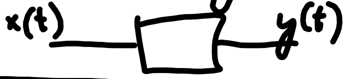
Cont.-time systems



Freq. chars?

Theory

$$b_0 x(t) + b_1 \frac{dx(t)}{dt} + \dots + b_q \frac{d^q x(t)}{dt^q} = a_0 y(t) + a_1 \frac{dy(t)}{dt} + \dots + a_p \frac{d^p y(t)}{dt^p}$$

$$\sum_{k=0}^{q} b_k \frac{d^k x(t)}{dt^k} = \sum_{k=0}^{p} a_k \frac{d^k y(t)}{dt^k}$$

differential equation

↳ get rid of it A.S.A.P.

Laplace transform

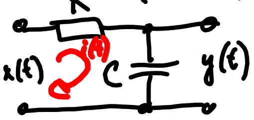
F.T.: $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$

L.T.: $H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$

$s = j\omega$

"s" is a complex variable

Example



$i(t) = \frac{x(t) - y(t)}{R}$

$i(t) = C \frac{dy(t)}{dt}$

$\frac{x(t) - y(t)}{R} = C \frac{dy(t)}{dt}$

$x(t) - y(t) = RC \frac{dy(t)}{dt}$

$x(t) = y(t) + RC \frac{dy(t)}{dt}$ differential equation

$b_0 = 1$

$a_0 = 1 \quad a_1 = RC$

↓ L.T.

"Dictionary" of L.T.:

$x(t) \to X(s)$

$a\,x(t) \to a\,X(s)$

$\frac{dx(t)}{dt} \to X(s) \cdot s$

$\frac{d^k x(t)}{dt^k} \to X(s) \cdot s^k$

$X(s) = Y(s) + RC\,Y(s)\,s$

$X(s) = Y(s)[1 + RCs]$

$b_0 X(s) + b_1 X(s) s + \dots + b_q X(s) s^q = a_0 Y(s) + a_1 Y(s) \cdot s + \dots + a_p Y(s) s^p$

$X(s)[b_0 + b_1 s + \dots + b_q s^q] = Y(s)[a_0 + a_1 s + \dots + a_p s^p]$

polynomials

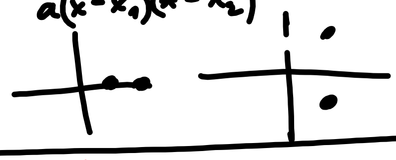
Transfer function (system function)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s + \dots + b_q s^q}{a_0 + a_1 s + \dots + a_p s^p} = \frac{\sum_{k=0}^{q} b_k s^k}{\sum_{k=0}^{p} a_k s^k}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1 + RCs}$$

$ax^2 + bx + c = 0$

$a(x - x_1)(x - x_2)$



Getting to $H(j\omega)$

$s \to j\omega$

$$H(j\omega) = \frac{\sum_{k=0}^{q} b_k (j\omega)^k}{\sum_{k=0}^{p} a_k (j\omega)^k}$$

$$H(j\omega) = \frac{1}{1 + RCj\omega}$$

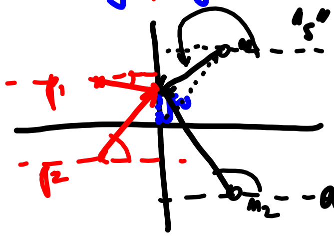
Polynomials rewritten with roots

$$H(j\omega) = \frac{b_0 (j\omega - m_1)(j\omega - m_2)\dots(j\omega - m_q)}{a_0 (j\omega - p_1)(j\omega - p_2)\dots(j\omega - p_p)}$$

