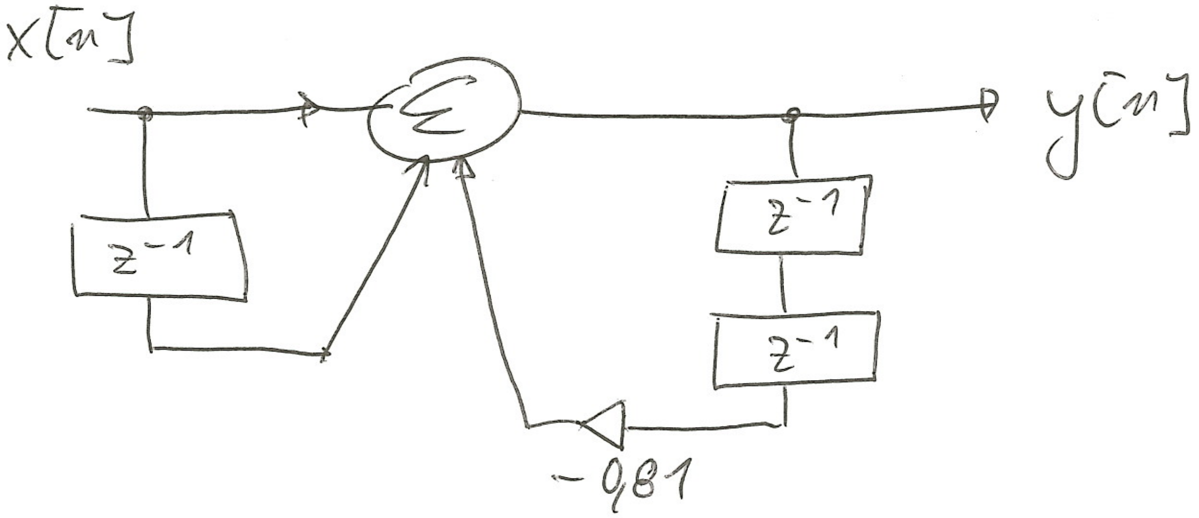
ISS Theoretical exercise for lab #6 ①

~~6 (2012/13) or 5 (2013/14 +) :-)~~

A digital filter is given by its scheme



① Determine its difference equation:

just rewrite the scheme into equation:

$$y[n] = x[n] + x[n-1] - 0{,}81\, y[n-2]$$

(you can remind the students that the general form is $y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k]$)

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② Perform Z-transform of this equation and write transfer function of the filter.

3 rules of z-transform:
1) $x[n] \longrightarrow X(z)$
2) constant $\longrightarrow$ constant
3) $x[n-k] \longrightarrow X(z)\, z^{-k}$

$$Y(z) = X(z) + X(z) z^{-1} - 0{,}81\, Y(z) z^{-2}$$

3 now group things belonging to X and Y:

$$Y(z) + 0{,}81\, Y(z) z^{-2} = X(z) + X(z) z^{-1}$$

$$Y(z)\left[1 + 0{,}81 z^{-2}\right] = X(z)\left[1 + z^{-1}\right]$$

it is now easy to find: $H(z) = \dfrac{Y(z)}{X(z)}$ just by re-shuffling the terms...

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 + 0{,}81 z^{-2}}$$ this is it!

$$H(z) = \frac{1 + z^{-1}}{1 + 0{,}81 z^{-2}}$$

general form of this is $H(z) = \dfrac{\sum_{k=0}^{Q} b_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}}$.

4 Find coefficients b_k of the numerator and a_k of the denominator.

This is pretty easy by looking at this.

$b_0 = 1 \quad b_1 = \cancel{0{,}81}\ 1 \quad a_2 = 0{,}81$.

They are visible also in the scheme and difference equation, a_k's with negative sign!

5 Convert $H(z)$ to the form with zeros and poles.

First, we need to convert the thing into the form with positive powers of z:

$$H(z) = \frac{1+z^{-1}}{1+0{,}81\,z^{-2}} = \frac{z^{-1}(z+1)}{z^{-2}(z^2+0{,}81)} = \frac{z(z+1)}{z^2+0{,}81}$$

(6) Let us work on looking for roots of the polynomials... numerator!

another one is $z_2 = 0$

$$z + 1 = 0$$

$$z_1 = -1 \quad \text{... root of numerator!}$$

(7) $z^2 + 0{,}81 = 0$ ... ouff, this looks like quadratic equation...

$$z_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 0{,}81}}{2 \cdot 1} =$$

$$= \frac{\pm\sqrt{-4 \cdot 0{,}81}}{2} \quad \text{simplifying...} \quad \frac{\pm\sqrt{4}\,\sqrt{-0{,}81}}{2} =$$

$$= \pm\sqrt{-0{,}81} \quad \text{this is a complex number!}$$

$z_1 = +0{,}9j \qquad z_2 = -0{,}9j \leftarrow$ roots of the denominator.

$$\cancel{H(z) = \frac{z^{-1}(z-(-1))(z-0)}{z^{-2}(z-0{,}9j)(z+0{,}9j)}}$$

~~trying to get rid of this...~~ $\cancel{\frac{z^{-1}}{z^{-2}} = z}$

8 ~~this z will actually create one more zero, as z = z - 0~~

$$H(z) = \frac{(z-0)(z-(-1))}{(z-0{,}9j)(z+0{,}9j)}$$

... general form is

$$H(z) = b_0 z^{p-q} \frac{\prod_{k=1}^{q} (z - n_k)}{\prod_{k=1}^{p} (z - p_k)}$$

