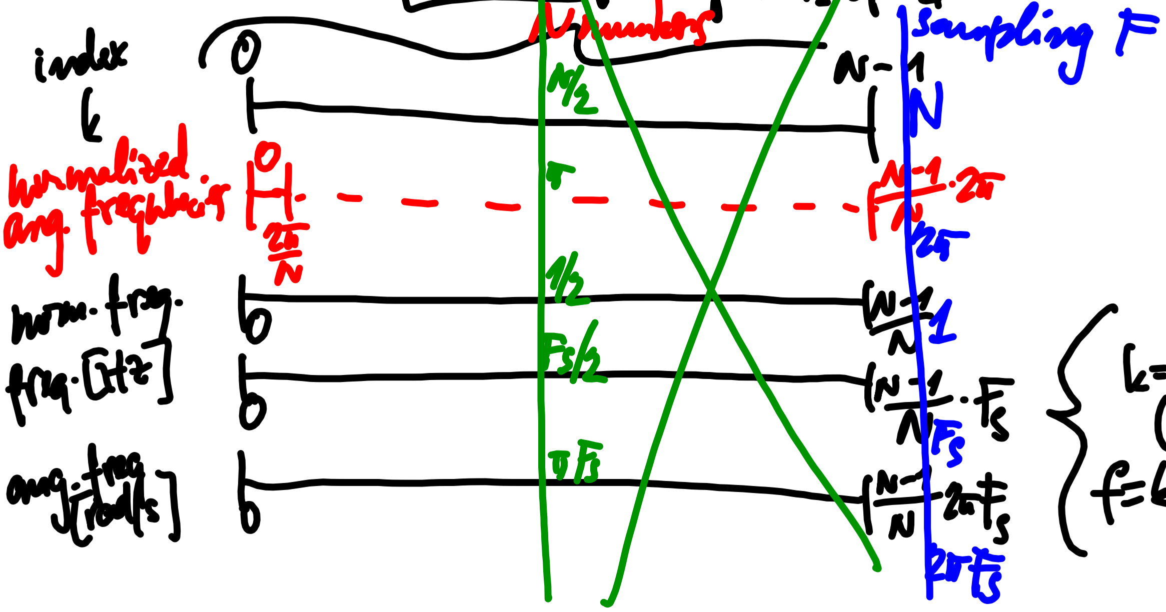
DFT: N samples ($\mathbb{R}$) in time $\longrightarrow$ N samples ($\mathbb{C}$) in frequency

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\boxed{\frac{2\pi}{N}}kn}$$

IDFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\boxed{\frac{2\pi}{N}}kn}$$

fft for N powers of 2



k=np.arange(0,N-1)

f=k/N*Fs

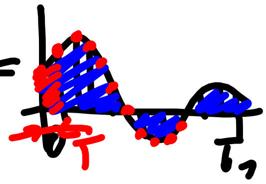
Using DFT for analysis of cont. time periodic signals.

I can sample with sampl. per. T
$NT = T_1$

analog world: fund. freq: $\omega_1 = \frac{2\pi}{T_1}$

spectrum = coefficients of Fourier series c_k "sitting" at $k\omega_1$

$$c_k = \frac{1}{T_1}\int_0^{T_1} x(t)\, e^{-jk\omega_1 t}\, dt = \frac{1}{N\cancel{T}} \sum x[n]\, e^{-jk\frac{2\pi}{N\cancel{T}} n\cancel{T}}\, \cancel{T} =$$



$\sum x[n]\cdot T$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}X[k]$$

CONDITIONS: 1) $f_{max} < \frac{F_s}{2}$ ≈ non-zero coeff. must have max index of $k = \frac{N}{2}$

2) I can compute only coefficients c_k for $k \in [-\frac{N}{2}, \dots \frac{N}{2}-1]$

positive k's: $c_k = \frac{1}{N}X[k]$

negative k's: $c_k = \frac{1}{N}X[N-k]$

3) $T_1 = N\cdot T$ BAD! Hard to fulfill! I don't know T_1 must use some guess!