m_k - nulls

p_k - pôles

$\arg\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

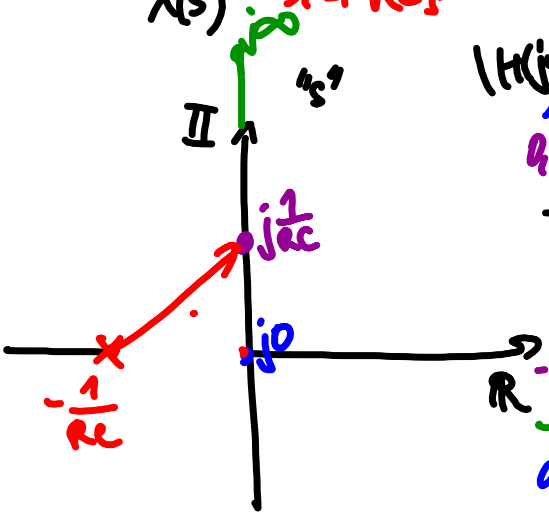


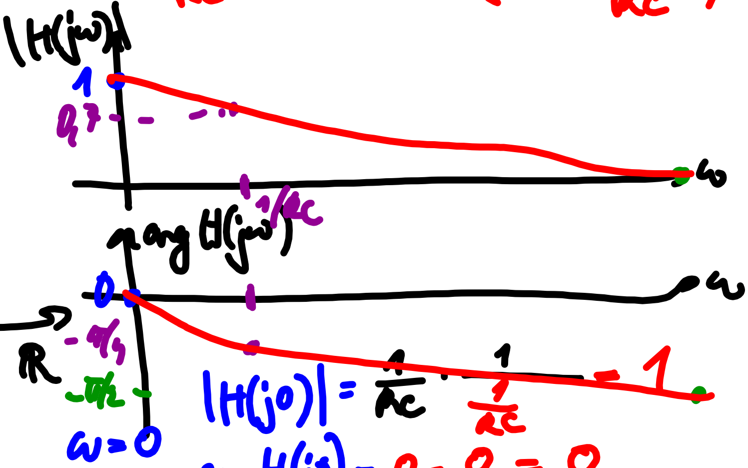
for a given ω:

$$|H(j\omega)| = \frac{b_0}{a_0} \cdot \frac{\text{product of lengths of black vectors}}{\text{product of lengths of red vectors}}$$

$\arg H(j\omega) =$ sum of angles of black vectors − sum of angles of red vectors

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{1+RCs}=\frac{1}{RC\left(\frac{1}{RC}+s\right)}=\frac{1}{RC\left(s-\left(-\frac{1}{RC}\right)\right)}$$





$|H(j0)| = \frac{1}{RC}\cdot\frac{1}{\frac{1}{RC}} = 1$

$\arg H(j0) = 0-0=0$

$\omega=\infty$ $|H(j\infty)| = \frac{1}{RC}\cdot\frac{1}{\infty} = 0$

$\arg H(j\infty) = 0-\pi/2 = -\pi/2$

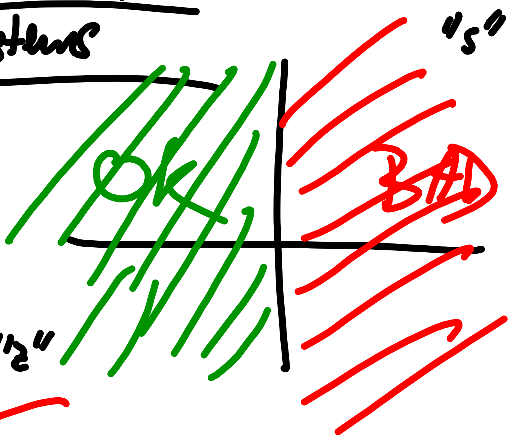
$\omega=\frac{1}{RC}$ $\left|H\left(j\frac{1}{RC}\right)\right| = \frac{1}{RC}\cdot\frac{1}{\sqrt{2}\,\frac{1}{RC}} = 0{,}7$

$\arg H\left(j\frac{1}{RC}\right) = 0-\frac{\pi}{4} = -\pi/4$

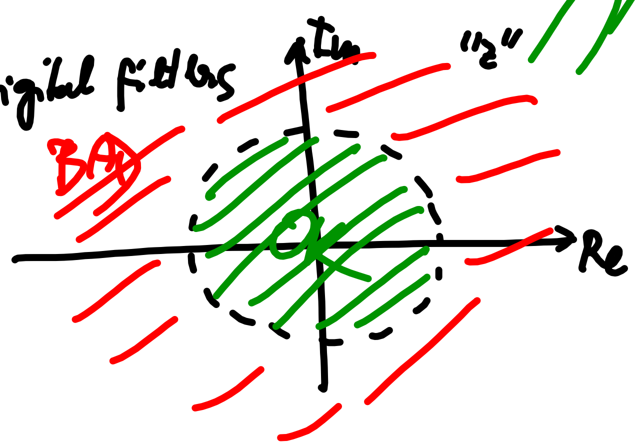
Reason for having $j\omega$ in the arg. of F.T. ≈ Laplace t.

Stability of systems

$\mathrm{Re}(p_i) < 0$



Digital filters



System - want frequency response

scheme → differential equation

← impulse response h(t) → Laplace transform

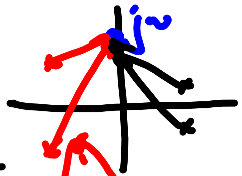
impulse response h(t) → F.T. → H(jω)

Laplace transform → transfer function H(s)

