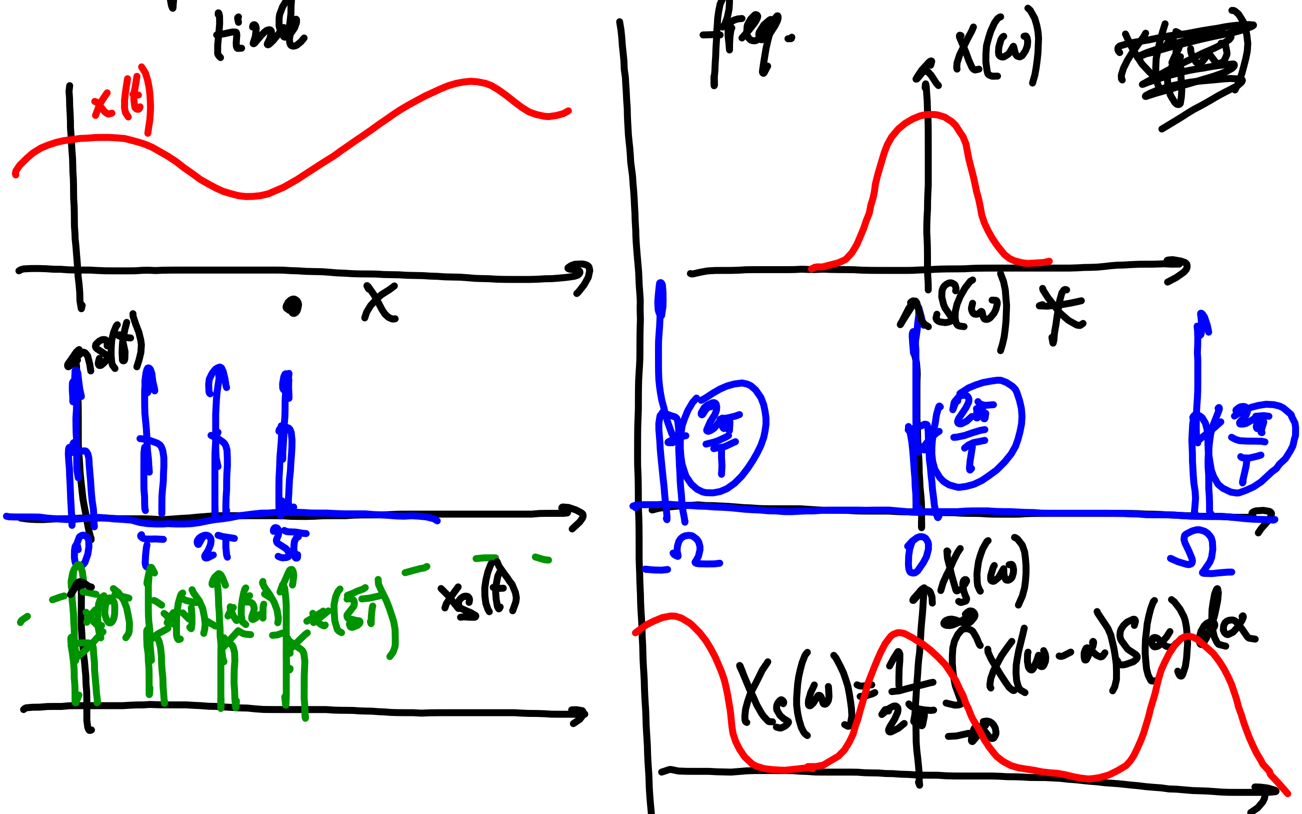
Sampling

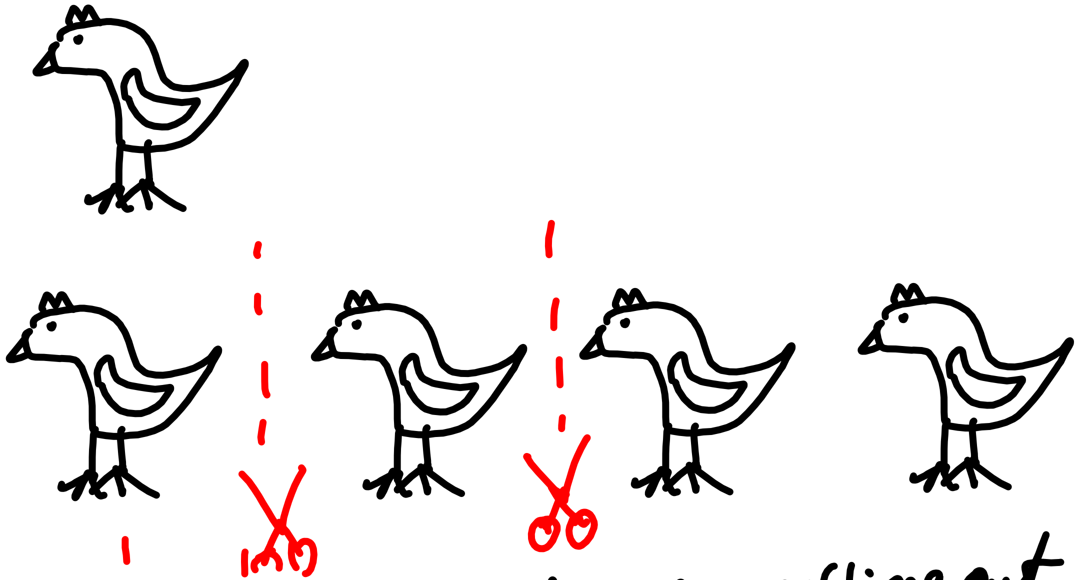


Condition for ideal sampling

$\boxed{2\omega_{max} < \Omega}$ $\Leftrightarrow$ $\boxed{2f_{max} < F_s}$

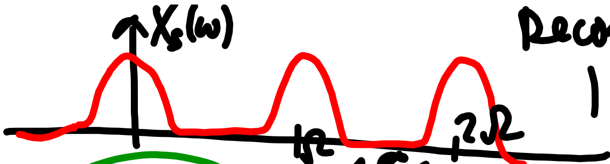
Shannon / Котельников / Nyquist / sampling theorem

if not fulfilled $\Rightarrow$ <u>aliasing</u>.



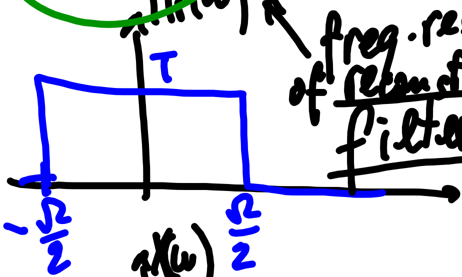
Anti-aliasing: cutting out everything $< -\frac{\Omega}{2}$ and $> \frac{\Omega}{2}$