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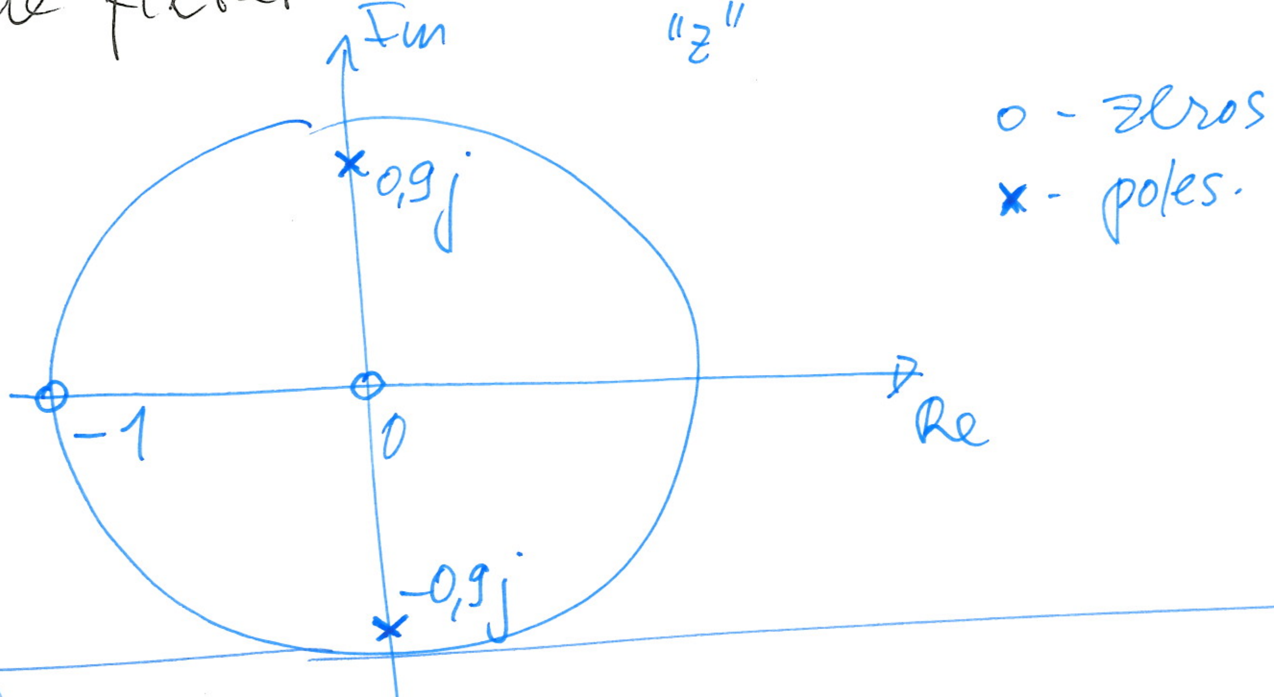
9 Draw the zeros and poles to complex plane "z" and check the stability of the filter.



10 For stability, poles must be inside unit circle, or $|p_k| < 1$. This is true ⇒ **STABLE**

6) Using zeros and poles, estimate the magnitude and phase of the <u>frequency response</u> of the filter in three (normalized angular) frequencies: $\omega_1 = 0$, $\omega_2 = \pi/2$, $\omega_3 = \cancel{\pi}\ 0{,}999\pi$

Try to draw the complete freq. response and compare with the one computed by Matlab.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

we look for all z's and replace them with $e^{j\omega}$!

$e^{j\omega}$ lies on the unit circle.

$$H(e^{j\omega}) = \frac{\overbrace{(e^{j\omega}-0)}^{\spadesuit}\overbrace{(e^{j\omega}-(-1))}^{\clubsuit}}{\underbrace{(e^{j\omega}-0{,}9j)}_{\heartsuit}\underbrace{(e^{j\omega}-(-0{,}9j))}_{\diamondsuit}}$$

