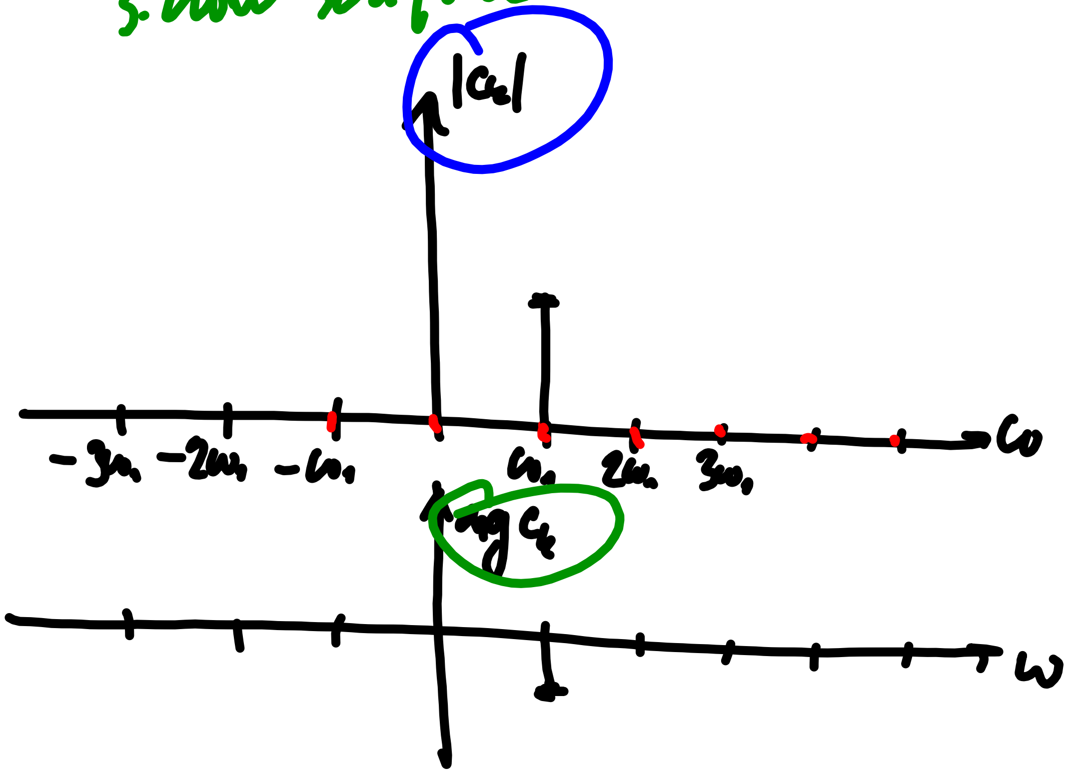
⋮

9 - - -

10 - - -

Spectrum representation of signal in frequency

1. where → discrete frequencies $k\omega_1$
2. how much
3. how shifted.



Ex#1: Complex exp:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t}$$

match with what we have :)

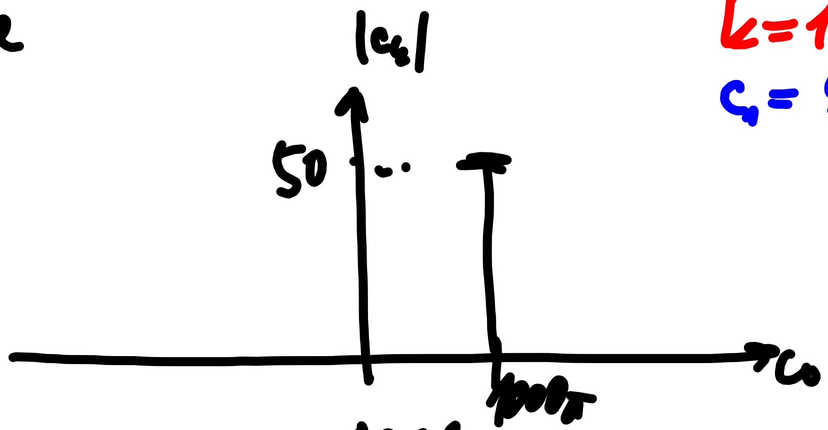
$$x(t) = 50 e^{j(1000\pi t + \pi/2)} = 50 e^{j\pi/2} \cdot e^{j1000\pi t}$$