Reconstruction in time

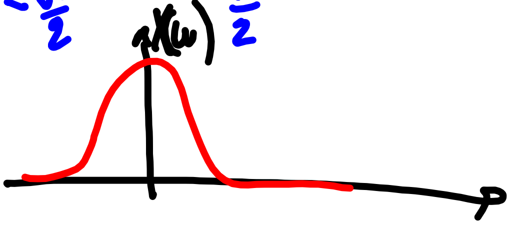


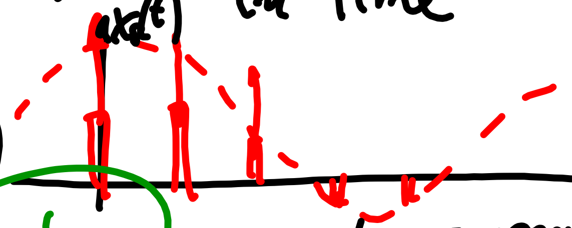
multiplication

$X_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\Omega)$



freq. responce of reconstruction filter



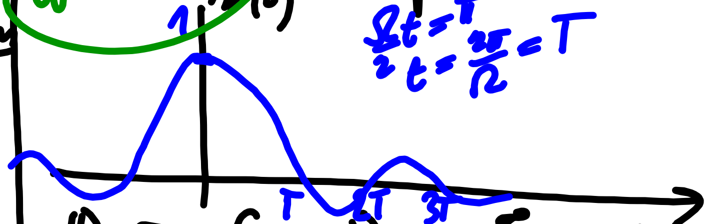


convolution

$h_r(t)$ - impulse responce

$\frac{\Omega}{2} t = \pi$

$t = \frac{2\pi}{\Omega} = T$



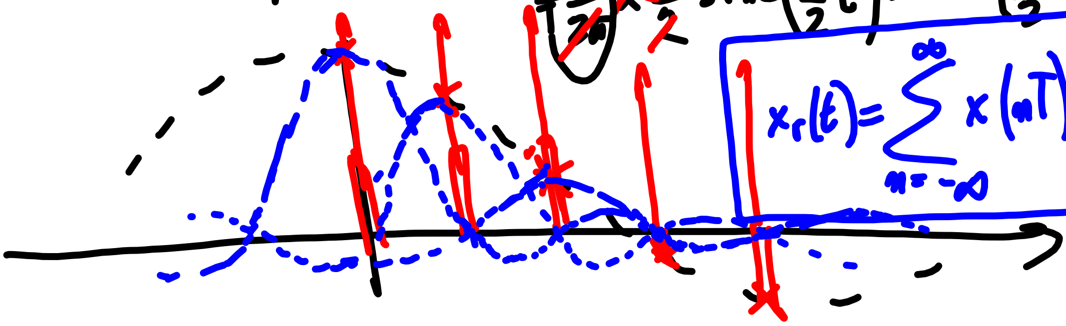
$h_r(t) = IFT\{H_r(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(\omega) e^{j\omega t} d\omega =$

$= \frac{1}{2\pi} T \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{j\omega t} d\omega$

$\int_{-b}^{b} e^{jxy} dy = 2b\,\mathrm{sinc}(bx)$

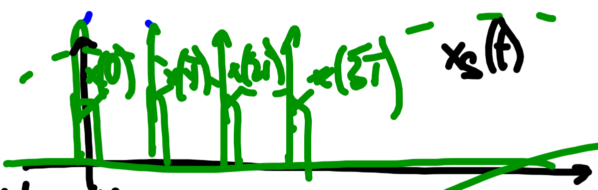
$= \frac{1}{2\pi} \times \frac{\Omega}{2}\,\mathrm{sinc}\left(\frac{\Omega}{2}t\right) = \mathrm{sin}\left(\frac{\Omega}{2}t\right)$

$\Omega = \frac{2\pi}{T}$



$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{\Omega}{2}(t-nT)\right)}$$

Times and frequencies



theory

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

practice

$x[n] \rightarrow$ C++, Python, Matlab, Java ....

n	0	1	2	3	....
	5	-2	3		

F_s kHz

$x(nT) \rightarrow x[n] \leftarrow$ discrete-time signal

true time [s]

[ ] discrete-time

$n = \frac{nT}{T}$ ← normalization of time

T - sampling period

$\boxed{\text{sound}(x, F_s)}$

$$f' = \frac{f\,[Hz]}{F_s\,[Hz]} \qquad \frac{F_s}{F_s}$$

normalized frequency

$$\omega' = \frac{\omega\,[rad/s]}{F_s\,[Hz]} \qquad \frac{rad/s}{1/s} \qquad \frac{2\pi F_s}{F_s}$$

normalized angular frequency

0 ——— F_s ——→ f

0 ——— $\Omega = 2\pi F_s$ ——→ ω

0 ——— 1 ——→ f' normalized freq.

0 ——— 2π ——→ ω' normalized angular freq.

Example: tone on 440 Hz with $F_s = 8000$ Hz

$$T = \frac{1}{8000}$$

$$x(t) = \cos(440 \cdot 2\pi \cdot t)$$

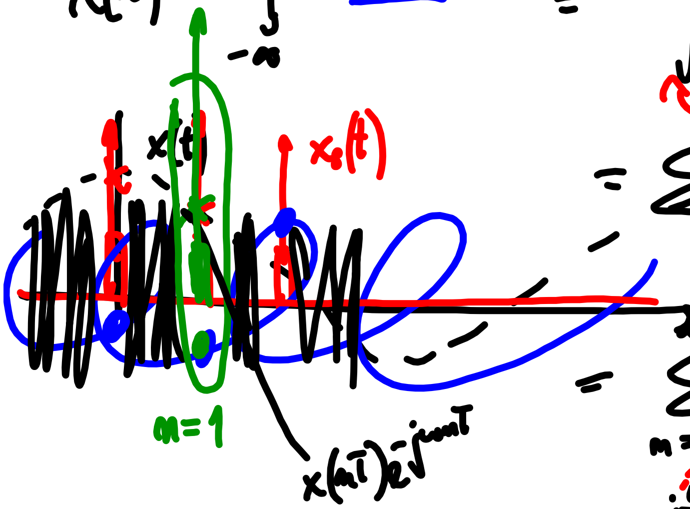
$$x(nT) = \cos(440 \cdot 2\pi \cdot nT) = \cos\Big(\underbrace{2\pi \underbrace{\frac{440}{8000}}_{f'}}_{\omega'} n\Big)$$

Spectral analysis of discrete signals.