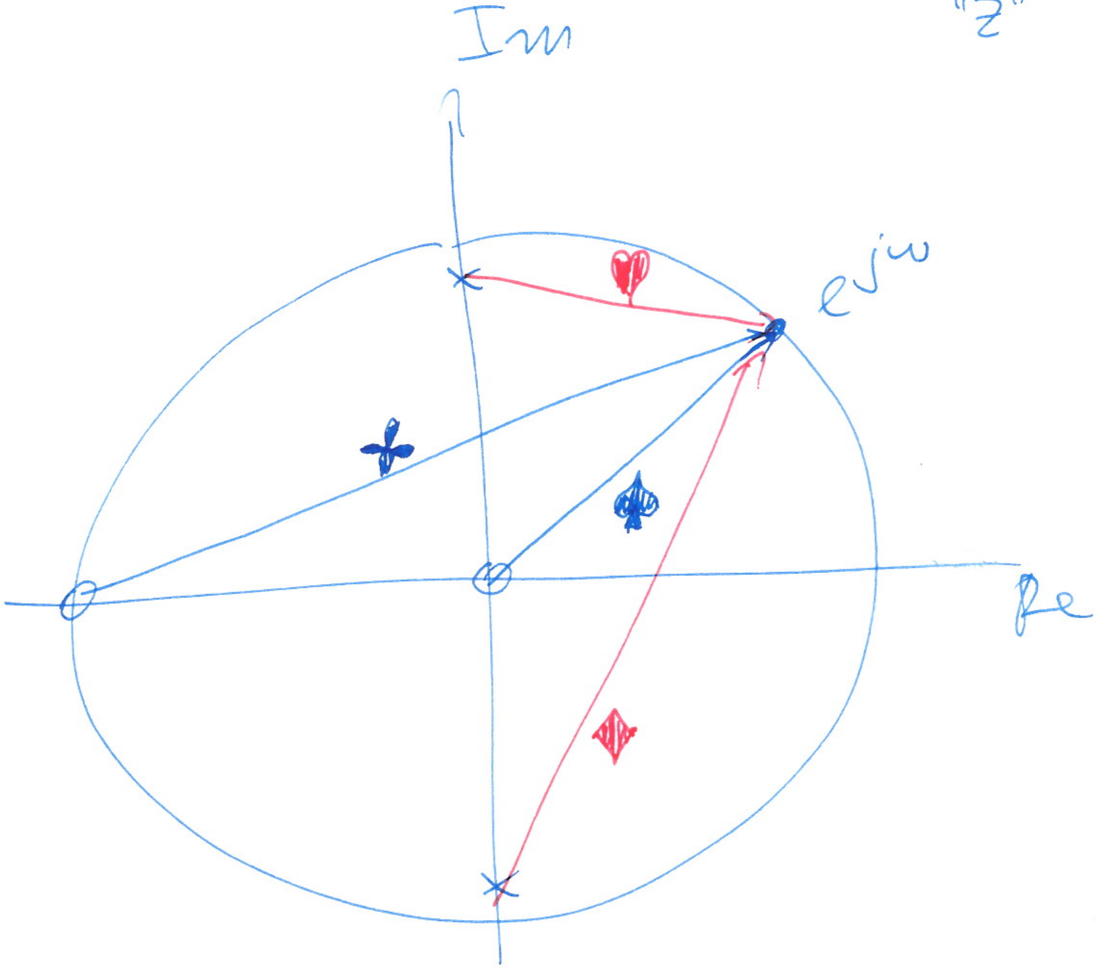
see next page!

Each bracket in this is a <u>complex number</u> given by a difference of $e^{j\omega}$ and the respective zero (or pole).

The <u>magnitude</u> of the result will be

$$|H(e^{j\omega})| = \frac{|\spadesuit|\cdot|\clubsuit|}{|\heartsuit|\cdot|\diamondsuit|}$$

we work with absolute values.


Im
"z"
$e^{j\omega}$
Re

The angle will be

$$\arg H(e^{j\omega}) = \arg ♠ + \arg ♣ - \arg ♥ - \arg ♦$$

↑ arguments.

These are the standard rules for multiplication and division of complex numbers.

In the complex plane, we can imagine ♠, ♣, ♥ and ♦ as vectors starting in respective zero or pole and ending in $e^{j\omega}$. $|.|$ is the length of such vector and arg. is the angle of this vector respective to the real axis.

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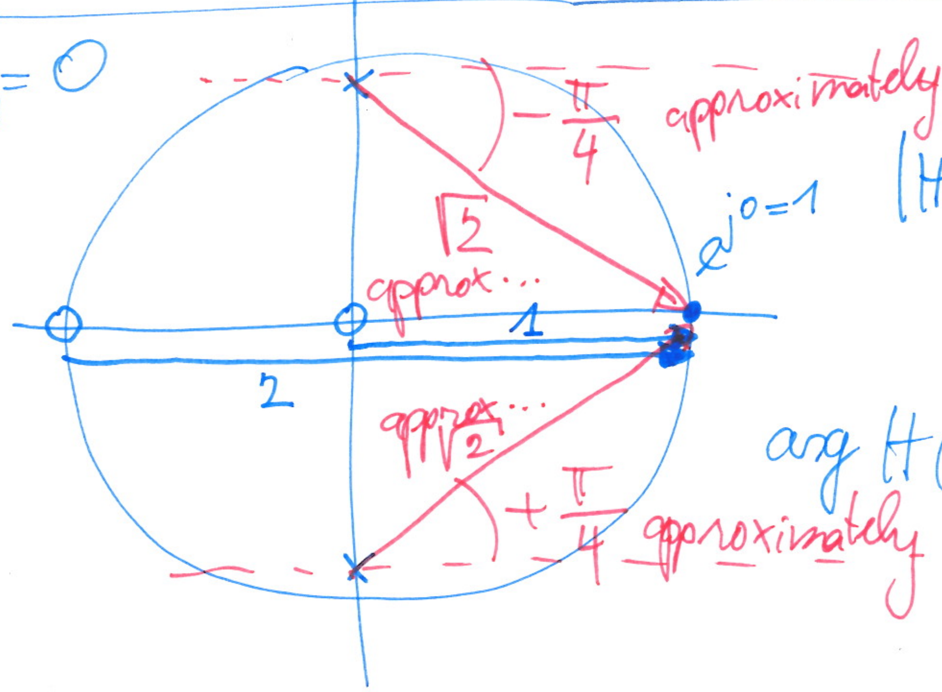
$\omega_1 = 0$



$$|H(e^{j0})| = \frac{1 \cdot 2}{\sqrt{2} \cdot \sqrt{2}} = \underline{\underline{1}}$$

$$\arg H(e^{j0}) = 0 + 0 - \left(-\frac{\pi}{4}\right) - \frac{\pi}{4} = \underline{\underline{0}}$$