Matlab example $x[n] = 3\cos\left(\frac{2\pi}{N}n - \frac{\pi}{2}\right)$

theoretical $c_1 = 1{,}5\, e^{-j\pi/2}$ $c_{-1} = 1{,}5\, e^{+j\pi/2}$

DFT: OK: $X[1] = 256\cdot 1{,}5$

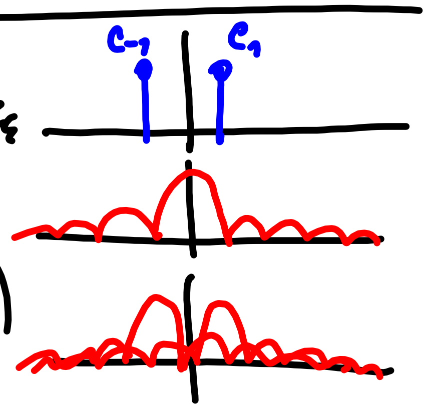
$x[n] = 3\cos\left(\frac{2\pi\cdot 1{,}1}{N}n - \pi/2\right) \rightarrow$ not OK.

WHY???

x(t) → X(jω) or c_k

w(t) → W(jω)

x(t)·w(t) → X(jω) * W(jω)



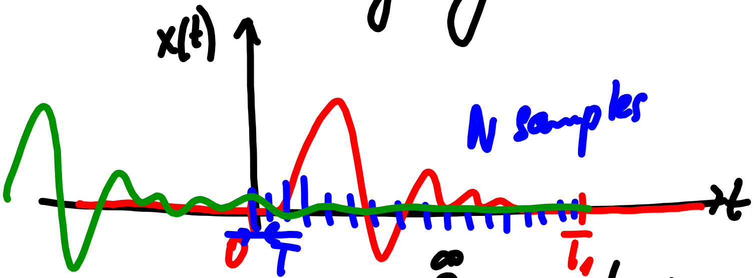
Increasing the freq. resolution of DFT?

DFT: eats N samples → produce ~~N~~ samples!
more than N

zero padding bourrage des zéros

$N = 256 \rightarrow N = 256$ values in spectrum
visualize just 129 !
use zero padding to have a nicer output
Remember: no better information !!!!

Use of DFT for analysis of non-periodic analog signals



I created an interval $T_1 = NT$, must compute with freq. $k\omega_1$

$\omega_1 = \frac{2\pi}{T_1}$

analog: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \sum x[n] e^{-j k \frac{2\pi}{\cancel{NT}} \cdot n\cancel{T}} \cdot T =$

$= T \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$

$$X\left(jk\frac{2\pi}{NT}\right) = T \cdot X[k]$$

CONDITIONS:

1) $f_{max} < \frac{F_s}{2}$
2) I obtain $X(j\omega)$ only for interval freq. [-1/2 sampling freq. till 1/2 sampling freq.
3) I obtain only samples on freq. axis
4) The signal must be limited in time from 0 till T_1

Green signal:

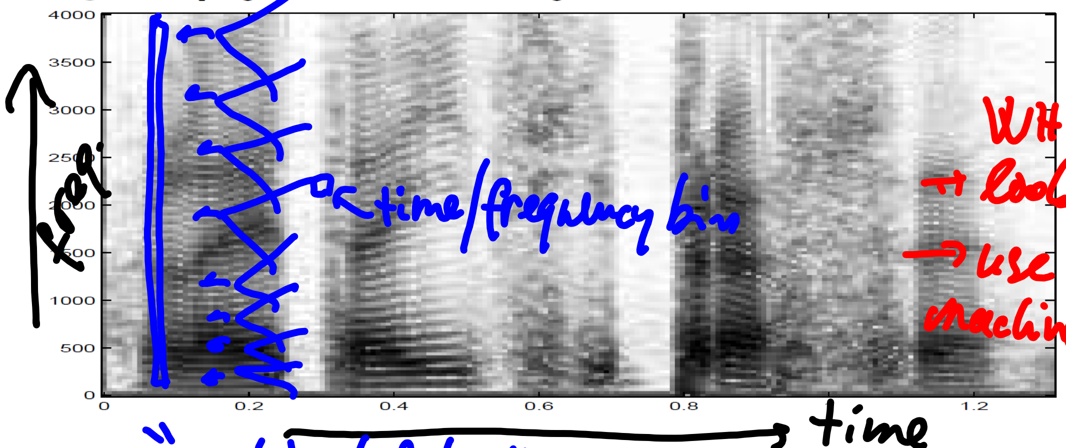
1. $y(t) \stackrel{!}{=} x(t-\tau)$ 2. Do ↑ 3. Correct: $X(j\omega) = Y(j\omega) \cdot e^{+j\omega\tau}$