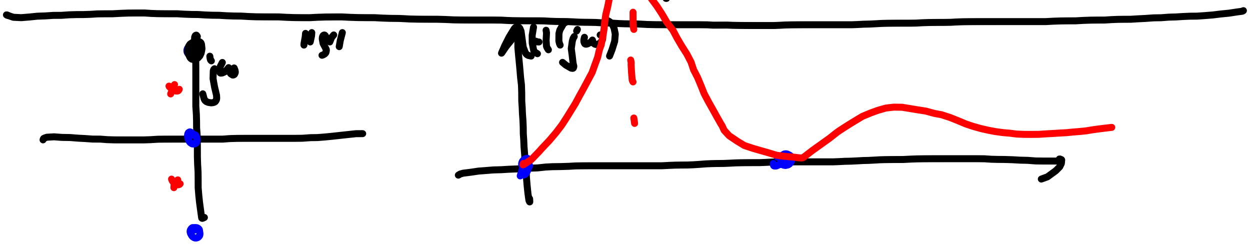
transfer function H(s) → factorization of polynomials → H(s) represented as nulls/poles

H(s) → H(jω): numerically $s=j\omega$

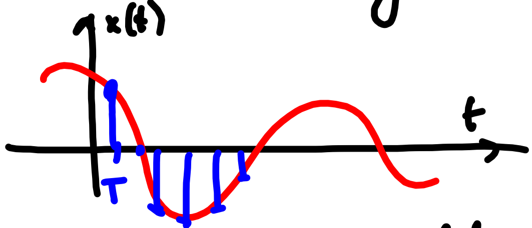
by hand



→ stability.



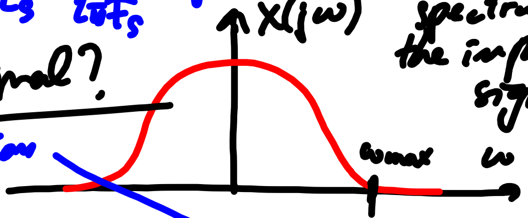
Sampling cont. time → discrete time



T · sampling period [s]

$F_s = \frac{1}{T}$ sampling frequency

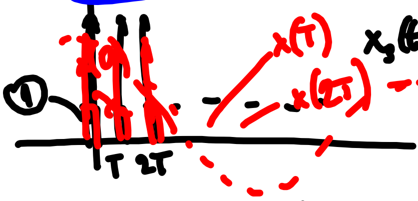
$\Omega_s = 2\pi F_s$ angular sampling freq.



spectrum of the input signal

Spectrum of sampled signal?

$\boxed{x_s(t) = x(t)\,s(t)}$ multiplication → convolution in spectrum

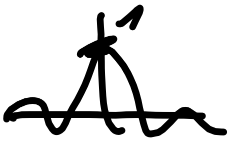


$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)$

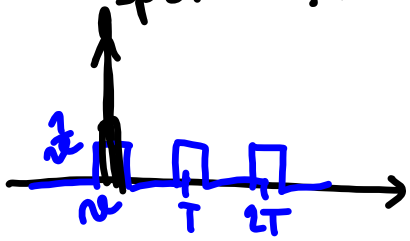
Spectrum of sampling signal

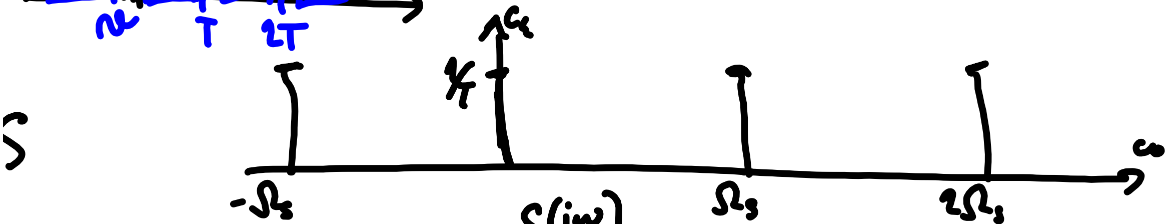
F. series

$c_k = \frac{Dw}{T}\,\text{sinc}\left(\frac{w}{2}k\Omega_s\right) = \frac{1}{T}\cdot\text{sinc}\left(\frac{w}{2}k\Omega_s\right)$

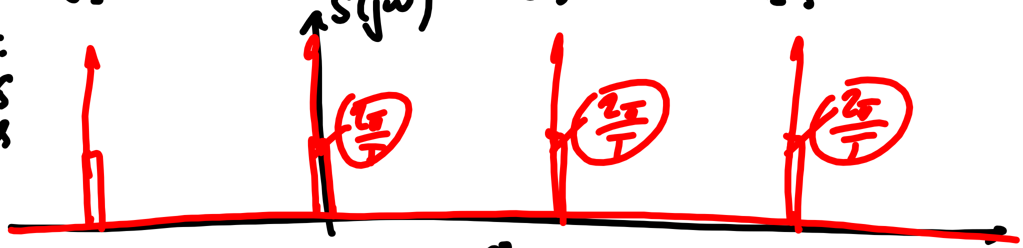


$c_0 = \frac{1}{T}\text{sinc}(0k\Omega_s) = \frac{1}{T}$





F.S. → F.T. plot them as Dirac impulses with sizes $2\pi c_k$



$$X_s(\omega) = X(\omega) * S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\hat\omega)\,S(\omega-\hat\omega)\,d\hat\omega$$