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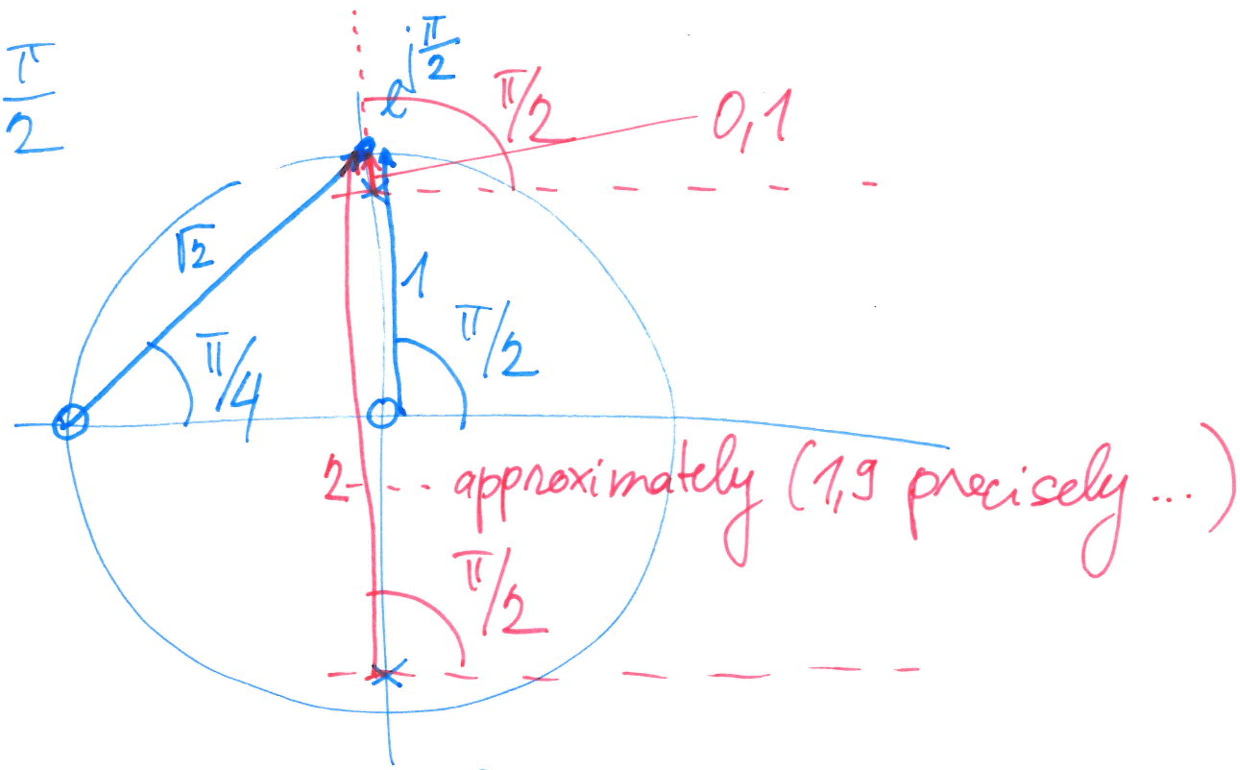
$\omega_2 = \frac{\pi}{2}$



$$\left|H\left(e^{j\frac{\pi}{2}}\right)\right| = \frac{1\cdot\sqrt{2}}{0{,}1\cdot 2} = 10\cdot\frac{1}{\sqrt{2}} = \underline{\underline{7}}$$

$$\arg H\left(e^{j\frac{\pi}{2}}\right) = \pi/2 + \pi/4 - \pi/2 - \pi/2 = \underline{\underline{-\frac{\pi}{4}}}$$

