Systems: - causality $x(t)$ $x(<t)$
$y(<t)$

- linearity $x_1(t) \to y_1(t)$
$x_2(t) \to y_2(t)$
$a x_1(t) + b x_2(t) \to a y_1(t) + b y_2(t)$

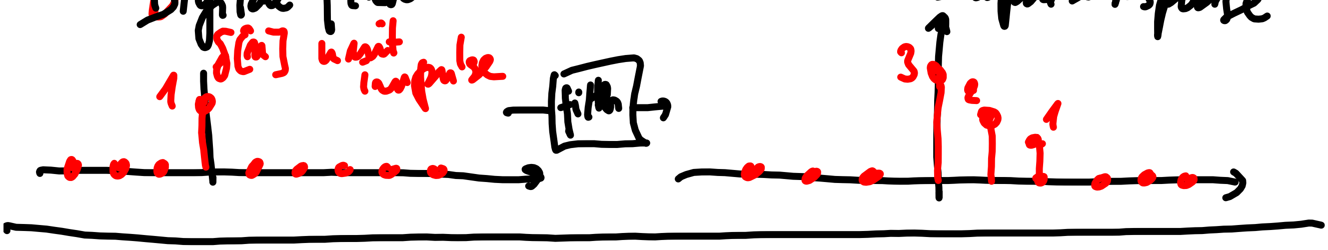
- time invariance
- stability BI → BO

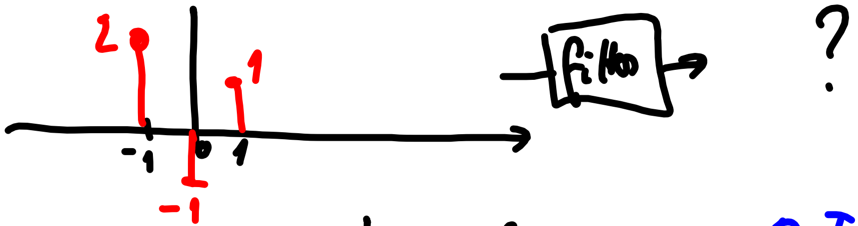
LTI

Difficult signal → $\boxed{\text{system}}$ → ?

1. Σ elementary signals
2. for ...
process an el. signal
3. Σ of all outputs

Digital filter Impulse response





1. Dissect the input signal

2. Indep. process of inputs

3. Sum all the inputs

n	-2	-1	0	1	2	3	4	5	6	7
x[n]		2	-1	1						
2δ[n+1]		2								
		6	4	2						
-1δ[n]			-1							
			-3	-2	-1					
1δ[n-1]				1						
				3	2	1				
		6	1	3	1	1				