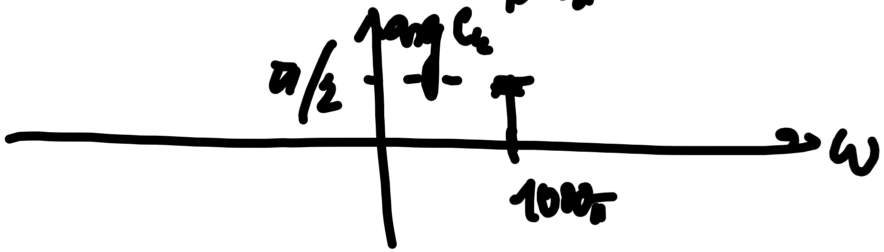
$\omega_1 = 1000\pi$

$f_1 = 500\,\text{Hz}$ ...

$k = 1$

$c_1 = 50 e^{j\pi/2}$





$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Ex#2: cos.

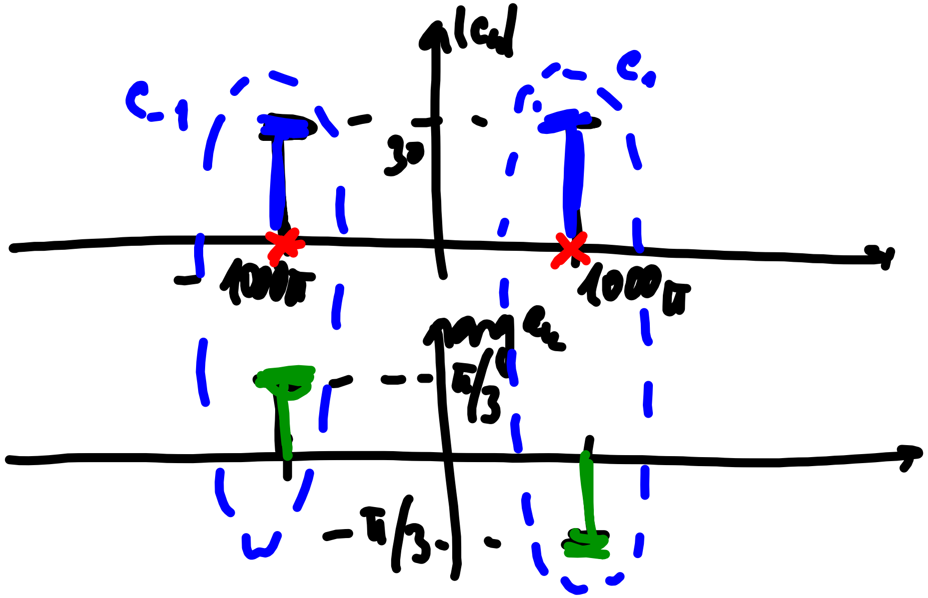
Check: $c_k = c_{-k}^*$

$$x(t) = 60\cos(1000\pi t - \pi/3) = 30 e^{-j\pi/3} e^{j1000\pi t} + 30 e^{j\pi/3} e^{-j1000\pi t}$$