Spectrogram:

"frames"

→ DFT → visualize $0 \dots \frac{F_s}{2}$

long-term: `specgram(s,256,8000,hamming(256),200);`



"Filterbank features"

WHY?
→ look at it
→ use for machine learning

Summary of DFT

- takes N samples in time ($\mathbb{R}$)

→ outputs N samples in freq. ($\mathbb{C}$)

- for $x[n] \in \mathbb{R}$ it's symmetrical

$$X[k] = X^*[N-k]$$

that's why we store / visualize / analyze just $X[0] \dots X[N/2]$

- CAN use DFT to compute e.g. on $X(j\omega)$ but...!!!

- circular shift of signal $x[n] \longrightarrow X[k]$

$$x[\mathrm{mod}_N(n-m)] \longrightarrow X[k] \cdot e^{-jk\frac{2\pi}{N}m}$$

- circular convolution $x_1[n] \, (N) \, x_2[n] \longrightarrow X_1[k] X_2[k]$

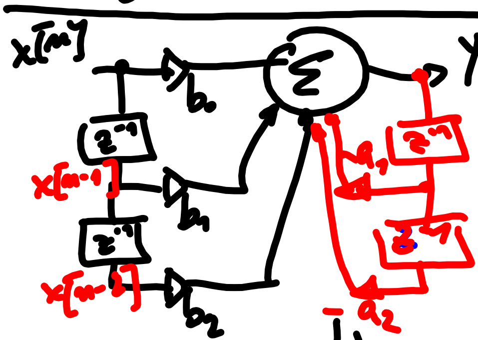
Digital filters

≈ discrete-time systems, discrete systems, numerical filters, filters.

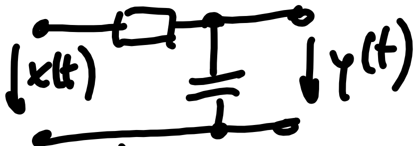
→ frequency response ?

→ stability ?

DIGITAL | ANALOG



Scheme:



Difference equation

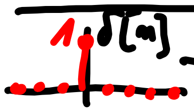
$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Differential equation

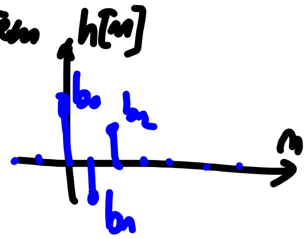
$$b_0 x(t) + b_1 \frac{dx(t)}{dt} + \dots + b_Q \frac{d^Q x(t)}{dt^Q} = a_0 y(t) + a_1 \frac{dy(t)}{dt} + \dots + a_P \frac{d^P y(t)}{dt^P}$$

by hand   aut. analyzers.

Impulse response

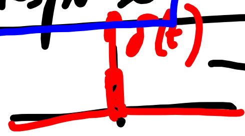


→ system

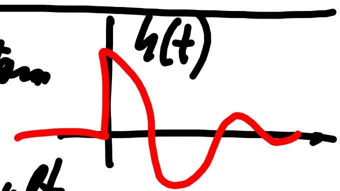


easy to generate ???

difficult for recursive/IIR filter



→ system



theoretical/difficult

Frequency response

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

→ implemented by DFT

$$H(j\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

(complex) frequency response

DIGITAL

$$y[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_Q x[n-Q] - a_1 y[n-1] - \dots - a_P y[n-P]$$

z-transform

$$x[n] \longrightarrow X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

"z"

dictionary:

$$x[n] \longrightarrow X(z)$$
$$a\,x[n] \longrightarrow a X(z)$$
$$x[n-1] \longrightarrow X(z) z^{-1}$$
$$x[n-k] \longrightarrow X(z) z^{-k}$$

Do z transform of difference equation:

$$Y(z) = b_0 X(z) + b_1 X(z) z^{-1} + \dots + b_Q X(z) z^{-Q} - a_1 Y(z) z^{-1} - \dots - a_P Y(z) z^{-P}$$

I want transfer function

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z)\left[1 + a_1 z^{-1} + \dots + a_P z^{-P}\right] = X(z)\left[b_0 + b_1 z^{-1} + \dots + b_Q z^{-Q}\right]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots b_Q z^{-Q}}{1 + a_1 z^{-1} + \dots a_P z^{-P}}$$