↑n

$y[1] = x[1]\cdot h[0] + x[0]h[1] + x[-1]\cdot h[2]$

$= x[n]h[0] + x[n-1]h[1] + x[n-2]h[2]$

$$y[n] = \underbrace{\sum_k x[n-k]h[k]}_{\text{convolution}}$$

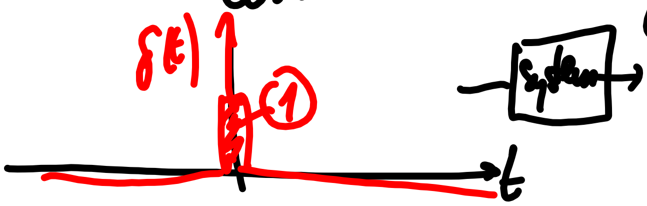
$$\sum_k x[k]h[n-k]$$

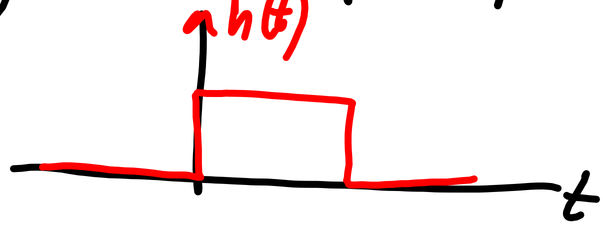
$$y[n] = x[n] * h[n] = h[n] * x[n]$$

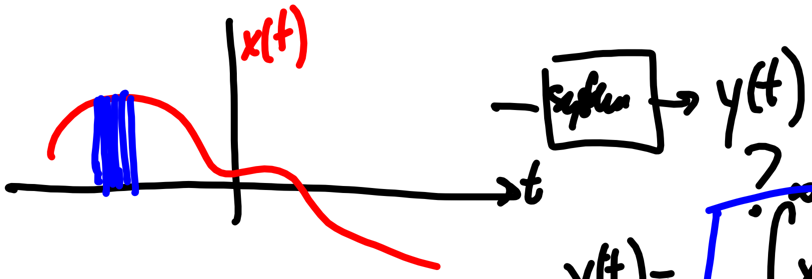
$$\boxed{y[n] = \sum_k x[n-k]\,h[k]}$$

Continuos time system

Impulse response







CONVOLUTION

$$y(t) = \boxed{\int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau}$$

$$\boxed{y(t) = x(t) * h(t)} = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

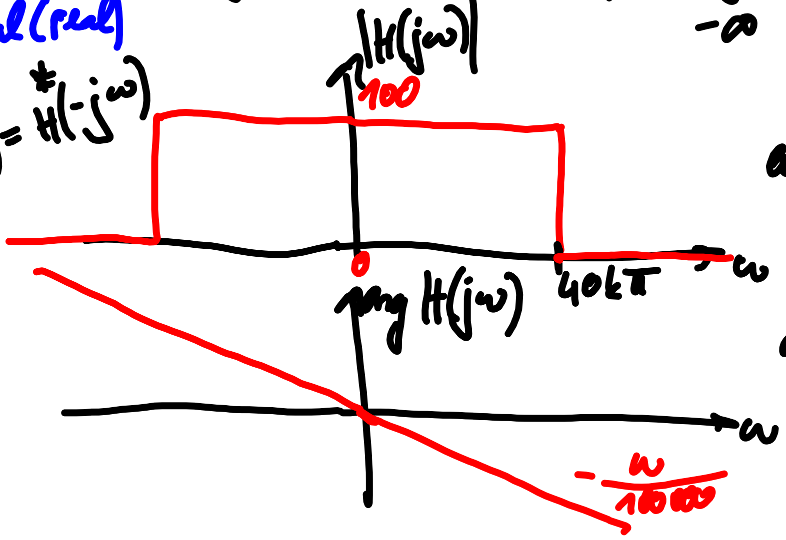
Frequency response of a cont.-time system

h(t) impulse response → frequency (spectrum)

$$H(j\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

signal (real)

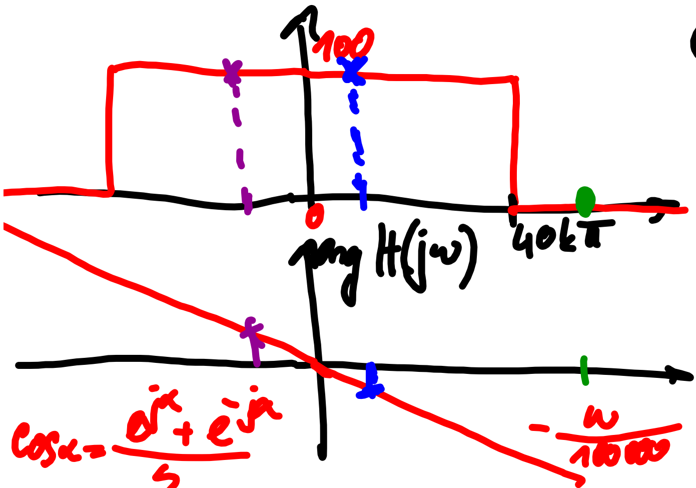
$$H(j\omega) = \overset{*}{H}(-j\omega)$$



abs.

ang.

$2\pi \cdot 20\,kHz = 40k\pi$ rad/s



$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$

Exercise 1:

$x(t) = 5\,e^{j\,10000\pi t}$

$H(j\,10000\pi) = 100 \cdot e^{-0{,}1\pi}$

multiply

$r_1 e^{j\varphi_1} \cdot r_2 e^{j\varphi_2} = r_1 r_2\, e^{j(\varphi_1+\varphi_2)}$