$\omega_0 = 1000\pi$

$k=1 \quad c_1 = 30e^{-j\pi/3}$

$k=-1 \quad c_{-1} = 30e^{+j\pi/3}$

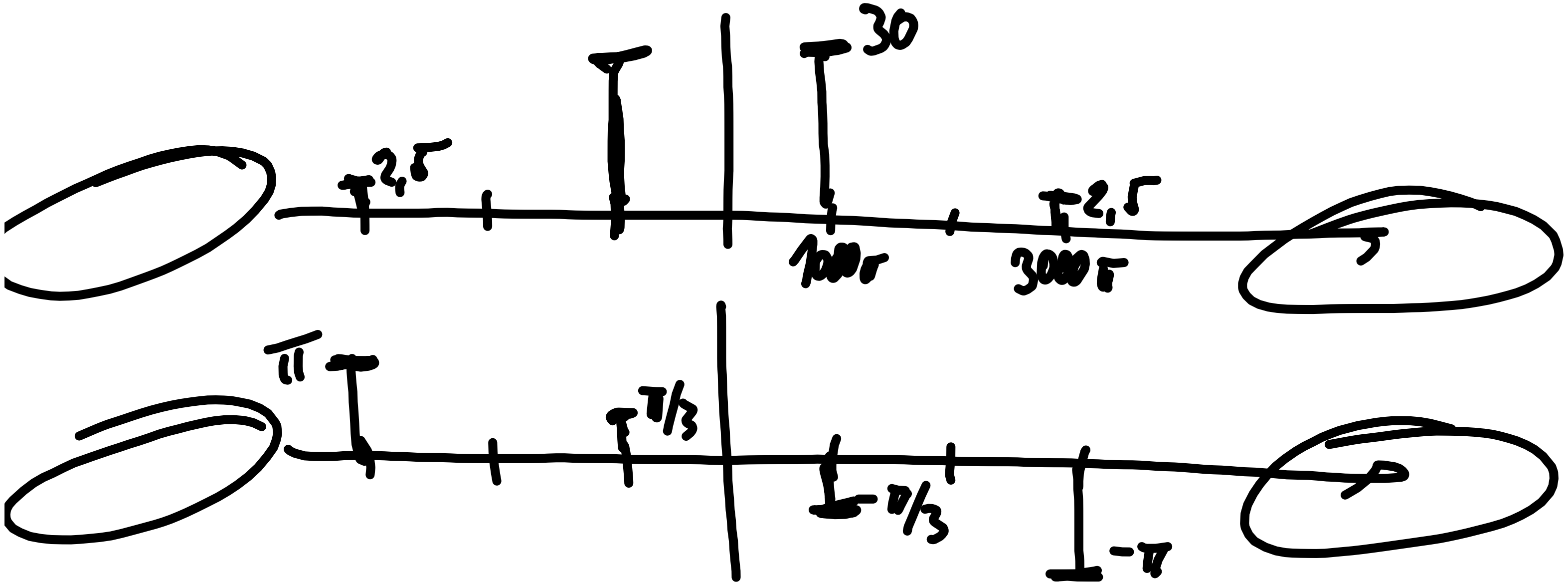


Ex#3: $x(t) = 60\cos(1000\pi t - \pi/3) + 5\cos(3000\pi t - \pi)$

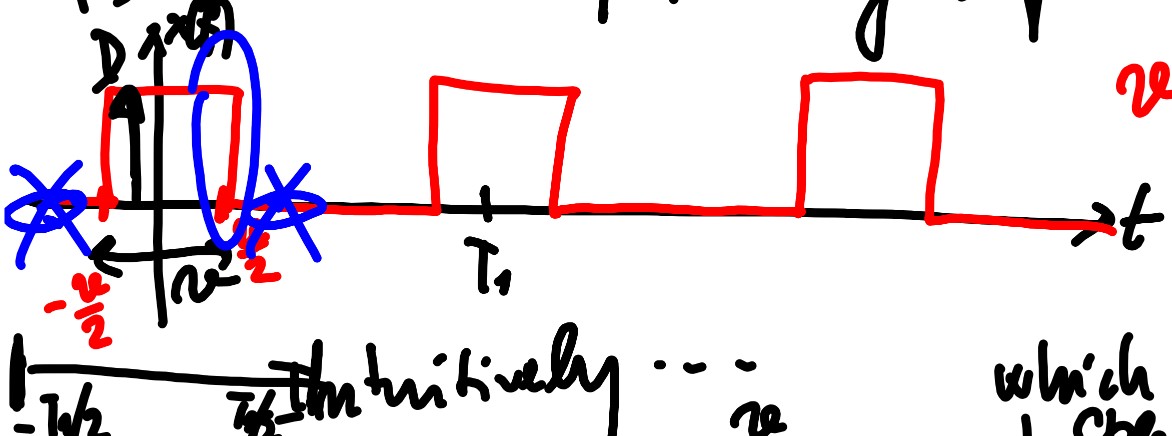
$\omega_0 = 1000\pi$

$k=1 \quad c_1 = 30e^{-j\pi/3} \qquad k=-1 \quad c_{-1} = 30e^{j\pi/3}$

$k=3 \quad c_3 = 2.5e^{-j\pi} \qquad k=-3 \quad c_{-3} = 2.5e^{+j\pi}$



FS of a series of rectangular pulses



τ width of pulse

$\frac{\tau}{T_1}$ "duty cycle"