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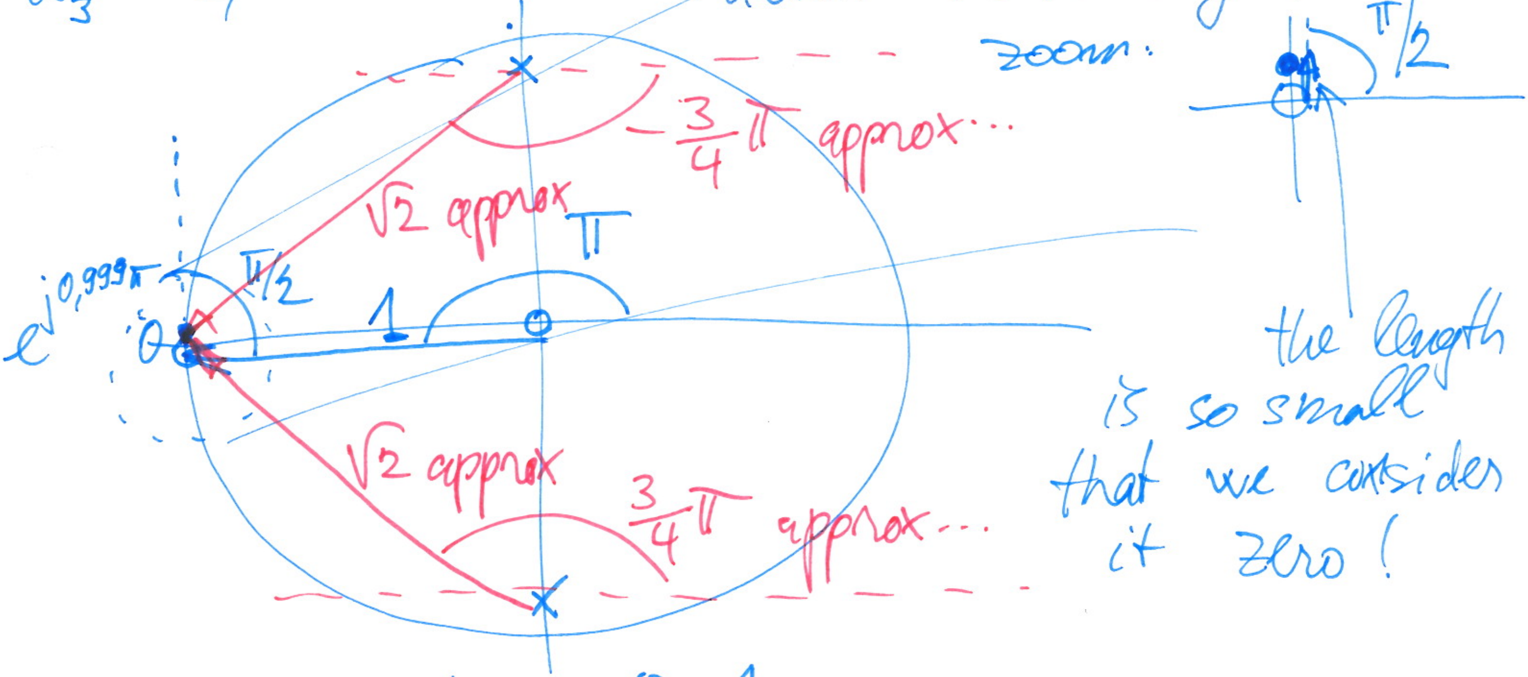
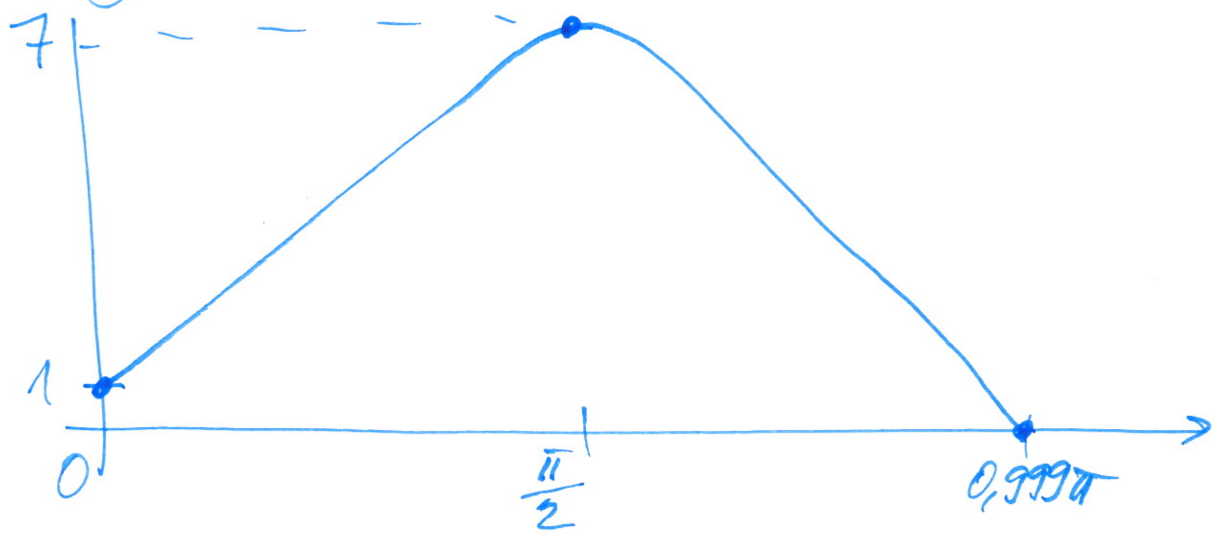
$\omega_3 = 0{,}999\pi$ — not π, because for we could not determine the angle...