$x[n] \longrightarrow$ spectrum $X(\ldots)$

$$X(\omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\omega t}\, dt = \int \sum x(nT)\,\delta(t-nT)\, e^{-j\omega nT}\, dt =$$

$$= \sum \underbrace{\cancel{\int} x(nT)\,\cancel{\delta(t-nT)}\, e^{-j\omega nT}\, dt} =$$



$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j\omega nT}$$

get rid of time!

$$= \sum x[n]\, e^{-j\frac{\omega}{F_s} n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega' n}$$

spectrum is periodic

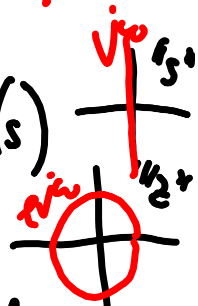
Discrete-time Fourier transform

$$\boxed{\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

normalized ang. freq.

Continuous time: F.T. $X(j\omega) \approx$ Laplace transform $X(s)$

DTFT $\tilde{X}(e^{j\omega}) \approx$ z-transform $X(z)$



@ FIT: we don't like that $\tilde{X}(e^{j\omega})$ is a function of ω

DTFT $\xrightarrow{\text{sampling in freq.}}$ DFT | Inverse DTFT

output = maybe constant · sum. op. · input $e^{\pm j\,\text{time freq.}}$

$$\boxed{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{X}(e^{j\omega})\, e^{+j\omega n}}$$

periodicity in norm. ang. frequencies is $2\pi = \frac{2\pi \cancel{F_s}}{\cancel{F_s}}$

Periodicity of DTFT ?

$$\tilde{X}(e^{j\omega}) = \boxed{\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

should be periodic with $k\,2\pi$

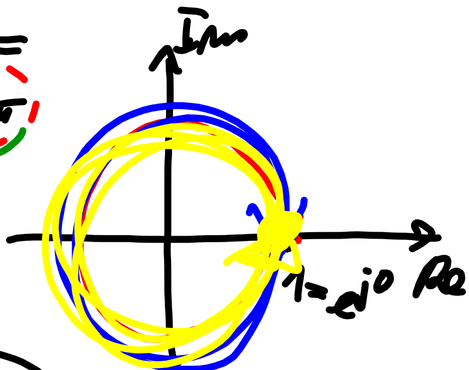
$$e^{a+b} = e^{a} e^{b}$$

$$\tilde{X}(e^{j(\omega+2k\pi)}) = \sum x[n]\, e^{-j(\omega+k2\pi)n} =$$

$$= \sum x[n]\, e^{-j\omega n} \underbrace{e^{-jkn2\pi}}_{1}$$

k - integer, n - integer } kn - integer

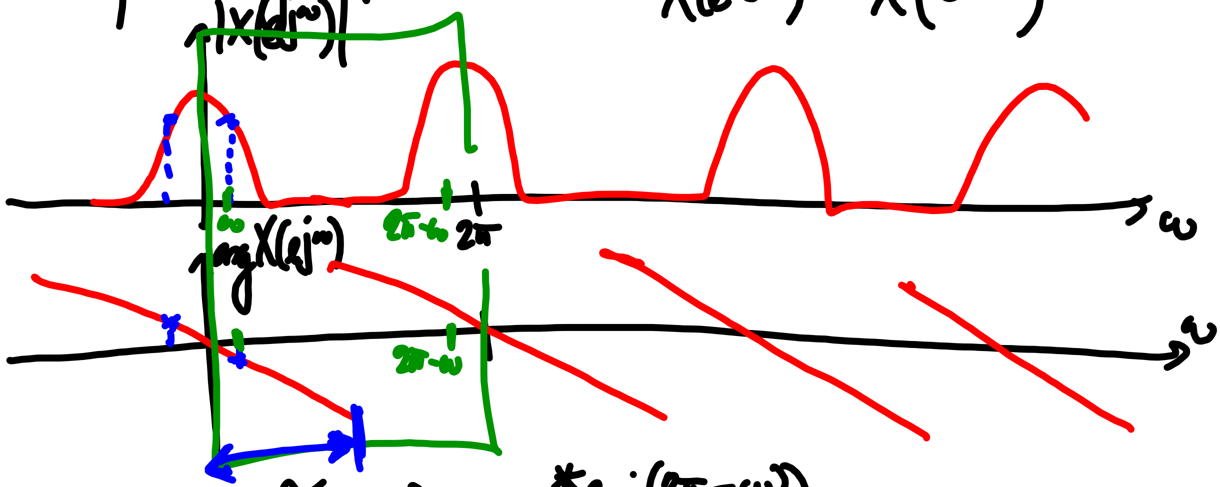
$$= \sum x[n]\, e^{-j\omega n} = \tilde{X}(e^{j\omega})$$