$y(t) = 500\, e^{j(10000\pi t - 0{,}1\pi)}$

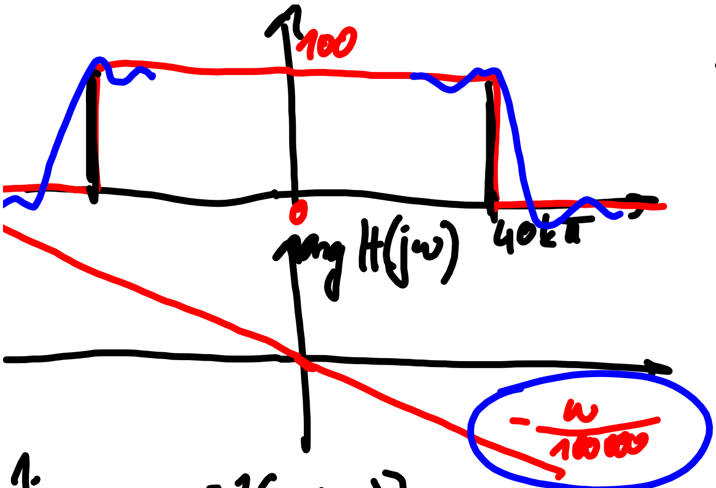
Exercise 2: $x(t) = 5e^{j\,60000\pi t}$

$y(t) = 0 \cdot x(t) = \emptyset$

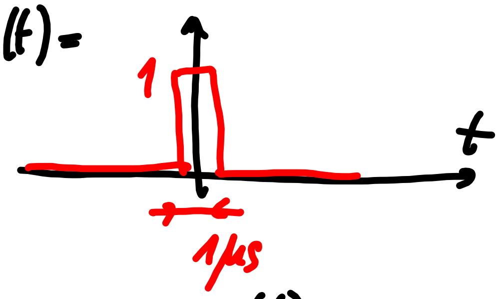
Exercise 3: $x(t) = 10\cos(10000\pi t) = 5e^{j\,10000\pi t} + 5e^{-j\,10000\pi t} =$

$500\, e^{j(10000\pi t - 0{,}1\pi)} + 500\, e^{-j(10000\pi t - 0{,}1\pi)}$

$= 1000 \cos(10000\pi t - 0{,}1\pi)$



$x(t) =$



?

1. $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$

$y(t) = x(t) * h(t)$

2. $\boxed{y(t) = x(t) * h(t)} \rightarrow \boxed{Y(j\omega) = X(j\omega)\cdot H(j\omega)}$

Sp: $\int_{-b}^{b} e^{-j\omega t} = 2b\,\mathrm{sinc}(b\omega)$

1. compute $X(j\omega)$

$X(j\omega) = \int x(t) e^{-j\omega t} dt =$

$= D\tau\,\mathrm{sinc}\left(\frac{\tau}{2}\omega\right) =$

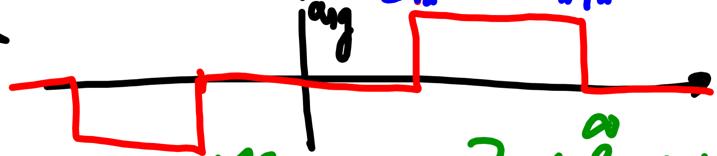
$= 1\cdot 10^{-6}\,\mathrm{sinc}(0{,}5\,10^{-6}\,\omega)$

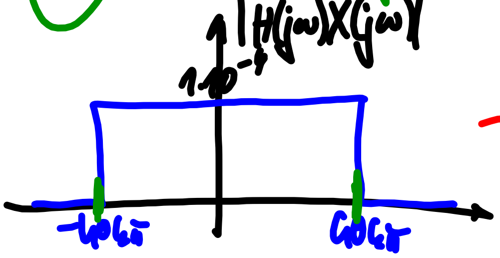
$0{,}5\cdot 10^{-6}\omega = \pi$

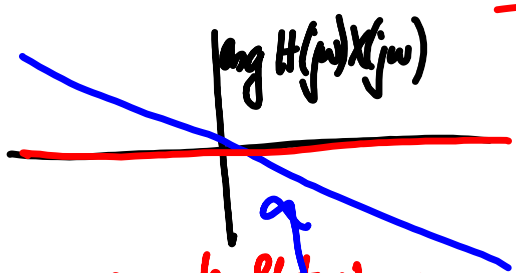
$\omega = 2\pi\cdot 10^{6}$

2. do the multiplication

3. convert out. spec. to signal







$y(t) = \mathcal{F}^{-1}\{\underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega) e^{+j\omega t} d\omega =$