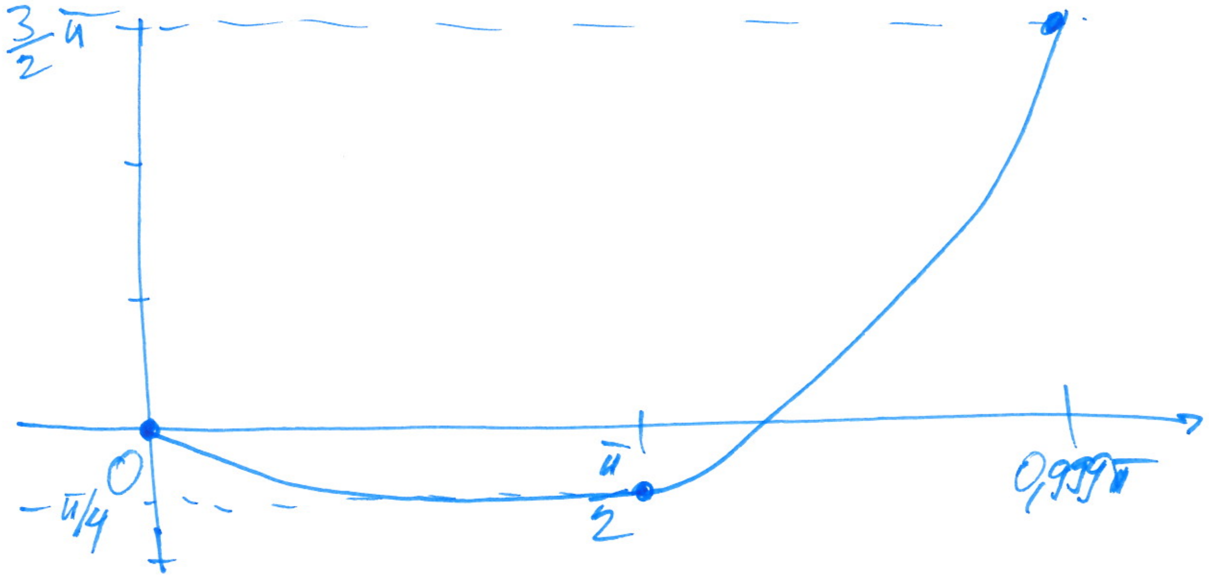


$$\left|H\left(e^{j0{,}999\pi}\right)\right| = \frac{0\cdot 1}{\sqrt{2}\cdot\sqrt{2}} = \underline{\underline{0}}$$

$$\arg H\left(e^{j0{,}999\pi}\right) = \pi + \frac{\pi}{2} - \left(-\frac{3}{4}\pi\right) - \frac{3}{4}\pi = \underline{\underline{\frac{3}{2}\pi}}$$

Let's try to draw it!





Now let the students look at the correct one (~~eps~~ PDF included).

Magnitude: not bad. clear band-pass filter.

Phase: well.... we'd need more points and consider that $\frac{3}{2}\pi \approx -\frac{\pi}{2}$.

16. See the Google sheet ... pretty standard formulae ...

$$a[n] = \frac{1}{\Omega}\sum_{\omega=1}^{\Omega} \xi_\omega[n]$$

17.
$$D[n] = \frac{1}{\Omega}\sum_{\omega=1}^{\Omega}\left(\xi[n] - a[n]\right)^2$$

18.
$$\sigma[n] = \sqrt{D[n]}$$

19. If the signal is stacionary, the values of parameters are the same.

$$a[7] = a[5], \quad D[7] = D[5], \quad \sigma[7] = \sigma[5]$$

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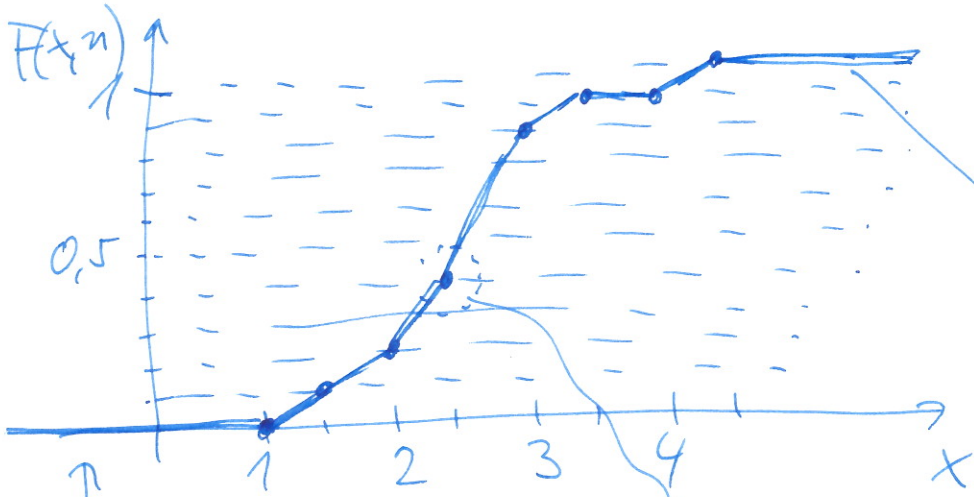
20. Counting the values "smaller than x" (remember the definition of CPDF...)

x	1	1,5	2	2,5	3	3,5	4	4,5
count	0	1	2	4	8	9	9	10
$F(x,n) = \frac{\text{count}}{\Omega}$	0	0,1	0,2	0,4	0,8	0,9	0,9	1,0