DTFT is periodic

Symmetry of DTFT

$$X(e^{-j\omega}) = X^{*}(e^{j\omega})$$



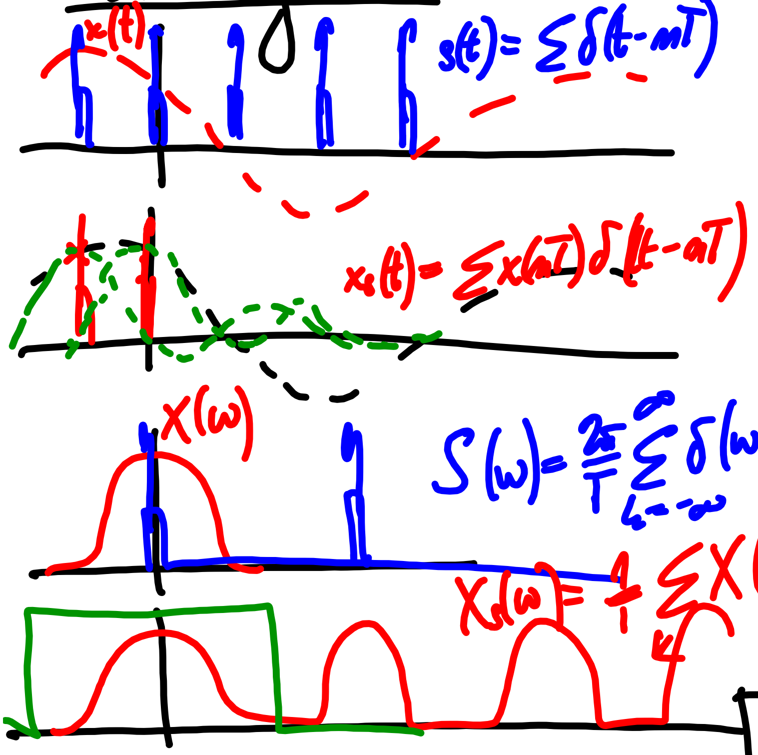
$$\tilde{X}(e^{j\omega}) = X^{*}(e^{j(2\pi-\omega)})$$

for $x[n] \in \mathbb{R}$, ok to know only $\tilde{X}(e^{j\omega})$ for $\omega \in [0, \pi]$, the rest can be obtained thru periodicity and symmetry.

Periodicity

norm. ang. freq.	ω	2π
norm. f.	f	1
ang. freq.	ω	$2\pi F_s$
freq.	f	F_s

Summary



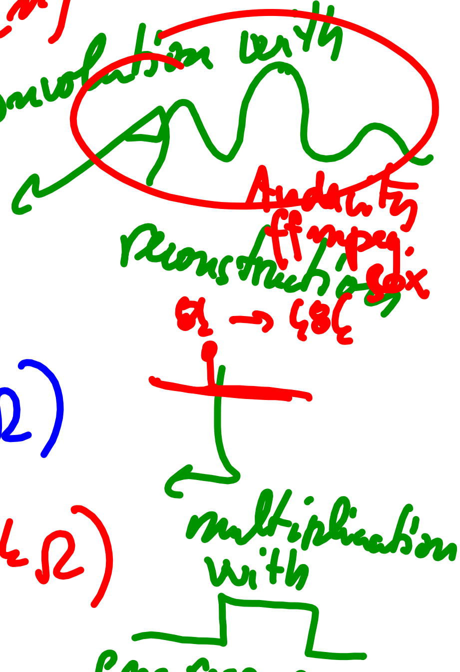
$s(t) = \sum \delta(t - nT)$

$x_s(t) = \sum x(nT)\,\delta(t - nT)$

$S(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\Omega)$

$X_s(\omega) = \frac{1}{T} \sum_k X(\omega - k\Omega)$

x[n]

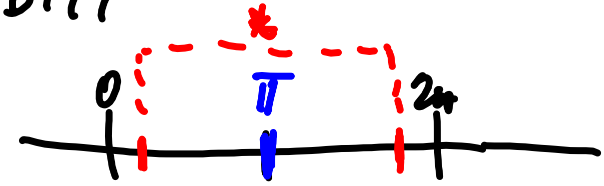


$2\omega_{max} < \Omega_s$

x[n] - freq. anal. DTFT

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{X}(e^{j\omega})\, e^{+j\omega n}\, d\omega$$



1. periodic with 2π
2. symmetrical $\tilde{X}(e^{j\omega}) = \tilde{X}^*(e^{-j\omega})$

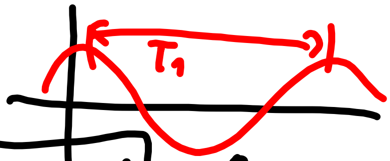
$$\tilde{X}(e^{j\omega}) = \tilde{X}^*(e^{j(2\pi - \omega)})$$

Playing with discrete-time signals...