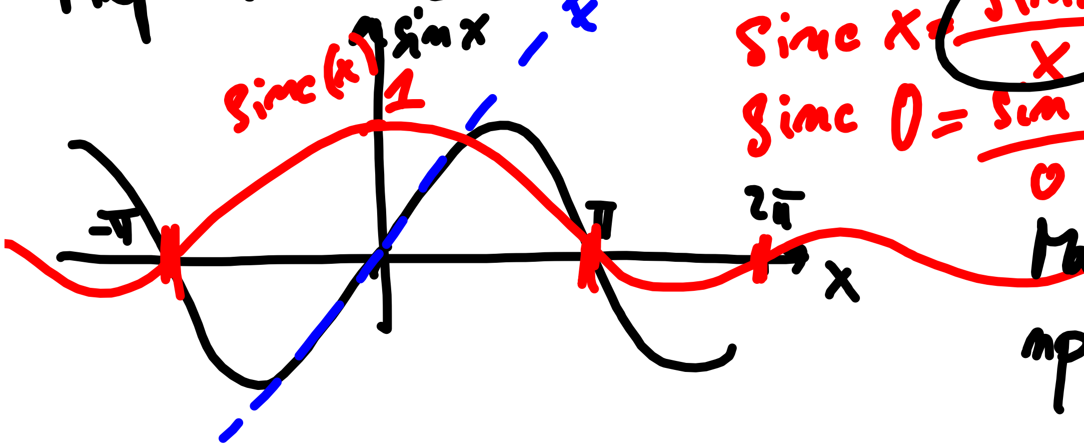
Intuitively ...

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)e^{-jk\omega_1 t}dt = \frac{1}{T_1}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} D e^{-jk\omega_1 t}dt$$

which frequency does its spectrum stop?

Never stops! $\pm\infty$!

Prepa #1: cardinal sine



$$\text{sinc } x = \frac{\sin x}{x}$$

$$\text{sinc } 0 = \frac{\sin 0}{0} = \frac{0}{0} = ??? = 1$$

Matlab/python $\frac{\sin \pi x}{\pi x}$

np.sinc(x/np.pi)

Prep FT2:

$$\int_{-b}^{b} e^{jxy}\,dy = \left[\frac{e^{jxy}}{jx}\right]_{-b}^{b} = \frac{e^{jxb}}{jx} - \frac{e^{-jxb}}{jx} =$$

"Sebesta's tool"

$$\sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$= \frac{2}{x}\,\frac{e^{jxb} - e^{-jxb}}{2j} = \frac{2b}{xb}\sin(xb) = 2b\,\mathrm{sinc}(xb)$$

$$\mathrm{sinc}\,\alpha = \frac{\sin\alpha}{\alpha}$$

$$c_k = \frac{1}{T_1}\int_{-\frac{\vartheta}{2}}^{\frac{\vartheta}{2}} D e^{-jk\omega_1 t}\,dt = \frac{D}{T_1}\int_{-\frac{\vartheta}{2}}^{\frac{\vartheta}{2}} e^{-jk\omega_1 t}\,dt = \begin{matrix}\frac{\vartheta}{2} = b\\ y = t\\ x = k\omega_1\end{matrix} = \frac{D}{T_1}\cancel{2}\frac{\vartheta}{\cancel{2}}\,\mathrm{sinc}\left(\frac{\vartheta}{2}k\omega_1\right)$$

$$= \frac{D\vartheta}{T_1}\,\mathrm{sinc}\left(\frac{\vartheta}{2}k\omega_1\right)$$