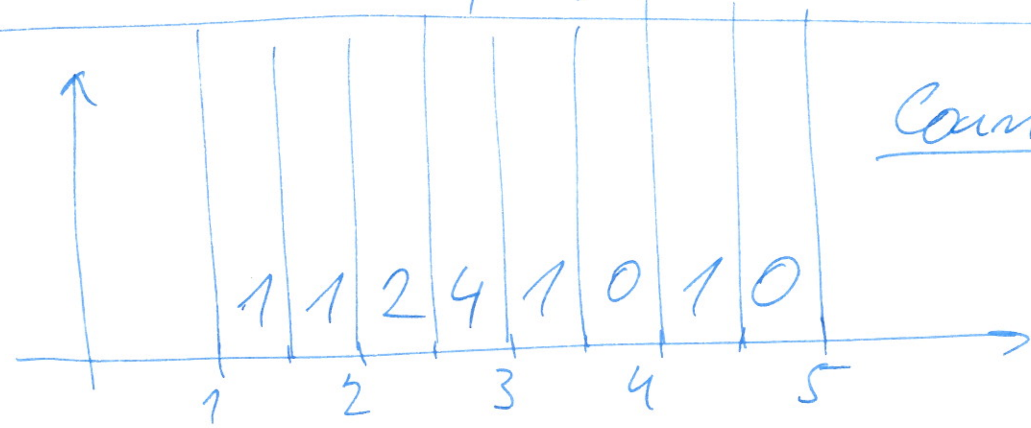
everything smaller than 4,5 … → are.

nothing smaller than 1 ⇒ zero

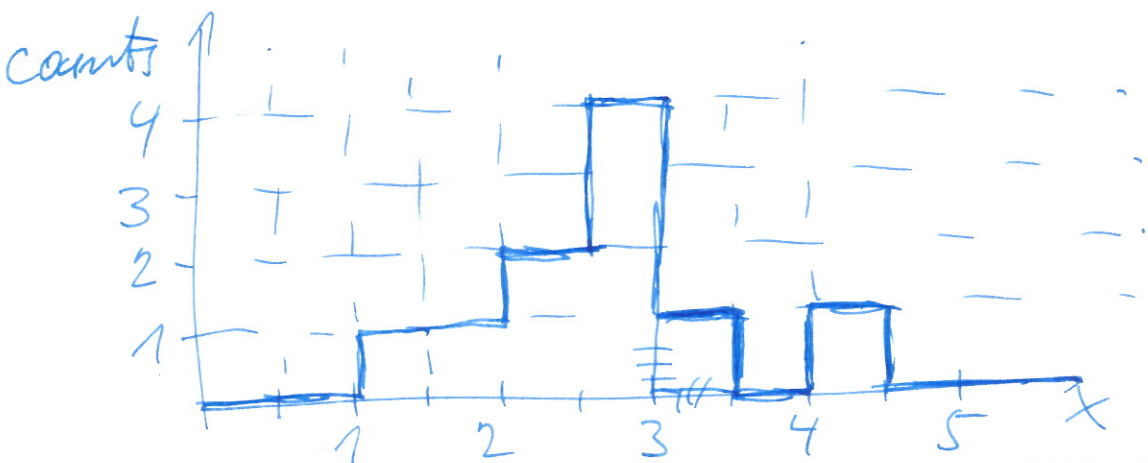
(21)

$$P\{\xi[5] \geq 2{,}5\} = 1 - P\{\xi[n] < 2{,}5\} =$$
$$= 1 - F(2{,}5;5) = 1 - 0{,}4 = \underline{\underline{0{,}6}}$$

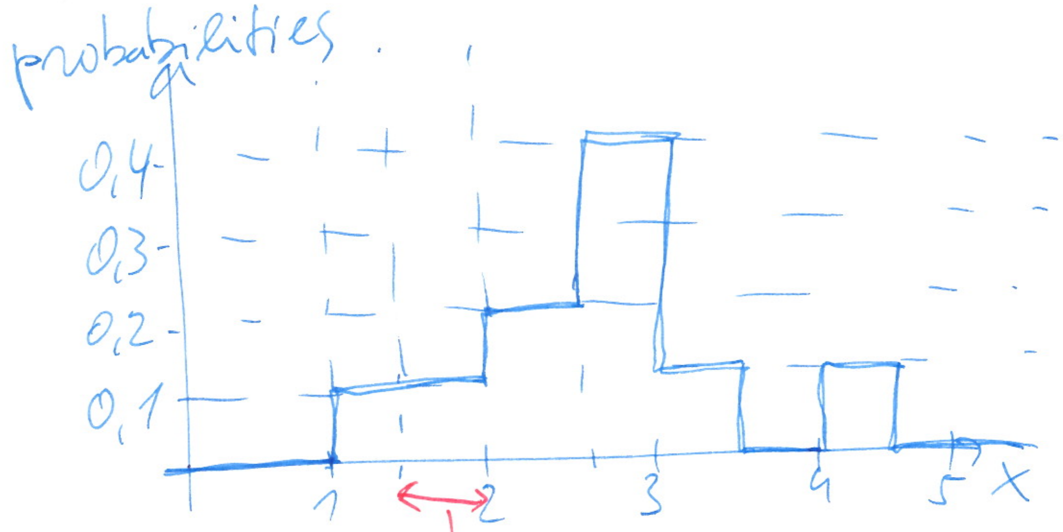
(22)