Cosine

cont. time

$x(t) = C_1 \cos(\omega_1 t + \varphi_1)$



$\boxed{T_1 = \frac{1}{f_1}} = \frac{2\pi}{\omega_1}$

2 different T_1's $\rightarrow$ 2 different f_1's

discrete

$x[n] = C_1 \underline{\cos}(\omega_1 n + \varphi_1)$ ← normalized ang. freq.

$\cancel{N_1 = \frac{2\pi}{\omega_1}}$ because i need N_1 integer!

Condition of periodicity:

$\omega_1(n + N_1) - \omega_1 n = k\, 2\pi$

$\cancel{\omega_1 n} + \omega_1 N_1 - \cancel{\omega_1 n} = k\, 2\pi$

$N_1 = k \frac{2\pi}{\omega_1}$

1. compute $\frac{2\pi}{\omega_1}$
2. tune k so that N_1 is integer!

$\omega = \frac{\pi}{8}$ $\quad N_1 = k\frac{2\pi}{\frac{\pi}{8}} = k\,16$ ☺ $\quad k = 1$ $\quad N_1 = 16$ ✓

$\omega = \frac{8\pi}{31}$ $\quad N_1 = k\frac{2\pi}{\frac{8\pi}{31}} = k\frac{31}{4}$ 😐 $\quad k = 4$ $\quad N_1 = 31$ ✓

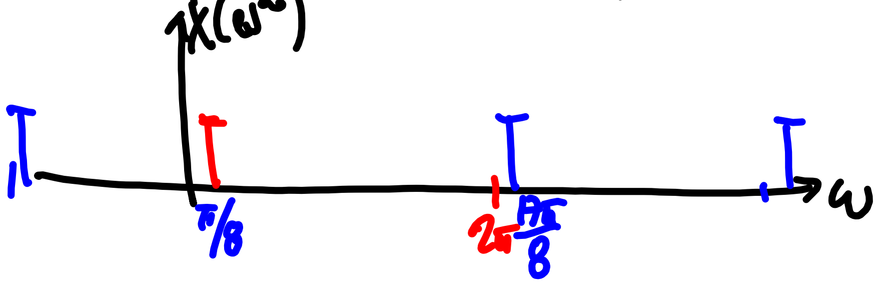
$\omega = \frac{1}{6}$ $\quad N_1 = k\frac{2\pi}{\frac{1}{6}} = k\,12\pi$ ☹ not periodic

$\omega = \frac{17\pi}{8}$ $\quad N_1 = k\frac{2\pi}{\frac{17\pi}{8}} = k\frac{16}{17}$ $\quad k = 17$ $\quad N_1 = 16$

S/N/k/S
$2\omega_{max} < \Omega$
normalized ang. freq.

$2\omega_{max} < 2\pi$

WHY? $x[n] = \cos\left(\frac{17\pi}{8}n\right) = \cos\left(\left(2\pi + \frac{\pi}{8}\right)n\right) = \cos\left(\cancel{2\pi n} + \frac{\pi}{8}n\right) = \cos\left(\frac{\pi}{8}n\right)$



Basic ops. with discrete signals

n	-3	-2	-1	0	1	2	3	4	5	...
$x[n]$	-1	6	8	-2	5	3	1	4	-7	

I want to have a signal of length $N=4$

$R_N[n]$: 1 1 1 1

n		0	1	2	3	
$y[n]$		-2	5	3	1	

$y[n] = R_N[n]\, x[n]$

window

$$R_N[n] = \begin{cases} 1 & \text{for } n \in [0, N-1] \\ 0 & \text{elsewhere.} \end{cases}$$

Periodization

$p[n] = y[\mathrm{mod}_4(n)]$

n	...				0	1	2	3	4	5	6	7	.
$p[n]$	-2	5	3	1	-2	5	3	1	-2	5	3	1	-2
$\mathrm{mod}_4 n$	0	1	2	3	0	1	2	3	0	1	2	3	0

Periodization with shift

Circular shift

TBD next time ...