$= \frac{1}{\cancel{2\pi}} 10^{-4}\, \cancel{2}\cdot\cancel{40k\pi}\,\mathrm{sinc}(40k\pi t) =$

a) consider flat phase

b) do it with the right phase $= 4\,\mathrm{sinc}(40k\pi t)$

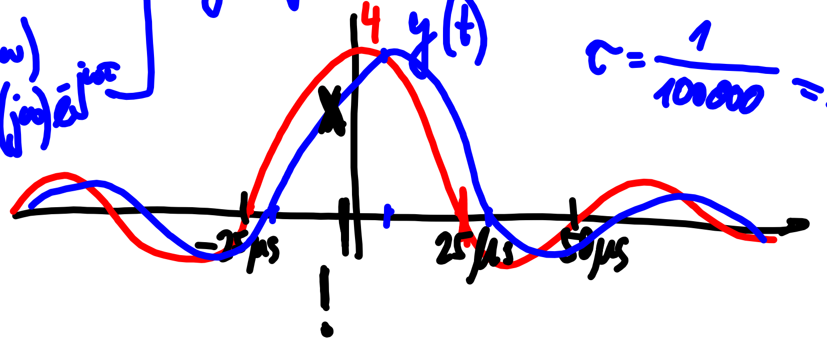
$40k\cancel{\pi} t = \cancel{\pi}$

$t = \frac{1}{40000} = 25\mu s$

$y(t) \quad Y(j\omega)$

$y(t-\tau) \quad Y(j\omega)e^{-j\omega\tau}$

$\tau = \frac{1}{100000} = 10\mu s$



Quest for frequency response

Theory

Differential equation

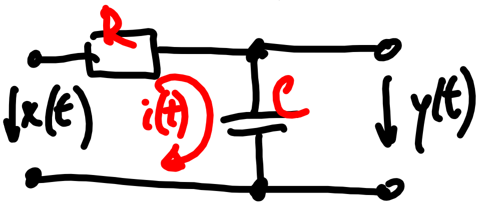
$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}}$$

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$a_1 = RC,\ a_0 = 1,\ b_0 = 1$

$\boxed{H(j\omega)}$???

Example



$$i(t) = \frac{x(t) - y(t)}{R}$$

$$i(t) = C\frac{dy(t)}{dt}$$

the same!

$$x(t) - y(t) = RC\frac{dy(t)}{dt}$$

$$\boxed{RC\frac{dy(t)}{dt} + y(t) = x(t)}$$

$\mathcal{L}$!

Laplace transform

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt$$

complex number

FT

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

$$RC\, Y(s) \cdot s + Y(s) = X(s)$$

"Laplace vocabulary"

$$x(t) \rightarrow X(s)$$

$$a\, x(t) \rightarrow a X(s)$$

$$\frac{dx(t)}{dt} \rightarrow X(s) s$$

$$\frac{d^k x(t)}{dt^k} \rightarrow X(s) s^k$$

why? Transfer function

$$H(s) = \frac{Y(s)}{X(s)}$$

$$\hookrightarrow Y(s)(RCs + 1) = X(s)$$

$$\frac{Y(s)}{X(s)} = \boxed{H(s) = \frac{1}{RCs + 1}}$$

$$\sum_{k=0}^{N} a_k Y(s) s^k = \sum_{k=0}^{M} b_k X(s) s^k$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum b_k s^k}{\sum a_k s^k}$$

fraction of polynomials!

We want freq. response !!!

$s \rightarrow j\omega$

$$H(j\omega) = \frac{\sum b_k (j\omega)^k}{\sum a_k (j\omega)^k}$$

$$H(j\omega) = \frac{1}{RCj\omega + 1}$$

freqs

mp. freqs

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum b_k s^k}{\sum a_k s^k} =$$

$$= const \cdot \frac{(s-n_1)(s-n_2) \cancel{\cdot \ldots \cdot (s-n_N)}}{(s-p_1)(s-p_2) \cancel{\cdot \ldots \cdot (s-p_N)}}$$