ANALOG

$$b_0 x(t) + \dots b_Q \frac{d^Q x(t)}{dt^Q} = a_0 y(t) + \dots + a_P \frac{d^P y(t)}{dt^P}$$

Laplace transform

$$x(t) \longrightarrow X(s)$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

dictionary of $\mathcal{L}$.t.:

$$x(t) \longrightarrow X(s)$$
$$a\,x(t) \longrightarrow a X(s)$$
$$\frac{d^k x(t)}{dt^k} \longrightarrow X(s) s^k$$

Laplacizing the differential eq.

$$X(s)(b_0 + b_1 s + \dots + b_Q s^Q) = Y(s)(a_0 + a_1 s + \dots + a_P s^P)$$

We want transfer function (system)!

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s + \dots + b_Q s^Q}{a_0 + a_1 s + \dots a_P s^P}$$

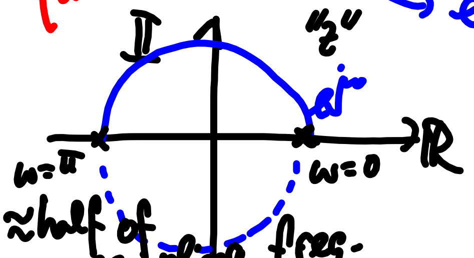
fraction of two polynomials!

Why did we derive transfer functions?

→ we want freq. response
— " — check of stability!

DIGITAL		ANALOG	
z-transf	DTFT	Laplace vs.	Fourier transf.
$H(z)=\sum x[n]z^{-n}$	$H(e^{j\omega})=\sum x[n]e^{-j\omega n}$	$H(s)=\int_{-\infty} x(t)e^{-st}dt$	$H(j\omega)=\int_{-\infty} x(t)e^{-j\omega t}dt$
have transfer function	want freq. response	have transf. function	want freq. response!

Yes! $z \rightarrow e^{j\omega}$



≈half of sampling freq.

$$H(z)=\frac{b_0+b_1z^{-1}+\dots+b_Qz^{-Q}}{1+a_1z^{-1}+\dots+a_Pz^{-P}}$$

$z \rightarrow e^{j\omega}$

np. freqz(b, a, N, 'whole')

N: number of points from 0 to Fs/2

"s" Yes! $s \rightarrow j\omega$

$$H(s)=\frac{b_0+b_1s+\dots+b_Qs^Q}{a_0+a_1s+\dots+b_Ps^P}$$

$s \rightarrow j\omega$

freqs(b, a, range of ω's?)

How usually seen:

$$y[n]=\sum_{k=0}^{Q}b_kx[n-k]-\sum_{k=1}^{P}a_ky[n-k]$$

transf.: $$H(z)=\frac{\sum_{k=0}^{Q}b_kz^{-k}}{1+\sum_{k=1}^{P}a_kz^{-k}}$$

freq.ch.: $$H(e^{j\omega})=\frac{\sum b_k(e^{j\omega})^{-k}}{1+\sum a_k(e^{j\omega})^{-k}}$$

FACTORIZATION OF TRANSFER FUNCTION

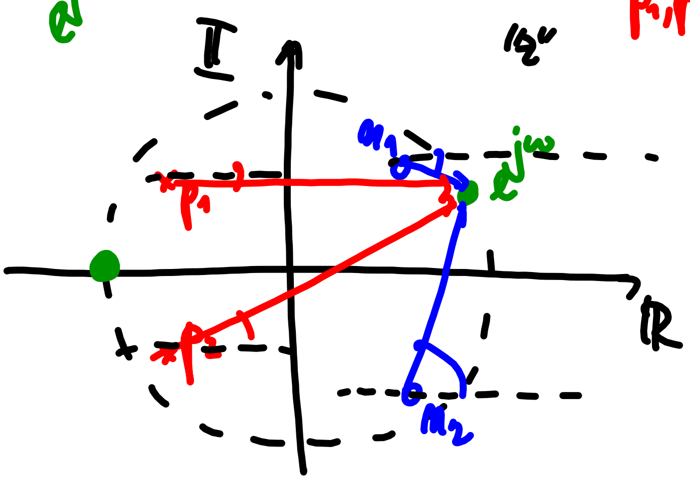
$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}\cdot\frac{z^2}{z^2}=\frac{b_0z^2+b_1z+b_2}{z^2+a_1z+a_2}$$