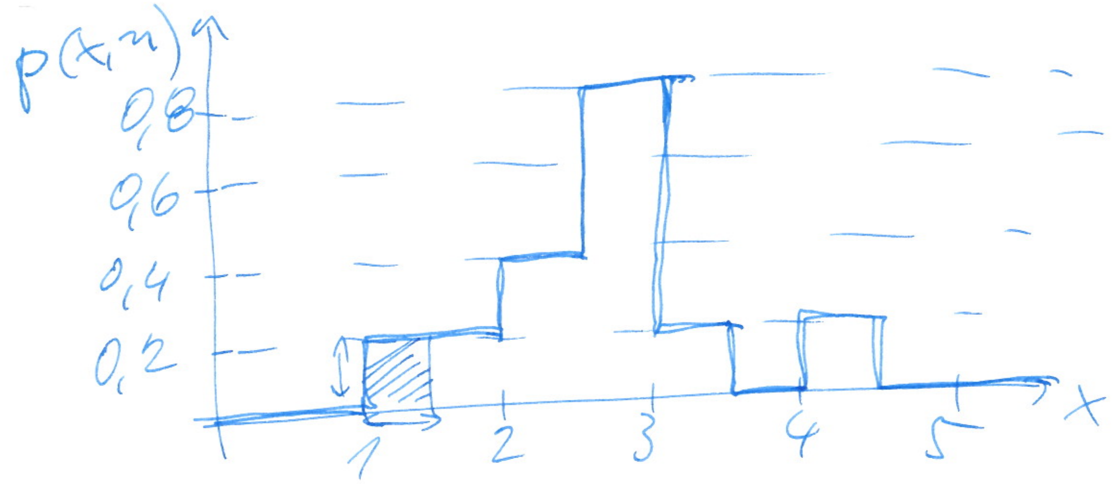
Counts



(23) Probabilities: divide by the number of realizations! By 10!



(24) For PDF you need to divide by the width of the interval: 0,5.



(25) just compute the surface under the curve...

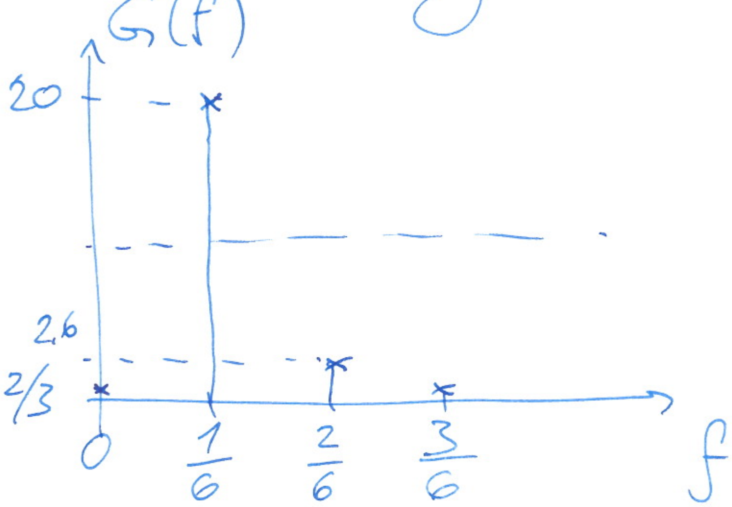
$0{,}2 \cdot 0{,}5 + 0{,}2 \cdot 0{,}5 + \ldots\ldots =$

$= (0{,}2 + 0{,}2 + 0{,}4 + 0{,}8 + 0{,}2 + 0{,}2) \cdot 0{,}5 =$

$= 2 \cdot 0{,}5 = \underline{1}$ ok!

(26) etc - see Google sheets!
to (36)

37 Converting k to normalized freq. - just division by N.



$$\frac{4}{6} = \frac{2}{3}$$

$$\frac{120}{6} = 20$$

$$\frac{16}{6} = 2{,}6$$

38 $f = \frac{1}{6}$.

39 The signal seems to have period $N = 8$

$$\underbrace{2 \quad 4 \quad 2 \quad 0 \quad -2 \quad -4 \mid -2 \quad 0}_{0}$$

So that its peak should be at $f = \frac{1}{N} = \frac{1}{8}$. We don't have resolution to see it properly but it's not too far from $\frac{1}{6}$.

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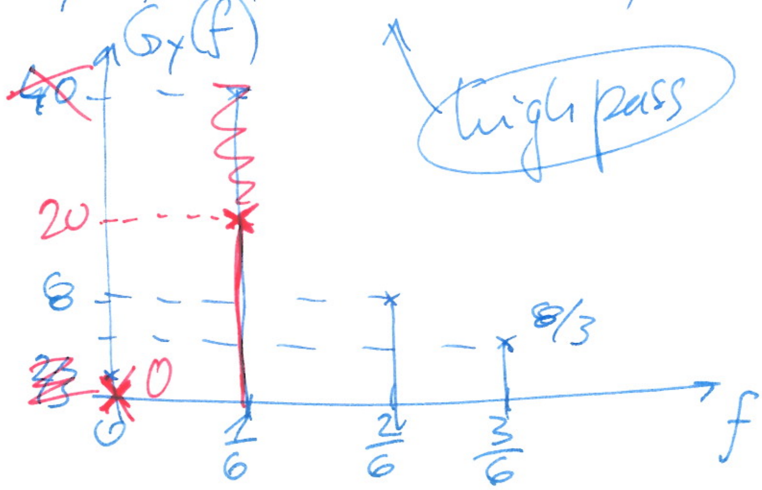
40 $G_Y(f) = |H(f)|^2 \cdot G_X(f)$

high pass



~~0.~~ $\frac{2}{3} \longrightarrow \cancel{\frac{2}{3}}$ 0

~~1.~~ $20 \longrightarrow \cancel{40}$ 20

3. $2{,}6 \longrightarrow 8$

4. $\frac{2}{3} \longrightarrow \frac{8}{3}$

Maximums has the same position.