$x^2 - 2x + 1 = (x-1)(x-1)$

roots

$x^2 - 2x + 1 = 0$

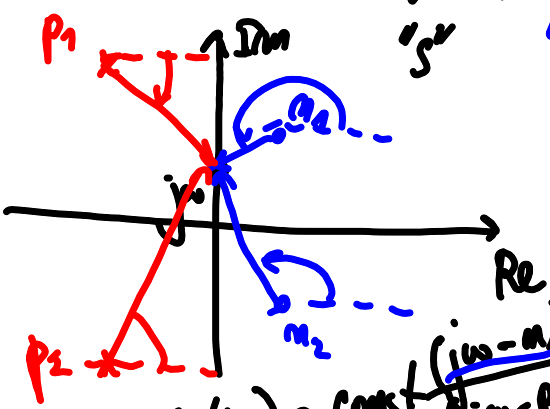
$x_{1/2} = \ldots$

np. roots

"s"

n_k - "zeros"

p_k - "poles"

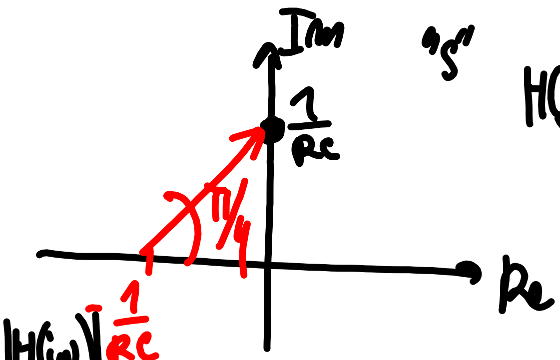


$$|H(j\omega)| = const \cdot \frac{\text{product of lengths of blue vectors}}{\text{product of lengths of red vectors}}$$

$$\text{ang } H(j\omega) = \text{sum of angles of blue vectors} - \text{sum of vectors of red vectors.}$$

$$H(j\omega) = const \cdot \frac{(j\omega - n_1)(j\omega - n_2)}{(j\omega - p_1)(j\omega - p_2)}$$

$$H(s)=\frac{1}{RCs+1}=\frac{1}{RC}\,\frac{1}{\left(s+\frac{1}{RC}\right)}=\frac{1}{RC}\,\frac{1}{\left(s-\left(-\frac{1}{RC}\right)\right)}$$



$H(j\omega)$

$\omega=0 \quad |H(j0)|=\cancel{\frac{1}{RC}}\cdot\frac{1}{\cancel{\frac{1}{RC}}}=1$

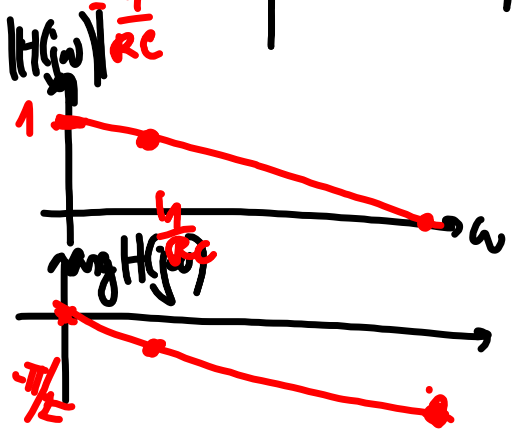
$\arg=-0=0$

$\omega=\infty \quad |H(j\infty)|=\frac{1}{RC}\cdot\frac{1}{\infty}$

$\arg=-\pi/2$

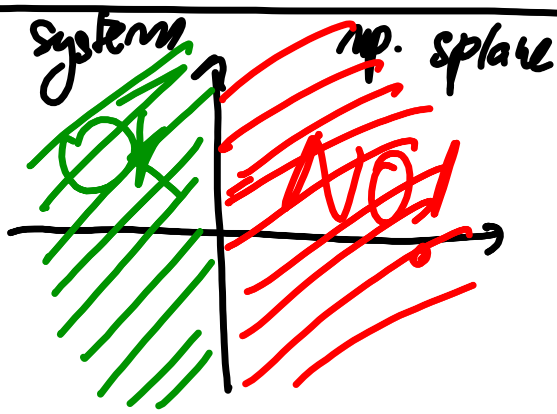
$\omega=\frac{1}{RC} \quad |H(j\tfrac{1}{RC})|=\cancel{\frac{1}{RC}}\cdot\frac{1}{\frac{\sqrt{2}}{\cancel{RC}}} \doteq 0{,}7$