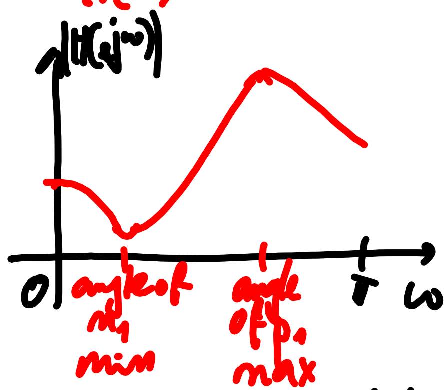
factorize 'em!

$$=\frac{b_0(\cancel{z}-n_1)(\cancel{z}-n_2)}{(\cancel{z}-p_1)(\cancel{z}-p_2)}\qquad e^{j\omega}$$

n_1, n_2 - roots of numerator, zeros $H(z)=0$

p_1, p_2 - roots of denominator. pôles $H(z)=\infty$





Why? (1) Frequency response: computing $|H(e^{j\omega})|$ and $\arg H(e^{j\omega})$ imagine brackets as vectors in "z" plane!

$$|H(e^{j\omega})|=b_0\frac{\text{product of lengths of blue vectors}}{\text{product of lengths of red vectors}}.$$

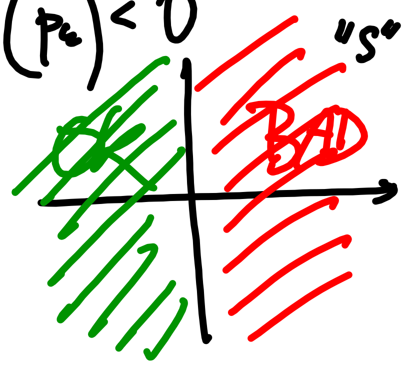
$$\arg H(e^{j\omega}) = \text{sum of angles of blue vectors} - \text{sum of angles of red vectors}.$$

STABILITY

check, that all pôles are inside unit circle $\boxed{|p_i|<1}$

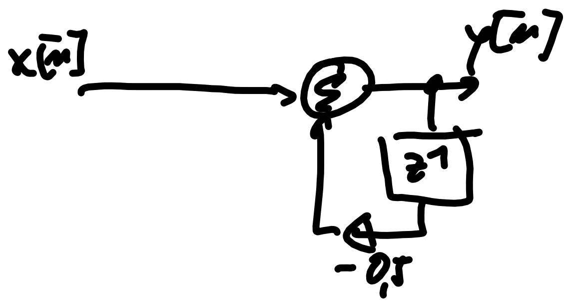
ANALOG

$\text{Re}(p_i)<0$



Example:

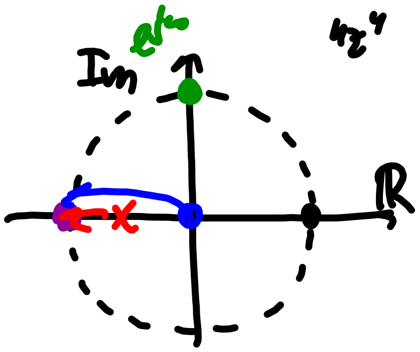
$y[n] = x[n] - 0{,}5\, y[n-1]$



$\mathcal{Z}$: $Y(z) = X(z) - 0{,}5\, Y(z) z^{-1}$

$\underline{Y(z)(1+0{,}5z^{-1})} = \underline{X(z)}$

$$H(z) = \frac{1}{1+0{,}5z^{-1}} \cdot \frac{z}{z} = \frac{z}{z+0{,}5} = \frac{z-0}{z-(-0{,}5)}$$



$\omega = 0$: $|H| = \frac{1}{1{,}5} = 0{,}66$

$\text{ang}\, H = 0 - 0 = 0$

$\omega = \pi/2$: $|H| = \frac{1}{\sqrt{0{,}5^2 + 1^2}} = 0{,}89$

$\text{ang}\, H = \pi/2 - \pi/4 = \frac{\pi}{4}$

$\omega = \pi$: $|H| = \frac{1}{0{,}5} = 2$

$\text{ang}\, H = \pi - \pi = 0$