Static Analysis and Verification

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Lattices and Fixpoints
A Brief Introduction

Partial Orders

- A tuple $(A, \leq_A)$ is a poset (partially-ordered set) iff A is a set and $\leq_A \subseteq A \times A$ is a partial order (i.e., a reflexive, transitive, and antisymmetric binary relation) on A.

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    1. $\forall b \in B.\ a \leq_A b$ ("lower bound") and
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❖ An example: For $(2^{\{a,b,c\}}, \subseteq)$, $\sqcup\{\emptyset, \{a\}, \{b\}\} = \{a, b\}$, and $\{a\} \sqcap \{b, c\} = \emptyset$.

Lattices

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- Given a poset $(A, \leq_A)$, a set $B \subseteq A$ is a chain iff $\forall b, b' \in B.\ b \leq_A b' \ \vee\ b' \leq_A b$.
  - E.g., $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ is a chain wrt. $(2^{\{a,b,c\}}, \subseteq)$.

Functions on Lattices

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    - 0 is the least fixpoint of f and ∞ is the greatest fixpoint of f.

Knaster–Tarski Theorem

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❖ A dual result holds for the greatest fixpoint.

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    - Moreover, a $\sqcup$-continuous function is monotone, and hence one in fact computes the supremum of the ascending chain $\bot_A \leq_A f(\bot_A) \leq_A f(f(\bot_A)) \leq \ldots$.

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- If f is $\sqcap$-continuous, the greatest fixpoint of f is $\nu f = \sqcap\{f^i(\top_A) \mid i \geq 0\}$.
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- **Theorem.** For finite complete lattices, every monotonic function is $\sqcap$- and $\sqcup$-continuous.

- **Corollary.** On finite lattices, the Kleene fixpoint theorem is applicable, hence,
  - to compute the least fixpoint, start with $\bot_A$ and iteratively apply f till $f^i(\bot_A) = f^{i+1}(\bot_A) = \mu f$,
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