$\arg=-\pi/4$



Stability of system

$Re\{p_k\}<0$



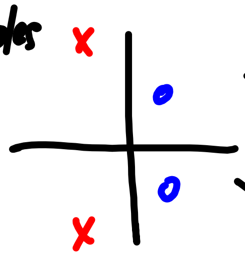
Cont. time systems summary

differential equation → impulse response $h(t)$ $\xrightarrow{F.T.}$ Frequency response $H(j\omega)$

differential equation $\xrightarrow{\text{Laplace transf.}}$ transfer function $H(s)$

transfer function $H(s)$ $\xrightarrow{s = j\omega}$ Frequency response $H(j\omega)$

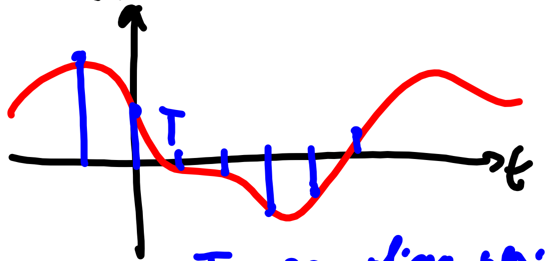
transfer function $H(s)$ $\xrightarrow{\text{factorization} \to \text{zeros/poles}}$



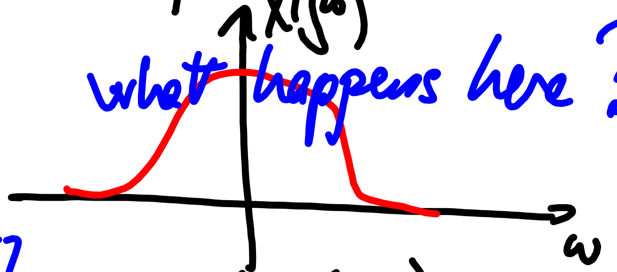
by hand "graphically" → Frequency response $H(j\omega)$

→ stability poles are on the left

SAMPLING.



Spectrum



what happens here?

T - sampling period [s]

$F = \frac{1}{T}$ sampling freq [Hz] — 8 kHz (telephone), 16 kHz (wide-band), 48 kHz Hi-Fi, 25 frames/s video

$\Omega = 2\pi F$ angular sampl. freq. [rad/s]

sampling signal s(t) — sampled signal $x_s(t)$

$$s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

$$x_s(t) = x(t)\, s(t)$$

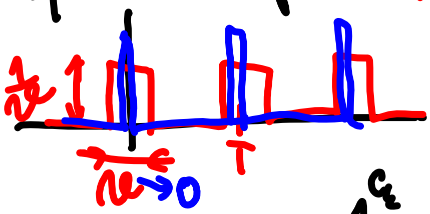
$$x_s(t) = \sum_{m=-\infty}^{\infty} x(mT)\, \delta(t - mT)$$

Spectrum?

$$X_s(j\omega) = X(j\omega) * S(j\omega) \;?$$

spectrum of periodic seq. of rectangular pulses.

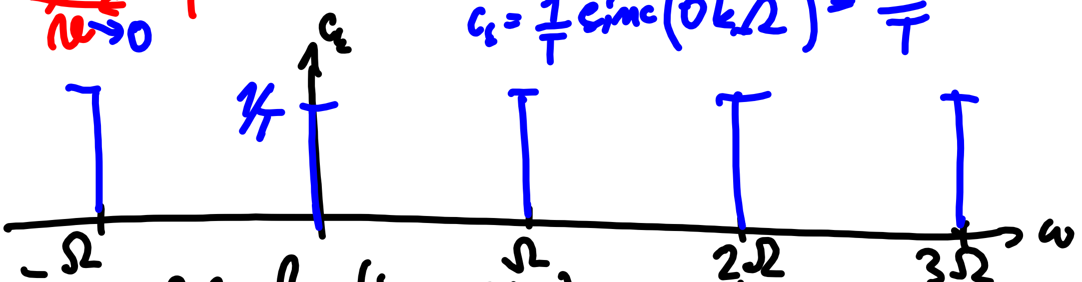
$D = \frac{1}{\vartheta}$



Fourier series:

$$c_k = \frac{1}{T} \cdot \mathrm{sinc}\left(\frac{\vartheta}{2} k\Omega\right)$$

$$c_k = \frac{1}{T}\, \mathrm{sinc}(0 k\Omega) = \frac{1}{T}$$



F.S. → spec. function: $S(j\omega)$ everywhere zero, except for $k\Omega$, Dirac pulses with area $2\pi c_k$