Shannon - Kotelnikov - Nyquist - sampling theorem

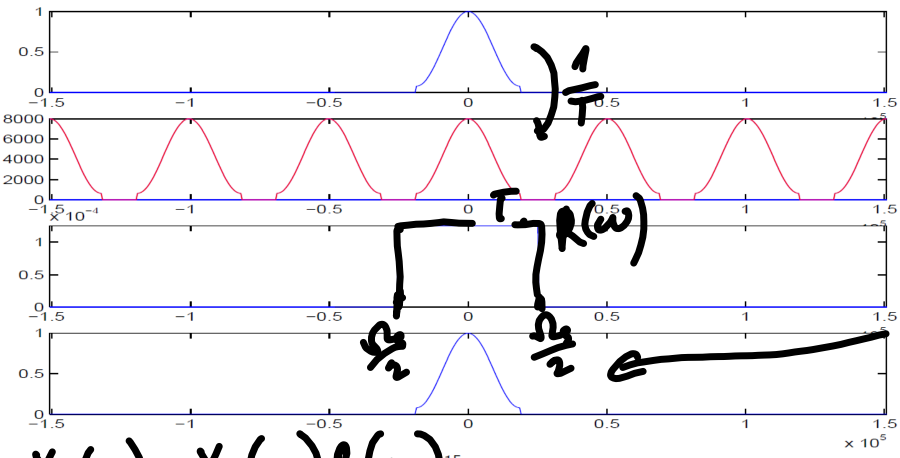
$$\boxed{2\omega_{max} < \Omega_s}$$

Aliasing if not fulfilled.

Countering this:
- increase F_s :(
- use anti-aliasing filter

Reconstruction

in freq:



$X_r(\omega) = X_s(\omega) R(\omega)$

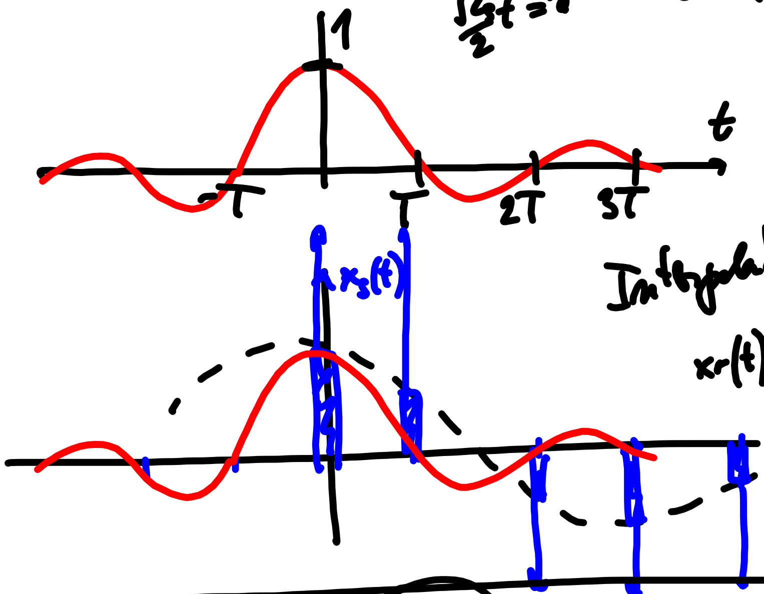
in time: $x_r(t) = x_s(t) * r(t)$ ← impulse response of the reco. filter.

$$r(t) = \text{IFT}\{R(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega) e^{+j\omega t} d\omega$$

$$\int_{-b}^{b} e^{\pm jxy} dy = 2b\,\text{sinc}(bx)$$

$r(t) = \text{sinc}\left(\frac{\Omega_s}{2} t\right)$ impulse resp of the reco. filter.

$\frac{\Omega_s}{2} t = \pi$ $\quad t = \frac{2\pi}{\Omega_s} = \frac{\cancel{2\pi}}{\cancel{2\pi} F_s} = \frac{1}{F_s} = T$



$x_r(t) = x_s(t) * r(t)$

Interpolation by cardinal sines!

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\left(\frac{\Omega_s}{2}(t - nT)\right)$$

$x[nT] \longrightarrow x[n]$

$n = \frac{nT}{T}$ normalization of time

$f' = \frac{f}{F_s}$ normalized frequency

$\omega' = \frac{\omega}{F_s}$